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Homogeneous systems of higher-order ordinary differential equations

Mike Crampin

Abstract. The concept of homogeneity, which picks out sprays from the general run of systems of second-order ordinary differential equations in the geometrical theory of such equations, is generalized so as to apply to equations of higher order. Certain properties of the geometric concomitants of a spray are shown to continue to hold for higher-order systems. Third-order equations play a special role, because a strong form of homogeneity may apply to them. The key example of a single third-order equation which is strongly homogeneous in this sense states that the Schwarzian derivative of the dependent variable vanishes. This equation is of importance in the theory of the association between third-order equations and pseudo-Riemannian manifolds due to Newman and his co-workers.

1 Introduction

In the geometrical theory of systems of second-order ordinary differential equations an important role is played by a class of equations which are homogeneous in a certain sense. In the theory envisaged here a system of second-order equations in m dependent variables is represented by a vector field Γ of a special type on the tangent bundle $T(M)$ of an m-dimensional smooth manifold M. A tangent bundle comes equipped with a canonical vertical vector field Δ, the Liouville field, which is the infinitesimal generator of dilations of the fibres. Then the equations are homogeneous if the corresponding differential equation field Γ satisfies $[\Delta, \Gamma] = \Gamma$ (so one might say it is homogeneous of degree 1). Such differential equation fields, which may in fact be defined not on the whole of $T(M)$ but only on the slit tangent bundle ($T(M)$ with its zero section deleted), are often called sprays. Examples include the geodesic field of a Finsler space, and a fortiori that of a Riemannian space, or indeed of any affine connection. It should be mentioned that

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this paper deals only with equations which do not explicitly involve the independent variable; in a dynamical context such equations could be called autonomous, and the examples mentioned above are clearly of this type. For a textbook account of the geometry of sprays see [13].

The aim of the present paper is to propose a definition — or as it turns out, definitions — of homogeneity for systems of higher-order ordinary differential equations, where 'higher-order' means of course 'of order higher than the second'; and to establish under these definitions properties that parallel those enjoyed by sprays in the second-order case.

To describe the properties of sprays that will be generalized I first need to say somewhat more about the geometrical theory of systems of second-order differential equations. A standard reference here is [3]. The main aim of the theory may be described as the formulation of differential-geometric concomitants of the equation field Γ, or in other words associated geometric objects which do not depend on the choice of coordinates. (Despite what has just been said, I must admit to the rather frequent use of coordinates below — but only in order to simplify or speed up the exposition.) For my purposes here there are three important concomitants of a second-order differential field Γ. These involve in their specifications so-called vector fields along the tangent bundle projection $\tau : T(M) \to M$, or in other words sections of the pull-back bundle $\tau^*T(M) \to T(M)$. I make a short detour to mention a couple of useful features of this construction. There is a canonical section of $\tau^*T(M)$, given in local coordinates by $\dot{y}^i\partial/\partial y^i$, where (y^i), $i = 1, 2, \dots, m$ are coordinates on M and $(\dot{y}^i)$ the corresponding fibre cordinates on $T(M)$; this section is sometimes called the total derivative; I shall denote it by T. Given any section Y of $\tau^*T(M)$ there is a canonical associated vertical vector field on $T(M)$, its vertical lift, denoted by Y^V. The three geometric objects associated with a second-order differential equation field are

1. its *dynamical covariant derivative*: this is an $\mathbb{R}$-linear operator

$$\nabla : \operatorname{sect} \tau^*T(M) \to \operatorname{sect} \tau^*T(M)$$

satisfying the covariant-derivative-like property $\nabla(fY) = f\nabla Y + \Gamma(f)Y$ for $f \in C^\infty(T(M))$;

2. its *horizontal distribution*: this is a distribution on $T(M)$, complementary to the vertical distribution, which can be thought of as defining a nonlinear connection, and leads to a horizontal lift operation taking $Y \in \operatorname{sect} \tau^*T(M)$ to Y^H, a horizontal vector field on $T(M)$;

3. its *Jacobi endomorphism*: this is a $C^\infty(T(M))$-linear map

$$\Phi : \operatorname{sect} \tau^*T(M) \to \operatorname{sect} \tau^*T(M)\,,$$

so called because Jacobi's equation, which is the equation satisfied by variation fields along an integral curve of Γ, can be expressed as $\nabla^2 Y + \Phi(Y) = 0$.

The three are related as follows:

$$[\Gamma, Y^V] = -Y^H + (\nabla Y)^V, \qquad [\Gamma, Y^H] = (\nabla Y)^H + \Phi(Y)^V.$$

When Γ is a spray these objects have the following properties (see for example [5], [13]):

1. the horizontal distribution is invariant under Δ: if $Y \in \mathfrak{X}(M)$ then $[\Delta, Y^H] = 0$;

2. the Jacobi endomorphism is homogeneous of degree 2: $\mathcal{L}_\Delta \Phi = 2\Phi$ (where the Lie derivative is used in a sense to be explained fully later);

3. Γ is horizontal with respect to the horizontal distribution it defines, and in fact $\Gamma = T^H$;

4. $\Phi(T) = 0$.

In Section 3 below I describe how one may define analogues of the dynamical covariant derivative, the horizontal distribution and the Jacobi endomorphism for a differential equation field of any order. There are in fact several inequivalent ways of extending these concepts to equations of higher order, which are discussed in [1], [2], [4], [6], [8], [12]. The one adopted here is based on an approach first given for fourth-order equations in [1], and extended to equations of arbitrary order in [8]. It is to be preferred, in my opinion, because it gets most quickly to the generalized Jacobi equation.

