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A HYBRID MEAN VALUE RELATED TO CERTAIN HARDY SUMS AND KLOOSTERMAN SUMS

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Abstract. The main purpose of this paper is using the mean value formula of Dirichlet L-functions and the analytic methods to study a hybrid mean value problem related to certain Hardy sums and Kloosterman sums, and give some interesting mean value formulae and identities for it.

Keywords: Hardy sums, the Kloosterman sums, hybrid mean value, asymptotic formula, identity

MSC 2010: 11M20

1. Introduction

Let c be a natural number and d an integer prime to c. The classical Dedekind sums

$$S(d,c)=\sum_{a=1}^{c}\Big(\Big(\frac{a}{c}\Big)\Big)\Big(\Big(\frac{ad}{c}\Big)\Big), \tag{1}$$

with

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer}\end{cases}$$

describes the behavior of the logarithm of the eta-function (cf. [5]) under modular transformations. B. C. Berndt [1] gave an analogous transformation formula for the logarithm of the classical theta-function

$$\theta(z)=\sum_{-\infty}^{+\infty}\exp\left(\pi i n^2 z\right), \quad \mathrm{Im}(z)>0.$$

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Put $Vz = (az + b)(cz + d)$ with $a, b, c, d \in \mathbb{Z}$, $c > 0$ and $ad - bc = 1$. Then

$$\log \theta(Vz) = \log \theta(z) + \frac{1}{2} \log(cz + d) - \frac{1}{4}\pi\mathrm{i} + \frac{1}{4}\pi\mathrm{i} S_1(d, c), \tag{2}$$

where

$$S_1(d, c) = \sum_{j=1}^{c-1} (-1)^{j+1+[dj/c]}.$$

The sums $S_1(d, c)$ and certain related ones are sometimes called the Hardy sums. They are closely connected with Dedekind sums. Many authors studied various properties of $S(d, c)$, and obtained a series of interesting results. For example, L. Carlitz [2] obtained a reciprocity theorem of $S(d, c)$. J. B. Conrey et al. [3] studied the mean value distribution of $S(d, c)$, and proved the following important and interesting asymptotic formula:

$$\sum_{d=1}^{c}{}' |S(d, c)|^{2m} = f_m(c) \Big(\frac{c}{12}\Big)^{2m} + O((c^{9/5} + c^{2m-1+1/(m+1)}) \log^3 c), \tag{3}$$

where $\sum_d'$ denotes the summation over all d such that $(d, c) = 1$, and

$$\sum_{n=1}^{\infty} \frac{f_m(n)}{n^s} = 2 \frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s + 4m - 1)}{\zeta^2(s + 2m)} \zeta(s),$$

$\zeta(s)$ being the Riemann zeta-function.

Jia Chaohua [4] improved the error term in (3) to $O\left(c^{2m-1}\right)$ provided $m \geqslant 2$. Zhang Wenpeng [7] improved the error term of (3) for $m = 1$, and proved the asymptotic formula

$$\sum_{d=1}^{c}{}' |S(d, c)|^2 = \frac{5}{144} c\varphi(c) \cdot \frac{\prod_{p^\alpha \| c} \left((1 + 1/p)^2 - 1/p^{3\alpha+1}\right)}{\prod_{p|c} (1 + 1/p + 1/p^2)} + O\Big(c \exp\Big(\frac{4 \ln c}{\ln \ln c}\Big)\Big),$$

where $p^\alpha \| c$ denotes that $p^\alpha \mid c$ and $p^{\alpha+1} \nmid c$.

