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EXISTENCE AND NON-EXISTENCE OF SIGN-CHANGING SOLUTIONS FOR A CLASS OF TWO-POINT BOUNDARY VALUE PROBLEMS INVOLVING ONE-DIMENSIONAL p-LAPLACIAN

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Cordially dedicated to Professor Manabu Naito on his 60th birthday

Abstract. We consider the boundary value problem involving the one dimensional p-Laplacian, and establish the precise intervals of the parameter for the existence and non-existence of solutions with prescribed numbers of zeros. Our argument is based on the shooting method together with the qualitative theory for half-linear differential equations.

Keywords: boundary value problem, half-linear differential equation, Sturm comparison theorem, half-linear Prüfer transformation

MSC 2010: 34B15, 34C10

1. Introduction

In this paper we consider the existence and non-existence of sign-changing solutions for the one-dimensional p-Laplacian boundary value problem

$$(|u'|^{p-2}u')' + \lambda a(x)f(u) = 0, \quad 0 < x < 1, \tag{1.1}$$

$$u(0) = u(1) = 0, \tag{1.2}$$

where $p > 1$ and $\lambda > 0$ is a parameter. Problems of the form (1.1)–(1.2) describe some nonlinear phenomena in mathematical sciences and have been studied in recent years by many authors (see [1], [2], [6], [7], [9], [11], [13], [14] and references therein).

In (1.1) we assume that a satisfies

$$a \in C^1[0,1], \quad a(x) > 0 \quad \text{for } 0 \leqslant x \leqslant 1,$$

and that f satisfies $f \in C(\mathbb{R})$, $sf(s) > 0$ for $s \neq 0$, f is locally Lipschitz continuous on $\mathbb{R} \setminus \{0\}$; moreover, there exist limits f_0 and f_∞ with $f_0, f_\infty \in [0, \infty]$ such that

$$f_0 = \lim_{|s| \to 0} \frac{f(s)}{|s|^{p-2}s} \quad \text{and} \quad f_\infty = \lim_{|s| \to \infty} \frac{f(s)}{|s|^{p-2}s}.$$

By a solution u of (1.1) we mean a function $u \in C^1[0,1]$ with $|u'|^{p-2}u' \in C^1[0,1]$ which satisfies (1.1) at all points in $(0,1)$. For each $k \in \mathbb{N}$ we denote by S_k^+ (S_k^-) the set of all solutions u for (1.1)–(1.2) which have exactly $k-1$ zeros in $(0,1)$ and satisfy $u'(0) > 0$ (respectively, $u'(0) < 0$).

Let λ_k be the k-th eigenvalue of

$$\begin{cases} (|\varphi'|^{p-2}\varphi')' + \lambda a(x)|\varphi|^{p-2}\varphi = 0, & 0 < x < 1, \\ \varphi(0) = \varphi(1) = 0, \end{cases} \tag{1.3}$$

and let φ_k be an eigenfunction corresponding to λ_k. It is known that

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \lambda_{k+1} < \ldots, \qquad \lim_{k \to \infty} \lambda_k = \infty,$$

and that φ_k has exactly $k-1$ zeros in $(0,1)$. (See, e.g., [3], [4], [8].) For convenience, we put $\lambda_0 = 0$.

By [12, Theorem 1], if there exists an integer $k \in \mathbb{N}$ such that either

$$\lambda f_0 < \lambda_k < \lambda f_\infty \quad \text{or} \quad \lambda f_\infty < \lambda_k < \lambda f_0,$$

then $S_k^+ \neq \emptyset$ and $S_k^- \neq \emptyset$. As a consequence, in the case $f_0 \neq f_\infty$, if either

$$\lambda \in (\lambda_k/f_\infty, \lambda_k/f_0) \quad \text{or} \quad \lambda \in (\lambda_k/f_0, \lambda_k/f_\infty) \tag{1.4}$$

for some $k \in \mathbb{N}$, then $S_k^+ \neq \emptyset$ and $S_k^- \neq \emptyset$. Here, we agree that $1/0 = \infty$ and $1/\infty = 0$.