In Section 4 I give a definition of homogeneity for a differential equation field of any order, and show that properties 1 and 2 above hold, mutatis mutandis, for homogeneous fields with this definition. There is a stronger sense of homogeneity, but it applies only to equations of the third order. I explain this in Section 5, and show that the remaining two properties then hold.

Finally in Section 5 I obtain the most general strongly homogeneous third-order differential equation in a single dependent variable. This turns out to be essentially the vanishing of the Schwarzian derivative of the dependent variable. Single third-order differential equations are of interest because of their association in certain cases with pseudo-Riemannian spaces, an association rediscovered as a result of an approach to problems in general relativity initiated by E. T. Newman and his co-workers in [9]. The equations which feature in these studies are picked out by the vanishing of a certain invariant called the Wuenschmann invariant: this has a simple expression in terms of the Jacobi endomorphisms, as has been shown in [8], and also more recently in [4]. It turns out that each member of the class of strongly homogeneous third-order differential equations in a single dependent variable mentioned above has vanishing Jacobi endomorphisms, and ipso facto vanishing Wuenschmann invariant. The properties of pseudo-Riemannian spaces associated with these equations have been studied in [10], [11].

The paper proper begins with an account of the space which plays the role of the tangent bundle for equations of order n, namely the bundle of n-velocities.

2 The bundle of n-velocities

Let M be a smooth manifold of dimension m, with local coordinates (y^i), $i = 1, 2, \ldots, m$. Let $T^n(M)$ be the bundle of n-velocities on M: a point of $T^n(M)$ is an equivalence class of curves σ in M, defined on an open interval containing 0, under the equivalence relation $\sigma \equiv \rho$ if in one, and hence any, coordinate representation

$$\frac{d^r\sigma^i}{dx^r}(0) = \frac{d^r\rho^i}{dx^r}(0), \quad r = 0, 1, \dots, n.$$

Denote by $(y^i_r) = (y^i_0, y^i_1, \dots, y^i_n)$ the natural coordinates on $T^n(M)$, so that (y^i_r) are the coordinates of the equivalence class of the curve

$$y^i(x) = y^i_0 + xy^i_1 + \cdots + \frac{1}{r!}x^r y^i_r + \cdots + \frac{1}{n!}x^n y^i_n.$$

Evidently $T^n(M)$ is fibred over $T^r(M)$ for $r = 0, 1, \dots, n-1$, where $T^0(M) = M$. The corresponding projections are denoted by $\tau_r : T^n(M) \to T^r(M)$; we have $\tau_r(y^i_0, y^i_1, \dots, y^i_n) = (y^i_0, y^i_1, \dots, y^i_r)$. A vector $v \in T(T^n(M))$ such that $\tau_{0*}v = 0$ is said to be vertical; one such that $\tau_{r*}v = 0$ to be vertical over $T^r(M)$; on occasion, one vertical over $T^{n-1}(M)$ to be very vertical. At any point of $T^n(M)$, the space of vectors vertical over $T^r(M)$ is spanned by the coordinate vectors $\partial/\partial y^i_s$ for $s > r$. For $n \geq r \geq 1$, denote the vector sub-bundle of $T(T^n(M))$ consisting of vectors vertical over $T^{r-1}(M)$ by V_r, so that in particular the vector sub-bundle consisting of vertical vectors (vectors vertical over M) is V_1. Then

$$V_n \subset V_{n-1} \subset \cdots \subset V_1 \subset T(T^n(M))$$

is a filtration of $T(T^n(M))$. One can identify V_n, and V_{r-1}/V_r, with $\tau_0^*(T(M))$, the pull-back by $\tau_0 : T^n(M) \to M$ of the tangent bundle $T(M) \to M$. I shall denote by $\mathcal{V}_r$ the module of vector fields on $T^n(M)$ which are vertical over $T^{r-1}(M)$, or in other words, sections of $V_r \to T^n(M)$. Then

$$\mathcal{V}_n \subset \mathcal{V}_{n-1} \subset \cdots \subset \mathcal{V}_1 \subset \mathfrak{X}(T^n(M)).$$

The type $(1,1)$ tensor field S on $T^n(M)$ given by

$$S = \sum_{r=1}^{n} r\frac{\partial}{\partial y^i_r} \otimes dy^i_{r-1}$$

is called the vertical endomorphism. Evidently $S(\mathfrak{X}(T^n(M))) = \mathcal{V}_1$, and for $1 \leq r \leq n-1$, $S(\mathcal{V}_r) = \mathcal{V}_{r+1}$, while $S(\mathcal{V}_n) = \{0\}$.

The additive group $\mathbb{R}$ acts on $T^n(M)$ as follows. For any curve σ in M, and for $t \in \mathbb{R}$, we may define a curve σ_t by $\sigma_t(x) = \sigma(e^t x)$. This map of curves defines a map of n-velocities, given in coordinates by $(y^i_r) \mapsto (e^{rt}y^i_r)$. The corresponding (vertical) vector field on $T^n(M)$ is

$$\Delta = y^i_1\frac{\partial}{\partial y^i_1} + 2y^i_2\frac{\partial}{\partial y^i_2} + \cdots + ny^i_n\frac{\partial}{\partial y^i_n}.$$

It is the fundamental vector field of the action corresponding to the vector field $t\partial/\partial t$ on $\mathbb{R}$.