In this paper we find that there are some close relationships between $S_1(d, c)$ and the Kloosterman sums $S(m, n; c)$, which are defined as follows:

$$S(m, n; c) = \sum_{b=1}^{c}{}' e\Big(\frac{mb + n\bar{b}}{c}\Big),$$

where $e(y) = \mathrm{e}^{2\pi\mathrm{i}y}$ and $\overline{b}$ denotes the solution of the congruent equation $x \cdot b \equiv 1 \pmod c$. In fact, for any positive integer k and some special integers c, we can use the estimates for character sums and the mean value theorem of Dirichlet L-functions to get some interesting hybrid mean value formulae and identities for

$$\sum_{a=1}^{c}{}'\sum_{b=1}^{c}{}' S^2(m,a;c)S^2(m,b;c)S_1^k(2a\overline{b},c). \tag{4}$$

We will prove the following several assertions:

Theorem 1. *Let p be a prime with $p \equiv 1 \pmod 4$, then for any fixed positive integer k we have the identities*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} S^2(a,1;p)S^2(b,1;p)S_1^{2k-1}(2a\overline{b},p) = 0$$

and

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} S^2(a,1;p)S^2(b,1;p)S_1^{2k}(2a\overline{b},p)$$
$$= (p^3-2p^2-3p-1)\cdot\sum_{a=1}^{p-1} S_1^{2k}(2a,p) - p^2\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)S_1^{2k}(2a,p),$$

where $(\frac{a}{p})$ is the Legendre symbol.

Theorem 2. *Let p be a prime with $p \equiv 3 \pmod 4$, then for any fixed positive integer k we have the identities*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} S^2(a,1;p)S^2(b,1;p)S_1^{2k}(2a\overline{b},p)$$
$$= 4^k\cdot p^3 + (p^3-2p^2-3p-1)\cdot\sum_{a=1}^{p-1} S_1^{2k}(2a,p)$$

and

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} S^2(a,1;p)S^2(b,1;p)S_1^{2k-1}(2a\overline{b},p)$$
$$= 2^{2k-1}\cdot p^3 - p^2\cdot\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)S_1^{2k-1}(2a,p).$$

It is clear that from our theorems we can establish an asymptotic formula for (2). Especially for $k=1$, we have:

Corollary 1. *Let p be a prime, then we have the asymptotic formula*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} S^2(a,1;p)S^2(b,1;p)S_1^2(2a\overline{b},p) = 3\ p^5 + O\Big(p^4\cdot\exp\Big(\frac{3\ln p}{\ln\ln p}\Big)\Big),$$

where $\exp(y)=\mathrm{e}^y$.

Corollary 2. *Let p be a prime with $p\equiv 3 \bmod 4$, then we have the identity*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} S^2(a,1;p)S^2(b,1;p)S_1(2a\overline{b},p) = 2\ p^3 + \frac{16p^3}{\pi^2}\cdot\Big(\frac{2}{p}\Big)\cdot L^2(1,\chi_{2p}^2),$$

where χ_{2p}^2 denotes the product of the Legendre symbol and the principal character $\lambda_2 \bmod 2$.

2. Some lemmas

In this section, we give some lemmas which are necessary in the proof of our theorems.

Lemma 1. *Let $q>2$ be an integer, then for any integer a with $(a,q)=1$ we have the identity*

$$S(a,q)=\frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2,$$

where $\varphi(q)$ is the Euler function, $\sum\limits_{\substack{\chi \bmod d\\ \chi(-1)=-1}}$ denotes the summation over all odd characters modulo d, $L(s,\chi)$ denotes the Dirichlet L-function corresponding to χ modulo d.

P r o o f. For the proof of Lemma 1 see [8]. □

Lemma 2. *Let p be an odd prime, then we have the identity*

$$S_1(2,p)=\sum_{j=1}^{p-1}(-1)^{j+1+[2j/p]}=\begin{cases}2, & \text{if } p\equiv 3 \bmod 4;\\ 0, & \text{if } p\equiv 1 \bmod 4.\end{cases}$$