In this paper we will consider the non-existence of solutions with prescribed numbers of zeros, and also investigate the existence of solutions in the case $f_0 = f_\infty \in (0, \infty)$. To this end we define f_* and f^* by

$$f_* = \inf_{s \in \mathbb{R} \setminus \{0\}} \frac{f(s)}{|s|^{p-2}s} \quad \text{and} \quad f^* = \sup_{s \in \mathbb{R} \setminus \{0\}} \frac{f(s)}{|s|^{p-2}s},$$

respectively. Then it follows that $f_0, f_\infty \in [f_*, f^*]$.

Theorem 1.1. *Assume that* $\lambda \in (0, \lambda_k/f^*) \cup (\lambda_k/f_*, \infty)$ *for some* $k \in \mathbb{N}$. *Then* $S_k^+ = \emptyset$ *and* $S_k^- = \emptyset$.

Corollary 1.1. *Assume that* $\lambda_{k-1}/f_* < \lambda_k/f^*$ *for some integer* $k \in \mathbb{N}$. *If* $\lambda \in (\lambda_{k-1}/f_*,\ \lambda_k/f^*)$, *then the problem* (1.1)–(1.2) *has no nontrivial solution.*

Remark 1.1. Let us consider, for instance, the case where

$$f_* = f_0 < f_\infty = f^*. \tag{1.5}$$

In this case, by (1.4) and Theorem 1.1, we find that $S_k^+ \neq \emptyset$ and $S_k^- \neq \emptyset$ if $\lambda \in (\lambda_k/f_\infty, \lambda_k/f_0)$, and that $S_k^+ = S_k^- = \emptyset$ if $\lambda \in (0, \lambda_k/f_\infty) \cup (\lambda_k/f_0, \infty)$. Hence, λ_k/f_∞ and λ_k/f_0 are critical values for the existence of solutions in S_k^+ and S_k^-. For example, if $f(s)/|s|^{p-2}s$ is nondecreasing, then (1.5) holds.

Next, let us consider the existence of solutions in the case $f_0 = f_\infty \in (0, \infty)$. In this case we require that

$$\frac{f(s)}{|s|^{p-2}s} \not\equiv \text{constant} \quad \text{for any interval } (-\delta, \delta) \text{ with } \delta > 0. \tag{1.6}$$

It is clear that we have $f_* < f^*$, if (1.6) holds.

Theorem 1.2. *Assume that* $f_0 = f_\infty = f^* \in (0, \infty)$ *and* (1.6) *holds. Let* $k \in \mathbb{N}$.

(i) *If* $\lambda = \lambda_k/f^*$ *then* $S_k^+ = \emptyset$ *and* $S_k^- = \emptyset$.

(ii) *There exists* $\delta_k \in (\lambda_k/f^*, \lambda_k/f_*)$ *such that, if* $\lambda \in (\lambda_k/f^*, \delta_k)$, *then the problem* (1.1)–(1.2) *has at least four solutions* u_k^+, v_k^+, u_k^-, *and* v_k^- *such that* $u_k^+, v_k^+ \in S_k^+$ *and* $u_k^-, v_k^- \in S_k^-$.

Theorem 1.3. *Assume that* $f_0 = f_\infty = f_* \in (0, \infty)$ *and* (1.6) *holds. Let* $k \in \mathbb{N}$.

(i) *If* $\lambda = \lambda_k/f_*$ *then* $S_k^+ = \emptyset$ *and* $S_k^- = \emptyset$.