Vector fields Δ^r on $T^n(M)$, $r = 1, 2, \ldots, n$, are defined as follows:

$$\Delta^{r+1} = S(\Delta^r), \quad \Delta^1 = \Delta.$$

Then $S(\Delta^n) = 0$. It may be shown that these vector fields satisfy

$$[\Delta^r, \Delta^s] = \begin{cases} (r-s)\Delta^{r+s-1} & \text{for } r+s-1 \leq n \\ 0 & \text{otherwise.} \end{cases}$$

In particular, they form an n-dimensional Lie algebra, say $\mathfrak{D}$. The generators of this algebra may be related to vector fields on $\mathbb{R}$ in a way that extends the relation between Δ and $t\partial/\partial t$ described above. Let $\mathfrak{p}$ be the Lie algebra of vector fields on $\mathbb{R}$ whose coefficients are (formal) power series in t, and let $\mathfrak{p}^n$ be the subalgebra of those vector fields which vanish to order n at 0, that is, whose coefficient begins with t^{n+1}. Then $\mathfrak{p}^0$ is the subalgebra of vector fields which vanish at 0; for $n > 0$, $\mathfrak{p}^n$ is an ideal in $\mathfrak{p}^0$; and $\mathfrak{D}$ is anti-isomorphic to $\mathfrak{p}^0/\mathfrak{p}^n$. In fact the map $\mathfrak{p}^0 \to \mathfrak{X}(T^n(M))$ by $t^r\partial/\partial t \mapsto S^{r-1}(\Delta)$ for $r \geq 1$ is an anti-homomorphism, with kernel $\mathfrak{p}^n$. Now the group D of diffeomorphisms of $\mathbb{R}$ which leave 0 fixed acts on $T^n(M)$ by reparametrization; the action leaves the fibres invariant and induces the identity on M. We may think of $\mathfrak{p}^0$ as playing the role of the Lie algebra of D, and the vector fields Δ^r as the fundamental vector fields of the action.

The space $\mathfrak{q}$ of all quadratic vector fields on $\mathbb{R}$ is a Lie algebra, and no space of all polynomial vector fields of some higher degree has that property. Then $\mathfrak{p}^0/\mathfrak{p}^2$ is isomorphic to $\mathfrak{q}^0$, the subalgebra of $\mathfrak{q}$ consisting of quadratic vector fields that vanish at 0. Correspondingly, $\{\Delta^1, \Delta^2\}$ span a 2-dimensional Lie algebra of vector fields on $T^n(M)$, which is anti-isomorphic to $\mathfrak{q}^0$.

3 Systems of $(n+1)$st-order differential equations

A differential equation field of order $n+1$ is a vector field Γ on $T^n(M)$ of the form

$$\Gamma = y_1^i \frac{\partial}{\partial y_0^i} + y_2^i \frac{\partial}{\partial y_1^i} + \cdots + y_n^i \frac{\partial}{\partial y_{n-1}^i} + f^i \frac{\partial}{\partial y_n^i}.$$

The integral curves of such a vector field, projected onto M, are the solutions of the system of $(n+1)$st-order ordinary differential equations

$$y_{n+1}^i = f^i(y^j, y_1^j, \ldots, y_n^j), \qquad y_r^i = \frac{d^r y^i}{dx^r}.$$

The vector field Γ is a geometrical expression for the system of differential equations. One geometrical approach to the study of systems of ordinary differential equations is to work with the corresponding vector field: this is the approach adopted here.

Notice that Γ is a differential equation field if and only if $S(\Gamma) = \Delta$.

If Γ is a differential equation field of order $n+1$ and $\hat{\Gamma}$ is another vector field on $T^n(M)$) then $\hat{\Gamma}$ is also a differential equation field of order $n+1$ if and only if differs from Γ by a very vertical vector field.

Associated with a differential equation field Γ there is a linear differential operator ∇ with properties reminiscent of those of a covariant derivative; accordingly, it is called the dynamical covariant derivative. It acts on sect $\tau_0^*(T(M))$, and satisfies $\nabla(fY) = f\nabla Y + \Gamma(f)Y$ for $f \in C^\infty(T^n(M))$ and $Y \in \operatorname{sect} \tau_0^*(T(M))$. In terms of coordinate fields ∇ is given by

$$\nabla\left(\frac{\partial}{\partial y^i}\right) = -\frac{1}{n+1}\frac{\partial f^j}{\partial y_n^i}\frac{\partial}{\partial y^j} = \Gamma_i^j\frac{\partial}{\partial y^j}.$$

One can easily check from this formula that ∇ is well-defined, in the sense that Γ_i^j transforms appropriately under a coordinate transformation $\bar{y}^i = \bar{y}^i(y^j)$ on M. Then (taking account of the Leibniz-like rule)

$$\nabla Y = \nabla\left(Y^i\frac{\partial}{\partial y^i}\right) = (\Gamma(Y^i) + Y^j\Gamma_j^i)\frac{\partial}{\partial y^i} = (\nabla Y^i)\frac{\partial}{\partial y^i}.$$

The dynamical covariant derivative may of course be made to act on other geometrical objects. In particular, it acts on $T(M)$-valued 1-forms on $T^n(M)$, that is to say, sections of $T^*(T^n(M)) \otimes \tau_0^*(T(M)) \to T^n(M)$, as follows: for $\theta \in \operatorname{sect} T^*(T^n(M))$ and $Y \in \operatorname{sect} \tau_0^*(T(M))$ set

$$\nabla(\theta \otimes Y) = \mathcal{L}_\Gamma\theta \otimes Y + \theta \otimes \nabla Y.$$

For convenience of calculation one may proceed as follows. Any section of

$$T^*(T^n(M)) \otimes \tau_0^*(T(M))$$

can be expressed as $\theta^i \otimes \partial/\partial y^i$ where (θ^i) is an m-tuple of 1-forms on $T^n(M)$, and the index i may be thought of as tensorial so far as coordinate transformations on M are concerned. Set

$$\nabla\left(\theta^i \otimes \frac{\partial}{\partial y^i}\right) = (\nabla\theta^i) \otimes \frac{\partial}{\partial y^i}.$$