P r o o f. For all integers $1 \leqslant j \leqslant p-1$, it is clear that $[2j/p] = 0$ if $1 \leqslant j \leqslant \frac{1}{2}(p-1)$, and $[2j/p] = 1$ if $\frac{1}{2}(p+1) \leqslant j \leqslant p-1$. So we have

$$\begin{aligned} S_1(2,p) &= \sum_{j=1}^{p-1} (-1)^{j+1+[2j/p]} \\ &= \sum_{j=1}^{(p-1)/2} (-1)^{j+1} + \sum_{j=(p+1)/2}^{p-1} (-1)^{j+1+1} \\ &= \sum_{j=1}^{(p-1)/2} (-1)^{j+1} + \sum_{j=1}^{(p-1)/2} (-1)^{p-j} \\ &= 2 \sum_{j=1}^{(p-1)/2} (-1)^{j+1}. \end{aligned}$$

If $\frac{1}{2}(p-1)$ is an even number, then $\sum\limits_{j=1}^{(p-1)/2} (-1)^{j+1} = 0$. If $\frac{1}{2}(p-1)$ is an odd number, then $\sum\limits_{j=1}^{(p-1)/2} (-1)^{j+1} = 1$. This proves Lemma 2. □

Lemma 3. *Let $k > 0$ and $(h,k) = 1$. Then we have the identity*

$$S_1(h,k) = -8S(h+k, 2k) + 4S(h,k).$$

P r o o f. This formula is an immediate consequence of (5.9) and (5.10) in [6]. □

Lemma 4. *Let p be an odd prime, then for any integer h with $(h,p) = 1$ we have the identity*

$$S_1(h,p) = \begin{cases} -\dfrac{16p}{\pi^2(p-1)} \sum\limits_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(h)|L(1,\chi\lambda_2)|^2, & \text{if } 2 \mid h, \\ 0, & \text{if } 2 \nmid h, \end{cases}$$

where λ_2 denotes the principal character mod 2.

P r o o f. It is clear that the divisors of $2p$ are $1, 2, p$ and $2p$. So from Lemma 1, Lemma 3 and the fact that for any character $\chi \bmod p$, $\chi(h+p)=\chi(h)$, we have

$$
\begin{aligned}
S_1(h,p) &= -8S(h+p,2p)+4S(h,p) \\
&= -\frac{4}{\pi^2 p}\sum_{d\mid 2p}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h+p)|L(1,\chi)|^2 \\
&\quad + \frac{4}{\pi^2 p}\sum_{d\mid p}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h)|L(1,\chi)|^2 \\
&= -\frac{4}{\pi^2 p}\frac{(2p)^2}{\varphi(2p)}\sum_{\substack{\chi \bmod 2p\\ \chi(-1)=-1}}\chi(h+p)|L(1,\chi)|^2 \\
&= -\frac{16p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(h+p)\lambda_2(h+p)|L(1,\chi\lambda_2)|^2.
\end{aligned}
\tag{5}
$$

Note that $\chi(h+p)\lambda_2(h+p)=\chi(h)$ if $2\mid h$; and $\chi(h+p)\lambda_2(h+p)=0$ if $2\nmid h$. From (5) we immediately deduce Lemma 4. □

Lemma 5. *Let p be an odd prime, χ any non-real character modulo p, then we have the identities*

(a) $\displaystyle\sum_{a=1}^{p-1} S^2(a,1;p)=p^2-p-1,$

(b) $\displaystyle\sum_{a=1}^{p-1}\Big(\frac{a}{p}\Big)S^2(a,1;p)=p,$

(c) $\displaystyle\sum_{a=1}^{p-1}\chi(a)S^2(a,1;p)=\frac{\tau^4(\chi)}{\tau(\chi^2)},$

where $(\frac{a}{p})$ denotes the Legendre symbol $\bmod p$, *$\tau(\chi)=\sum_{a=1}^{p-1}\chi(a)e(\frac{a}{p})$ is the Gauss sum, and $e(y)=\mathrm{e}^{2\pi \mathrm{i}y}$.*