(ii) *There exists* $\delta_k \in (\lambda_k/f^*, \lambda_k/f_*)$ *such that, if* $\lambda \in (\delta_k, \lambda_k/f_*)$, *then the problem* (1.1)–(1.2) *has at least four solutions* u_k^+, v_k^+, u_k^-, *and* v_k^- *such that* $u_k^+, v_k^+ \in S_k^+$ *and* $u_k^-, v_k^- \in S_k^-$.

Remark 1.2. In Theorems 1.2 and 1.3, if $\lambda \in (0, \lambda_k/f^*) \cup (\lambda_k/f_*, \infty)$, then $S_k^+ = \emptyset$ and $S_k^- = \emptyset$ by Theorem 1.1.

In the proofs of Theorems 1.1, 1.2 and 1.3, we first consider the solution $u(x;\mu)$ of (1.1) satisfying the initial condition with a parameter $\mu \in \mathbb{R}$, and then we investigate the behavior of $u(x;\mu)$ as $\mu \to 0$ and $\mu \to \infty$. We will show the non-existence of solutions by employing variants of the Sturm comparison theorem for half-linear differential equations, and prove the existence of solutions with prescribed numbers of zeros by making use of the half-linear Prüfer transformation which involves the generalized trigonometric functions.

This paper is organized as follows. In Section 2 we give some variants of the Sturm comparison theorem, and in Section 3 we prove Theorem 1.1. In Section 4 we give the proofs of Theorems 1.2 and 1.3.

2. Comparison lemmas

Let us consider a pair of half-linear differential equations

$$(|u'|^{p-2}u')' + c(x)|u|^{p-2}u = 0, \quad 0 \leqslant x \leqslant 1, \tag{2.1}$$

and

$$(|U'|^{p-2}U')' + C(x)|U|^{p-2}U = 0, \quad 0 \leqslant x \leqslant 1, \tag{2.2}$$

where c, $C \in C[0,1]$ satisfy $C(x) \geqslant c(x)$ for $x \in [0,1]$. The Sturm comparison theorem for the half-linear differential equation is formulated as follows: [4, Theorem 1.2.4] (See also [3], [5] and [10].)

Lemma 2.1. *Assume that a nontrivial solution u of* (2.1) *satisfies $u(x_1) = u(x_2) = 0$ with some $0 \leqslant x_1 < x_2 \leqslant 1$. Then every nontrivial solution U of* (2.2) *has a zero in (x_1, x_2) or it is a multiple of the solution u on $[x_1, x_2]$. The latter possibility is excluded if $C(x) \not\equiv c(x)$ for $x \in [x_1, x_2]$.*

We will give some variants of Lemma 2.1.

Lemma 2.2. *Assume that a solution u of* (2.1) *satisfies $u(0) = u(1) = 0$ and has exactly $k-1$ zeros in $(0,1)$. Let U be a solution of* (2.2) *satisfying $U(0) = 0$ and $U'(0) \neq 0$. Then U possesses one of the following properties:*
(i) *U has at least k zeros in $(0,1)$;*
(ii) *U is a constant multiple of u on $[0,1]$ and $c \equiv C$ on $[0,1]$.*
In both cases (i) *and* (ii), *U has at least k zeros in $(0,1]$.*

Proof. In the case where $c \equiv C$ on $[0,1]$, it is clear that (ii) holds. Hence it suffices to show that (i) must hold in the case $c \not\equiv C$ on $[0,1]$. Let $\{x_i\}_{i=0}^{k}$ be zeros of u satisfying $0 = x_0 < x_1 < \ldots < x_{k-1} < x_k = 1$. Assume that $c \not\equiv C$ on $[x_{i_0-1}, x_{i_0}]$ for some $i_0 \in \{1, 2, \ldots, k\}$. Then Lemma 2.1 implies that U has at least one zero in (x_{i_0-1}, x_{i_0}). By Lemma 2.1, U has at least one zero in each interval $[x_{i-1}, x_i)$ for $i = i_0+1, i_0+2, \ldots, k$ and $(x_{i-1}, x_i]$ for $i = 1, 2, \ldots, i_0-1$. Thus U has at least k zeros in $(0,1)$. $\square$