This defines a new m-tuple of 1-forms on $T^n(M)$, $(\nabla\theta^i)$, and again the index is tensorial; explicitly,

$$\nabla\theta^i = \mathcal{L}_\Gamma\theta^i + \Gamma_j^i\theta^j.$$

It follows that for any vector field Z on $T^n(M)$

$$\theta^i([\Gamma, Z]) = \nabla(\theta^i(Z)) - (\nabla\theta^i)(Z).$$

This operation will now be applied repetitively, starting with $\theta^i = dy_0^i = \tau_{0*}(dy^i)$, in other words with the section of $T^*(T^n(M)) \otimes \tau_0^*(T(M))$ which, as a map, takes a vector v on $T^n(M)$ to $\tau_{0*}v$ considered as an element of $\tau_0^*(T(M))$ located at the same point of $T^n(M)$ as v. That is to say, we define 1-forms Θ_r^i, $0 \le r \le n$, by

$$\Theta_{r+1}^i = \nabla\Theta_r^i, \qquad \Theta_0^i = dy_0^i,$$

or in terms of $T(M)$-valued 1-forms, $\Theta_{r+1} = \nabla\Theta_r$. Now for $0 \le r < n$, $\mathcal{L}_\Gamma dy_r^i = dy_{r+1}^i$, whence for $0 \le r \le n$ we may write

$$\Theta_r^i = dy_r^i + \sum_{s=0}^{r-1}(C_r^s)_j^i dy_s^j$$

for some coefficients $(C^s_r)^i_j$ which are (local) functions on $T^n(M)$. These satisfy the recurrence relations

$$\begin{aligned}
(C^r_{r+1})^i_j &= (C^{r-1}_r)^i_j + \Gamma^i_j \\
(C^s_{r+1})^i_j &= \Gamma(C^s_r)^i_j + \Gamma^i_k(C^s_r)^k_j + (C^{s-1}_r)^i_j \quad 1 \le s \le r-1 \\
(C^0_{r+1})^i_j &= \Gamma(C^0_r)^i_j + \Gamma^i_k(C^0_r)^k_j.
\end{aligned}$$

In particular,

$$(C^r_{r+1})^i_j = (r+1)\Gamma^i_j.$$

From the formula $\Theta^i_r = dy^i_r + \sum_{s<r}(C^s_r)^i_j dy^j_s$ we see that $\{\Theta^i_r\}$, $0 \le r \le n$, is a local basis of 1-forms on $T^n(M)$. Let $\{Y^r_i\}$ be the dual basis. Then

$$Y^n_i = \frac{\partial}{\partial y^i_n}, \qquad Y^r_i = \frac{\partial}{\partial y^i_r} + \sum_{s=r+1}^{n}(D^r_s)^j_i \frac{\partial}{\partial y^i_s} \quad 0 \le r \le n-1,$$

for coefficients $(D^r_s)^j_i$ which may easily be expressed in terms of the $(C^s_r)^i_j$.

For $r = 0, 1, \ldots, n-1$ let $\mathcal{H}_r$ be the distribution on $T^n(M)$ spanned locally by the vector fields Y^r_i. We obtain in this way n m-dimensional distributions, each of which can be identified with $\operatorname{sect}\tau_0^*(T(M))$, such that $\mathcal{H}_r \cap \mathcal{H}_s = \{0\}$ for $r \ne s$. For $r \ge 1$, $\mathcal{H}_r$ is a complement to $\mathcal{V}_{r+1}$ in $\mathcal{V}_r$, while $\mathcal{H}_0$ is a complement to $\mathcal{V}_1$ in $\mathfrak{X}(T^n(M))$. If we set $\bar{\mathcal{H}}_r = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_r$ for $r < n$ then $\bar{\mathcal{H}}_r \oplus \mathcal{V}_{r+1} = \mathfrak{X}(T^n(M))$, and

$$\bar{\mathcal{H}}_0 \subset \bar{\mathcal{H}}_1 \subset \cdots \subset \bar{\mathcal{H}}_{n-1} \subset \mathfrak{X}(T^n(M))$$

is a filtration of $\mathfrak{X}(T^n(M))$ complementary to the filtration by vertical distributions. It is called a horizontal filtration, and the whole construction a multiconnection, a term coined in [12]. A vector field on $T^n(M)$ which lies in $\mathcal{H}_0$ is said to be horizontal with respect to the multiconnection, and $\mathcal{H}_0$ itself is the horizontal distribution.

For any section Y of $\tau_0^*(T(M))$ denote by Y^r the corresponding element of $\mathcal{H}_r$, $0 \le r \le n-1$, and by Y^n the corresponding element of $\mathcal{V}_n$ (which it is often convenient to think of as $\mathcal{H}_n$, however counter-intuitive this may be). Then

$$Y^r = Y^i Y^r_i \quad \text{where} \quad Y = Y^i \frac{\partial}{\partial y^i}.$$

Any vector field Z on $T^n(M)$ may be expresssed as a sum of its components in the $\mathcal{H}_r$, and each component identified with an element of $\operatorname{sect}\tau_0^*(T(M))$ which will be denoted by Z_r. Then $Z = \sum_{r=0}^{n}(Z_r)^r$, and $Z_r = \Theta_r(Z)$.