P r o o f. We only prove formula (c). Similarly, we can deduce (a) and (b). From the definition of Kloosterman sums and the properties of Gauss sums we have

$$
\sum_{a=1}^{p-1}\chi(a)S^2(a,1;p)=\sum_{a=1}^{p-1}\chi(a)\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}e\Big(\frac{a(m+n)+\overline{m}+\bar{n}}{p}\Big)
\tag{6}
$$

$$= \sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\sum_{a=1}^{p-1}\chi(a)e\Big(\frac{a(m+n)+\overline{m}+\bar{n}}{p}\Big)$$
$$= \tau(\chi)\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\overline{\chi}(m+n)e\Big(\frac{\overline{m}+\bar{n}}{p}\Big)$$
$$= \tau(\chi)\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\overline{\chi}(mn+n)e\Big(\frac{\overline{mn}+\bar{n}}{p}\Big)$$
$$= \tau(\chi)\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(n)\overline{\chi}(m+1)e\Big(\frac{n(\overline{m}+1)}{p}\Big)$$
$$= \tau^2(\chi)\sum_{m=1}^{p-1}\overline{\chi}\left((m+1)(\overline{m}+1)\right).$$

On the other hand, since χ is a non-real character modulo p, so $\chi^2 \neq \chi_p^0$, the principal character modulo p. Therefore, from the properties of the Gauss sums we also have

(7)
$$\tau^2(\chi) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\chi(b)\Big(\frac{a+b}{p}\Big)$$
$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(ab)\chi(b)\Big(\frac{ab+b}{p}\Big)$$
$$= \sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi^2(b)e\Big(\frac{b(a+1)}{p}\Big)$$
$$= \tau(\chi^2)\sum_{a=1}^{p-1}\chi(a)\overline{\chi}^2(a+1))$$
$$= \tau(\chi^2)\sum_{a=1}^{p-1}\overline{\chi}\left((a+1)(\overline{a}+1)\right).$$

Combining (6) and (7) we immediately deduce the identity

$$\sum_{a=1}^{p-1}\chi(a)S^2(a,1;p) = \frac{\tau^4(\chi)}{\tau(\chi^2)}.$$

This proves Lemma 5. □

3. Proof of the theorems

In this section we shall complete the proof of our theorems. First we prove Theorem 1. Note that if χ_1, χ_2, …, χ_{2k-1} are any $2k-1$ odd characters mod p, then $\chi_1\chi_2\dots\chi_{2k-1}$ is an odd character mod p. And if $p \equiv 1 \bmod 4$, then $\chi_1\chi_2\dots\chi_{2k-1}$ is not a real character mod p. So from Lemma 2, Lemma 4 and Lemma 5 we have

$$\begin{aligned}
&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} S^2(a,1;p)S^2(b,1;p)S_1^{2k-1}(2a\overline{b},p)\\
&= \Big(-\frac{16p}{\pi^2(p-1)}\Big)^{2k-1} \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k-1} \bmod p\\ \chi_{2k-1}(-1)=-1}} \Big|\sum_{a=1}^{p-1} \chi_1\chi_2\dots\chi_{2k-1}(a)S^2(a,1;p)\Big|^2\\
&\qquad \times \chi_1\chi_2\dots\chi_{2k-1}(2)\ |L(1,\chi_1\lambda_2)|^2\cdot|L(1,\chi_2\lambda_2)|^2\dots|L(1,\chi_{2k-1}\lambda_2)|^2\\
&= \Big(-\frac{16p}{\pi^2(p-1)}\Big)^{2k-1} \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k-1} \bmod p\\ \chi_{2k-1}(-1)=-1}} \Big|\frac{\tau^4(\chi_1\chi_2\dots\chi_{2k-1})}{\tau(\chi_1^2\chi_2^2\dots\chi_{2k-1}^2)}\Big|^2\\
&\qquad \times \chi_1\chi_2\dots\chi_{2k-1}(2)\ |L(1,\chi_1\lambda_2)|^2\cdot|L(1,\chi_2\lambda_2)|^2\dots|L(1,\chi_{2k-1}\lambda_2)|^2\\
&= p^3\Bigg(-\frac{16p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2)|L(1,\chi\lambda_2)|^2\Bigg)^{2k-1}\\
&= p^3 S_1^{2k-1}(2,p) = 0.
\end{aligned}$$

This proves the first formula of Theorem 1.

If $\chi_1, \chi_2, \dots, \chi_{2k}$ are any $2k$ odd characters mod p, then $\chi_1\chi_2\dots\chi_{2k}$ is an even character mod p. So if $\chi_1\chi_2\dots\chi_{2k}$ is a real character mod p, then it must be the principal character or the Legendre symbol mod p. Let χ_p^2 denote the Legendre symbol, then from Lemma 2, Lemma 4 and Lemma 5 we have

$$\begin{aligned}
&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} S^2(a,1;p)S^2(b,1;p)S_1^{2k}(2a\overline{b},p)\\
&= \Big(-\frac{16p}{\pi^2(p-1)}\Big)^{2k} \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p\\ \chi_{2k}(-1)=-1}} \Big|\sum_{a=1}^{p-1} \chi_1\chi_2\dots\chi_{2k}(a)S^2(a,1;p)\Big|^2\\
&\qquad \times \chi_1\chi_2\dots\chi_{2k}(2)\ |L(1,\chi_1\lambda_2)|^2\cdot|L(1,\chi_2\lambda_2)|^2\dots|L(1,\chi_{2k}\lambda_2)|^2
\end{aligned}$$

$$= \left(\frac{16p}{\pi^2(p-1)}\right)^{2k} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1\chi_2\cdots\chi_{2k}\neq\chi_0,\ \chi_p^2}} \left|\frac{\tau^4(\chi_1\chi_2\ldots\chi_{2k})}{\tau(\chi_1^2\chi_2^2\ldots\chi_{2k}^2)}\right|^2$$

$$\times \chi_1\chi_2\ldots\chi_{2k}(2)\ |L(1,\chi_1\lambda_2)|^2\cdot|L(1,\chi_2\lambda_2)|^2\ldots|L(1,\chi_{2k}\lambda_2)|^2$$

$$+ \left(\frac{16p}{\pi^2(p-1)}\right)^{2k} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1\chi_2\cdots\chi_{2k}=\chi_0}} (p^2-p-1)^2$$

$$\times |L(1,\chi_1\lambda_2)|^2\cdot|L(1,\chi_2\lambda_2)|^2\ldots|L(1,\chi_{2k}\lambda_2)|^2$$

$$+ \left(\frac{16p}{\pi^2(p-1)}\right)^{2k} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1\chi_2\cdots\chi_{2k}=\chi_p^2}} (p^2-p-1)^2$$

$$\times \chi_p^2(2)\ |L(1,\chi_1\lambda_2)|^2\cdot|L(1,\chi_2\lambda_2)|^2\ldots|L(1,\chi_{2k}\lambda_2)|^2$$

$$= p^3\left(\frac{16p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2)|L(1,\chi\lambda_2)|^2\right)^{2k}$$

$$+ \left(\frac{16p}{\pi^2(p-1)}\right)^{2k} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1\chi_2\cdots\chi_{2k}=\chi_0}} [(p^2-p-1)^2-p^3]$$

$$\times |L(1,\chi_1\lambda_2)|^2\cdot|L(1,\chi_2\lambda_2)|^2\ldots|L(1,\chi_{2k}\lambda_2)|^2$$

$$+ \left(\frac{p}{\pi^2(p-1)}\right)^{2k} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1\chi_2\cdots\chi_{2k}=\chi_p^2}} [p^2-p^3]$$

$$\times \chi_p^2(2)\ |L(1,\chi_1\lambda_2)|^2\cdot|L(1,\chi_2\lambda_2)|^2\ldots|L(1,\chi_{2k}\lambda_2)|^2$$

$$= p^3 S_1^{2k}(2,p) + \frac{(p^2-p-1)^2-p^3}{p-1}\sum_{a=1}^{p-1} S_1^{2k}(2a,p) - \frac{p^3-p^2}{p-1}\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)S_1^{2k}(2a,p)$$

$$= (p^3-2p^2-3p-1)\sum_{a=1}^{p-1} S_1^{2k}(2a,p) - p^2\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)S_1^{2k}(2a,p).$$

This proves the second formula of Theorem 1.