A Jacobi field for Γ is a vector field Z on $T^n(M)$ such that $\mathcal{L}_\Gamma Z = 0$. Thus along any integral curve γ of Γ, Z is the infinitesimal generator of variations of γ to nearby integral curves of Γ. We have

$$\Theta_r([\Gamma, Z]) = \nabla(\Theta_r(Z)) - (\nabla\Theta_r)(Z),$$

so that Z is a Jacobi field if and only if $Z_{r+1} = \nabla Z_r$ for $r = 0, 1, \ldots, n-1$ and $\nabla Z_n = (\nabla\Theta_n)(Z)$. Now the 1-forms $\nabla\Theta^i_n$ must be linearly dependent on the Θ^i_r with $0 \le r \le n$: say

$$\nabla\Theta^i_n + (\Phi_n)^i_j\Theta^j_n + (\Phi_{n-1})^i_j\Theta^j_{n-1} + \cdots + (\Phi_1)^i_j\Theta^j_1 + (\Phi_0)^i_j\Theta^j_0 = 0.$$

For each r, $(\Phi_r)^i_j$ are the components of an endomorphism Φ_r of $\operatorname{sect}\tau_0^*(T(M))$. In fact $\Phi_n = 0$: we have

$$\Theta^i_n = dy^i_n + n\Gamma^i_j dy^j_{n-1} + \sum_{s=0}^{n-2}(C^s_r)^i_j dy^j_s,$$

whence

$$\nabla\Theta^i_n \sim \mathcal{L}_\Gamma dy^i_n + (n+1)\Gamma^i_j dy^j_n \sim df^i - \frac{\partial f^i}{\partial y^j_n} dy^j_n \sim 0,$$

where $\sim$ indicates equality modulo terms in dy^i_r with $r < n$; the result follows. So we may write

$$\nabla\Theta_n + \Phi_{n-1}\circ\Theta_{n-1} + \cdots + \Phi_1\circ\Theta_1 + \Phi_0\circ\Theta_0 = 0.$$

Thus a Jacobi field Z is determined by a single element of $\operatorname{sect}\tau_0^*(T(M))$, say Y, with $Z_0 = Y$, $Z_1 = \nabla Y$, …, $Z_n = \nabla^n Y$; $Z = \sum_0^n (\nabla^r Y)^r$; and Y must satisfy

$$\nabla^{n+1}Y + \Phi_{n-1}(\nabla^{n-1}Y) + \cdots + \Phi_1(\nabla Y) + \Phi_0(Y) = 0.$$

This equation, or more precisely its restriction to any integral curve of Γ, is called Jacobi's equation for Γ, and the Φ_r are the Jacobi endomorphisms relative to the multiconnection.

Evidently Γ itself is a Jacobi field, with

$$\Gamma_0 = y^i_1 \frac{\partial}{\partial y^i},$$

which is a canonical element of $\operatorname{sect}\tau_0^*(T(M))$, and will be denoted by T. The Jacobi endomorphisms therefore satisfy

$$\nabla^{n+1}T + \Phi_{n-1}(\nabla^{n-1}T) + \cdots + \Phi_1(\nabla T) + \Phi_0(T) = 0.$$

Using the formula for $\Theta_r([\Gamma, Z])$ and the defining formula for the Jacobi endomorphisms one finds that for $Y \in \operatorname{sect}\tau_0^*(T(M))$

$$\begin{aligned}
[\Gamma, Y^0] &= (\nabla Y)^0 + \Phi_0(Y)^n \\
[\Gamma, Y^r] &= -Y^{r-1} + (\nabla Y)^r + \Phi_r(Y)^n, \qquad 1 \le r \le n-1 \\
[\Gamma, Y^n] &= -Y^{n-1} + (\nabla Y)^n.
\end{aligned}$$

4 Homogeneous differential equation fields

A differential equation field Γ will be said to be *homogeneous* if $[\Delta, \Gamma] = \Gamma$ ($\Delta = \Delta^1$). I show that, as for second-order fields, if Γ is homogeneous then its horizontal distribution is invariant under Δ, and in an appropriate sense the multiconnection defined above, and the Jacobi endomorphisms, are homogeneous.

For this purpose it is helpful to have a notion of Lie derivative with respect to a vector field on $T^n(M)$ acting on $\operatorname{sect}\tau_0^*(T(M))$. A general theory of Lie derivatives of vector fields along fibre bundle projections $\pi : E \to M$ by vector fields on E

projectable to M was given in [7]. Here we need only the case of the Lie derivative with respect to a π-vertical vector field, which is somewhat simpler. A section Y of $\pi^*(T(M)) \to E$ can be regarded as a linear operator $C^\infty(M) \to C^\infty(E)$ which obeys a Leibniz rule: for $\varphi_1, \varphi_2 \in C^\infty(M)$, $Y(\varphi_1\varphi_2) = Y(\varphi_1)\varphi_2 + \varphi_1 Y(\varphi_2)$. Then for any π-vertical vector field Z on E, $Z \circ Y$ is a linear operator $C^\infty(M) \to C^\infty(E)$, and it obeys the Leibniz rule because $Z(\varphi_1) = Z(\varphi_2) = 0$. This operator is defined to be $\mathcal{L}_Z Y$. Clearly $\mathcal{L}_Z Y$ depends $\mathbb{R}$-linearly on Y, and for $f \in C^\infty(E)$, $\mathcal{L}_Z(fY) = f\mathcal{L}_Z Y + Z(f)Y$. Moreover, if $Y \in \mathfrak{X}(M)$, $\mathcal{L}_Z Y = 0$. In fact the coordinate representation of $\mathcal{L}_Z$ is very simple:

$$\mathcal{L}_Z\left(Y^i \frac{\partial}{\partial y^i}\right) = Z(Y^i)\frac{\partial}{\partial y^i} :$$

in a sense, the foregoing discussion merely serves to justify the claim that when Z is vertical, differentiating the coefficients of Y along Z is a tensorial operation.

This Lie derivative operation can be extended to related geometric objects in the usual way. In particular, if Φ is an endomorphism of $\pi^*(T(M))$ then $(\mathcal{L}_Z\Phi)(Y) = \mathcal{L}_Z(\Phi(Y)) - \Phi(\mathcal{L}_Z Y)$; and again, one calculates $\mathcal{L}_Z\Phi$ by merely differentiating the components of Φ along Z.

In the case of interest $E = T^n(M)$ and $Z = \Delta$. A simple calculation shows that $[\Delta, Y^n] = (\mathcal{L}_\Delta Y - nY)^n$, and in particular if $Y \in \mathfrak{X}(M)$ then $[\Delta, Y^n] = -nY^n$. Here, as before, Y^n is the element of $\mathcal{V}_n$ corresponding to $Y \in \operatorname{sect}\tau_0^*(T(M))$; and we conclude (as indeed is otherwise obvious) that $[\Delta, \mathcal{V}_n] \subset \mathcal{V}_n$.

Theorem 1. *If Γ is homogeneous then $[\Delta, \mathcal{H}_r] \subset \mathcal{H}_r$ for $r = 0, 1, \dots, n-1$, and for any $Y \in \mathfrak{X}(M)$, $[\Delta, Y^r] = -rY^r$, where Y^r is the element of $\mathcal{H}_r$ corresponding to $Y \in \operatorname{sect}\tau_0^*(T(M))$. Furthermore, the Jacobi endomorphisms satisfy $\mathcal{L}_\Delta\Phi_r = (n+1-r)\Phi_r$.*

Proof. First of all, note that when it is expressed in terms of coordinates the homogeneity condition amounts to $\Delta(f^i) = (n+1)f^i$. It is an easy consequence that $\Delta(\Gamma^i_j) = \Gamma^i_j$, and hence that for any $Y \in \operatorname{sect}\tau_0^*(T(M))$, $[\mathcal{L}_\Delta, \nabla]Y = \nabla Y$. It further follows that for any $T(M)$-valued 1-form Θ, $[\mathcal{L}_\Delta, \nabla]\Theta = \nabla\Theta$. The $T(M)$-valued 1-forms Θ_r that define the multiconnection satisfy $\Theta_{r+1} = \nabla\Theta_r$ for $r = 0, 1, \dots, n-1$, and therefore

$$\mathcal{L}_\Delta\Theta_{r+1} = \nabla(\mathcal{L}_\Delta\Theta_r) + \nabla\Theta_r = \nabla(\mathcal{L}_\Delta\Theta_r) + \Theta_{r+1}.$$

Now $\mathcal{L}_\Delta\Theta_0 = 0$, whence $\mathcal{L}_\Delta\Theta_r = r\Theta_r$. Thus for any vector field Z,

$$\Theta_r([\Delta, Z]) = \mathcal{L}_\Delta(\Theta_r(Z)) - r\Theta_r(Z).$$

In particular, if $Y \in \operatorname{sect}\tau_0^*(T(M))$, for any $s = 0, 1, \dots, n$

$$\Theta_s([\Delta, Y^r]) = \mathcal{L}_\Delta(\Theta_s(Y^r)) - s\Theta_s(Y^r),$$

from which it follows that $[\Delta, Y^r] = (\mathcal{L}_\Delta Y)^r - rY^r \in \mathcal{H}_r$, and in particular that if $Y \in \mathfrak{X}(M)$ then $[\Delta, Y^r] = -rY^r$.

The Jacobi endomorphisms are determined by the formula

$$\nabla\Theta_n + \Phi_{n-1}\circ\Theta_{n-1} + \cdots + \Phi_1\circ\Theta_1 + \Phi_0\circ\Theta_0 = 0;$$

that is to say, for any $Y \in \operatorname{sect}\tau_0^*(T(M))$ and for $0 \le r \le n-1$

$$\Phi_r(Y) = -(\nabla\Theta_n)(Y^r).$$

Thus

$$\begin{aligned}
(\mathcal{L}_\Delta\Phi_r)(Y) &= -\mathcal{L}_\Delta((\nabla\Theta_n)(Y^r)) - \Phi_r(\mathcal{L}_\Delta Y)\\
&= -\mathcal{L}_\Delta(\nabla\Theta_n)(Y^r) - (\nabla\Theta_n)([\Delta, Y^r]) - \Phi_r(\mathcal{L}_\Delta Y)\\
&= -\nabla(\mathcal{L}_\Delta\Theta_n)(Y^r) - (\nabla\Theta_n)(Y^r) + r(\nabla\Theta_n)(Y^r)\\
&\qquad - (\nabla\Theta_n)((\mathcal{L}_\Delta Y)^r) - \Phi_r(\mathcal{L}_\Delta Y)\\
&= -n(\nabla\Theta_n)(Y^r) - (\nabla\Theta_n)(Y^r) + r(\nabla\Theta_n)(Y^r)\\
&\qquad - (\nabla\Theta_n)((\mathcal{L}_\Delta Y)^r) - \Phi_r(\mathcal{L}_\Delta Y)\\
&= (n+1-r)\Phi_r(Y)
\end{aligned}$$

as claimed. □

5 Strongly homogeneous third-order differential equation fields

In order for a differential equation field Γ to be worthy of the description homogeneous it must certainly satisfy $[\Delta, \Gamma] = \Gamma = [\Delta^1, \Gamma]$. But for $n > 1$ we have the whole algebra $\mathfrak{D}$ at our disposal, and one might imagine that one could impose some conditions on the brackets $[\Delta^r, \Gamma]$ for all $r = 1, 2, \ldots, n$. A little experimentation using coordinates suggests that such conditions would have to take the form

$$[\Delta^1, \Gamma] = \Gamma, \qquad [\Delta^r, \Gamma] = r\Delta^{r-1}, \quad r = 2, 3, \ldots, n.$$