Now we prove Theorem 2. If χ_1, χ_2, …, χ_{2k} are any $2k$ odd characters mod p, then $\chi_1\chi_2\ldots\chi_{2k}$ is an even character mod p. And if $p \equiv 3 \bmod 4$, then $\chi_1\chi_2\ldots\chi_{2k}$

is not a real non-principal character mod p. So by the method of proving the second formula of Theorem 1 we can easily deduce that

$$\begin{aligned}
&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} S^2(a,1;p)S^2(b,1;p)S_1^{2k}(2a\overline{b},p)\\
&= \Big(\frac{16p}{\pi^2(p-1)}\Big)^{2k} \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p\\ \chi_{2k}(-1)=-1}} \Big|\sum_{a=1}^{p-1} \chi_1\chi_2\dots\chi_{2k}(a) S^2(a,1;p)\Big|^2\\
&\qquad \times \chi_1\chi_2\dots\chi_{2k}(2)\ |L(1,\chi_1\lambda_2)|^2\cdot|L(1,\chi_2\lambda_2)|^2\dots|L(1,\chi_{2k}\lambda_2)|^2\\
&= \Big(\frac{16p}{\pi^2(p-1)}\Big)^{2k} \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p\\ \chi_{2k}(-1)=-1\\ \chi_1\chi_2\cdots\chi_{2k}\neq\chi_0}} \Big|\frac{\tau^4(\chi_1\chi_2\dots\chi_{2k})}{\tau(\chi_1^2\chi_2^2\dots\chi_{2k}^2)}\Big|^2\\
&\qquad \times \chi_1\chi_2\dots\chi_{2k}(2)\ |L(1,\chi_1\lambda_2)|^2\cdot|L(1,\chi_2\lambda_2)|^2\dots|L(1,\chi_{2k}\lambda_2)|^2\\
&\quad + \Big(\frac{16p}{\pi^2(p-1)}\Big)^{2k} \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p\\ \chi_{2k}(-1)=-1\\ \chi_1\chi_2\cdots\chi_{2k}=\chi_0}} (p^2-p-1)^2\\
&\qquad \times |L(1,\chi_1\lambda_2)|^2\cdot|L(1,\chi_2\lambda_2)|^2\dots|L(1,\chi_{2k}\lambda_2)|^2\\
&= p^3 S_1^{2k}(2,p) + (p^3-2p^2-3p-1)\sum_{a=1}^{p-1} S_1^{2k}(2a,p)\\
&= 4^k\cdot p^3 + (p^3-2p^2-3p-1)\sum_{a=1}^{p-1} S_1^{2k}(2a,p).
\end{aligned}$$

This proves the first formula of Theorem 2.

Similarly, we can deduce the second formula of Theorem 2.

Note that by virtue of the asymptotic formula (see [9])

$$\sum_{a=1}^{p-1} S_1^{2m}(a,p) = p^{2m}\frac{\zeta^2(2m)(1-\frac{1}{4^m})}{\zeta(4m)(1+\frac{1}{4^m})} + O\Big(p^{2m-1}\cdot\exp\Big(\frac{6\ln p}{\ln\ln p}\Big)\Big),$$

where $\zeta(s)$ is the Riemann zeta-function and $\exp(y)=\mathrm{e}^y$, from the second formula of Theorem 1 we may immediately deduce Corollary 1.

Corollary 2 follows from Lemma 4 and the second formula of Theorem 2.

This completes the proof of our theorems. □

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