Unfortunately, in general these conditions are inconsistent, for if $n+2 \ge r+s$ and $r \ne s$ we would have

$$[\Delta^r, [\Delta^s, \Gamma]] - [\Delta^s, [\Delta^r, \Gamma]] = (r-s)(r+s-1)\Delta^{r+s-2} \ne 0,$$

while if also $r+s > n+1$, $[\Delta^r, \Delta^s] = 0$, and Jacobi's identity would be violated. For $n > 2$ the values $r = n$, $s = 2$ satisfy both of the given inequalities (or indeed any r, s with $1 < r, s \le n$, $r \ne s$ and $r+s = n+2$). However, the conditions

$$[\Delta^1, \Gamma] = \Gamma, \qquad [\Delta^2, \Gamma] = 2\Delta^1$$

are consistent, and can be imposed whenever $n \ge 2$. The case of greatest interest is that with $n = 2$, in other words, the case of third-order differential equations.

Accordingly, a third-order differential equation field

$$\Gamma = y_1^i\frac{\partial}{\partial y_0^i} + y_2^i\frac{\partial}{\partial y_1^i} + f^i\frac{\partial}{\partial y_2^i}$$

is defined to be *strongly homogeneous* if it satisfies the conditions

$$[\Delta^1, \Gamma] = \Gamma, \qquad [\Delta^2, \Gamma] = 2\Delta^1,$$

where

$$\Delta^1 = y_1^i \frac{\partial}{\partial y_1^i} + 2y_2^i \frac{\partial}{\partial y_2^i}, \qquad \Delta^2 = 2y_1^i \frac{\partial}{\partial y_2^i}.$$

That is, Γ is strongly homogeneous if the assignment $\partial/\partial t \mapsto \Gamma$ extends the anti-isomorphism of Lie algebras $\mathfrak{q}^0 \to \mathfrak{D}$ to an anti-isomorphism of Lie algebras $\mathfrak{q} \to \mathfrak{D}^+$, where $\mathfrak{D}^+ = \langle\Gamma\rangle \oplus \mathfrak{D}$ with the brackets above. (The remarks in the opening paragraph of this section show that in general one cannot extend the anti-homomorphism $\mathfrak{p}^0 \to \mathfrak{X}(T^n(M)) : t^r\partial/\partial t \mapsto \Delta^r$ to an anti-homomorphism $\mathfrak{p} \to \mathfrak{X}(T^n(M))$ by a similar move.)

When expressed in terms of the coefficients f^i the conditions become

$$\Delta^1(f^i) = 3f^i, \qquad \Delta^2(f^i) = 6y_2^i.$$

Theorem 2. *Let Γ be a strongly homogeneous third-order differential equation field. Then Γ is horizontal with respect to the multiconnection defined by its dynamical covariant derivative; and the corresponding Jacobi endomorphisms satisfy $\Phi_r(T) = 0$, $r = 0, 1$.*

Proof. We have to show that $\Gamma \in \mathcal{H}_0$. Using the notation from the previous section, this is equivalent to $\Gamma_1 = \Gamma_2 = 0$. Now $\Gamma_0 = T$, $\Gamma_1 = \nabla T$, $\Gamma_2 = \nabla^2 T$. We have

$$\nabla T = \left(\Gamma(y_1^i) + y_1^j \Gamma_j^i\right) \frac{\partial}{\partial y^i},$$

and

$$\Gamma(y_1^i) + y_1^j \Gamma_j^i = y_2^i - \tfrac{1}{3} y_1^j \frac{\partial f^i}{\partial y_2^j}.$$

But $\Delta^2(f^i) = 6y_2^i$, which is to say that

$$y_1^j \frac{\partial f^i}{\partial y_2^j} = 3y_2^i,$$

and so $\nabla T = 0$, $\Gamma_1 = \Gamma_2 = 0$, and $\Gamma = T^0$.

From the formula $\Theta_r([\Gamma, Z]) = \nabla(\Theta_r(Z)) - (\nabla\Theta_r)(Z)$ with $Z = \Delta^1$, together with the fact that $\Delta_0^1 = 0$, using the homogeneity conditions in bracket form one finds that

$$\Delta_1^1 = \Gamma_0 = T, \quad \Delta_2^1 = \nabla\Gamma_0 + \Gamma_1 = 2\nabla T = 0,$$

so $\Delta^1 \in \mathcal{H}_1$, in fact $\Delta^1 = T^1$. The Jacobi endomorphisms are defined by the formula

$$\nabla\Theta_2 + \Phi_1 \circ \Theta_1 + \Phi_0 \circ \Theta_0 = 0.$$

It is easy to see that $\nabla\Theta_2$ vanishes on Γ and Δ^1, from which it follows that $\Phi_0(T) = \Phi_1(T) = 0$. $\square$

For completeness' sake it's worth pointing out that $\Delta^2_0 = \Delta^2_1 = 0$. Moreover, $\Delta^2_2 = T$, so $\Delta^2 = T^2 \in \mathcal{H}_2$, from which one can check independently that $\nabla\Theta_2(\Delta^2) = 0$.

As an instructive example of a strongly homogeneous third-order differential equation field I discuss the case of a single dependent variable, that is (in more natural notation) the equation

$$y''' = f(y, y', y'')$$

considered as defining a differential equation field on $T^2(\mathbb{R})$.

Theorem 3. *There is no strongly homogeneous third-order differential equation field defined on the whole of $T^2(\mathbb{R})$. If we restrict attention to the submanifold of $T^2(\mathbb{R})$ where $y' > 0$ then the most general strongly homogeneous third-order differential equation is*

$$y''' = \tfrac{3}{2}\frac{(y'')^2}{y'} + \kappa(y)(y')^3$$

where κ is an arbitrary smooth function of a single variable.

Proof. Denote by ϕ^1 and ϕ^2 the flows generated by Δ^1 and Δ^2 on $T^2(\mathbb{R})$. Then

$$\phi^1_t(y, y', y'') = (y, e^t y', e^{2t} y''), \qquad \phi^2_t(y, y', y'') = (y, y', y'' + 2ty').$$

Any point with $y' = 0$ is of course invariant under ϕ^2_t. From the condition $\Delta^1(f) = 3f$ it follows that $f(\phi^1_t(y, y', y'')) = e^{3t} f(y, y', y'')$. Along any integral curve of Δ^2, on the other hand, we have

$$\frac{df}{dt} = 6(y'' + 2ty'),$$

whence

$$f(\phi^2_t(y, y', y'')) = f(y, y', y'') + 6(ty'' + t^2 y').$$

This condition cannot be satisfied with $y' = 0$. For $y' > 0$, however, one can find values of s and t such that $(y, y', y'') = \phi^2_s(\phi^1_t(y, 1, 0))$, namely $t = \log y'$, $s = \frac{1}{2}y''/y'$. Then

$$\begin{aligned} f(y, y', y'') &= f(\phi^2_s(\phi^1_t(y, 1, 0))) = f(\phi^2_s(y, y', 0)) \\ &= (y')^3 f(y, 1, 0) + 6\left(\tfrac{1}{2}\frac{y''}{y'}\right)^2 y' \\ &= \tfrac{3}{2}\frac{(y'')^2}{y'} + \kappa(y)(y')^3 \end{aligned}$$

with $\kappa(y) = f(y, 1, 0)$. □

Since the equation is satisfied by $y(-x)$ if it is satisfied by $y(x)$, a similar result holds for $y' < 0$.

This result should not really come as a surprise, at least so far as the equation with $\kappa = 0$ is concerned. This equation says that the Schwarzian derivative of y is zero. Thus, or directly, the general solution is

$$y = \frac{ax+b}{cx+d}$$

(a factor may be taken to get the right number of arbitrary constants). Note that the solutions are invariant under reparametrizations

$$x \mapsto \frac{Ax+B}{Cx+D}, \quad AD - BC \neq 0,$$

and this is just the pseudo-group of local diffeomorphisms of $\mathbb{R}$ generated by the quadratic vector fields — the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}$ by fractional-linear or Möbius transformations.

The general analysis shows that since the differential equation field, even with κ nonzero, is strongly homogeneous it is horizontal. Moreover, the corresponding Jacobi endomorphisms Φ_0 and Φ_1 satisfy $\Phi_r(T) = 0$, which means in this case that they are both zero. It follows immediately that the Wuenschmann invariant, which in the present notation is $\nabla\Phi_1 - 2\Phi_0$ (see [4], [8]), vanishes.

Finally, it should be pointed out that the property of having solutions invariant under reparametrizations by fractional-linear transformations holds for all strongly homogeneous third-order systems. That is to say, if $\xi \mapsto (y^i(\xi))$ is a solution of such a system, and $x \mapsto \xi(x)$ is a fractional-linear transformation, then $x \mapsto (y^i(\xi(x)))$ is also a solution (where it is defined). For

$$\begin{aligned}
\frac{dy^i}{dx} &= \xi' \frac{dy^i}{d\xi} \\
\frac{d^2y^i}{dx^2} &= (\xi')^2 \frac{d^2y^i}{d\xi^2} + \xi'' \frac{dy^i}{d\xi} \\
\frac{d^3y^i}{dx^3} &= (\xi')^3 \frac{d^3y^i}{d\xi^3} + 3\xi'\xi'' \frac{d^2y^i}{d\xi^2} + \xi''' \frac{dy^i}{d\xi}.
\end{aligned}$$

On the other hand, the use of ϕ_t^1 and ϕ_t^2 as in the theorem leads to the result that for a strongly homogeneous system the functions f^i satisfy

$$f^i(y_0^j, ky_1^j, k^2y_2^j + ly_1^j) = k^3 f^i(y_0^j, y_1^j, y_2^j) + 3kly_2^i + \tfrac{3}{2}\frac{l^2}{k}y_1^i$$

for any $k, l \in \mathbb{R}$ with $k > 0$. It follows (taking $k = \xi'$ and $l = \xi''$) that

$$\begin{aligned}
\frac{d^3y^i}{dx^3} - f^i\left(y^i, \frac{dy^i}{dx}, \frac{d^2y^i}{dx^2}\right) = (\xi')^3 &\left(\frac{d^3y^i}{d\xi^3} - f^i\left(y^i, \frac{dy^i}{d\xi}, \frac{d^2y^i}{d\xi^2}\right)\right) \\
&+ \left(\xi''' - \tfrac{3}{2}\frac{(\xi'')^2}{\xi'}\right)\frac{dy^i}{d\xi},
\end{aligned}$$

and the final term vanishes if $\xi(x)$ is a fractional-linear function of x.

This is the analogue for strongly homogeneous third-order systems of the fact that the solutions of the second-order equations defined by a spray are invariant under affine reparametrizations.

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