Lemma 2.3. *Assume that a solution U of* (2.2) *satisfies $U(0) = U(1) = 0$ and has exactly $k-1$ zeros in $(0,1)$. Let u be a solution of* (2.1) *satisfying $u(0) = 0$ and $u'(0) \neq 0$. Then u possesses one of the following properties:*

(i) *u has at most $k-1$ zeros in $(0,1]$;*

(ii) *u is a constant multiple of U on $[0,1]$ and $c \equiv C$ on $[0,1]$.*

In both cases (i) *and* (ii), *u has at most $k-1$ zeros in $(0,1)$.*

Proof. We will show that u has at most $k-1$ zeros in $(0,1]$ when $c \not\equiv C$ on $[0,1]$. Let $\{x_i\}_{i=0}^k$ be zeros of U satisfying $0 = x_0 < x_1 < \ldots < x_{k-1} < x_k = 1$. Assume to the contrary that u has k zeros in $(0,1]$. Let $\{y_i\}_{i=0}^k$ be zeros of u satisfying $0 = y_0 < y_1 < \ldots < y_{k-1} < y_k \leqslant 1$. By applying Lemma 2.2 on the interval $(0, y_k)$, we conclude that the solution U has at least k zeros in $(0, y_k) \subset (0,1)$. This is a contradiction. Thus u has at most $k-1$ zeros in $(0,1]$, and (i) holds. □

We will need the following lemma [12, Lemma 3.3] in the proof of Theorem 1.1.

Lemma 2.4. *Let λ_k be the k-th eigenvalue of* (1.3), *and let $\{x_i\}_{i=0}^k$ be the zeros of the corresponding eigenfunction φ_k such that*

$$0 = x_0 < x_1 < x_2 < \ldots < x_{k-1} < x_k = 1. \tag{2.3}$$

Assume that $\tilde{\lambda} > \lambda_k$. Then for each $i \in \{1, 2, \ldots, k\}$ there is a solution w_i of the equation

$$(|w'|^{p-2}w')' + \tilde{\lambda} a(x)|w|^{p-2}w = 0 \tag{2.4}$$

which has at least two zeros in (x_{i-1}, x_i).

3. Proof of theorem 1.1

Let $\lambda > 0$. We denote by $u(x; \mu, \lambda)$ the solution of the problem (1.1) and

$$u(0) = 0 \quad \text{and} \quad u'(0) = \mu, \tag{3.1}$$

where $\mu \in \mathbb{R}$ is a parameter. By [12, Proposition 2.1] we obtain the following:

Lemma 3.1. *For each $\mu \in \mathbb{R}$ and $\lambda > 0$, the solution $u(x;\mu,\lambda)$ exists on $[0,1]$ and is unique. Furthermore, $u(x;\mu,\lambda)$ and $u'(x;\mu,\lambda)$ are continuous on $(x,\mu,\lambda) \in [0,1]\times\mathbb{R}\times(0,\infty)$, and the number of zeros of $u(x;\mu,\lambda)$ in $[0,1]$ is finite for each $\mu \in \mathbb{R}\setminus\{0\}$ and $\lambda > 0$.*

The *generalized sine function* $\sin_p$ is defined by the solution to the problem

$$(|S'|^{p-2}S')' + (p-1)|S|^{p-2}S = 0, \quad S(0) = 0 \text{ and } S'(0) = 1.$$

The function $\sin_p$ is defined on $\mathbb{R}$ and is periodic with period $2\pi_p$, where $\pi_p = (2\pi)/(p\sin(\pi/p))$. The *generalized cosine function* $\cos_p$ is defined by $\cos_p x = (\sin_p x)'$. For simplicity, we denote by $u(x;\mu)$ the solution of the problem (1.1) and (3.1) with fixed $\lambda > 0$. We define functions $r(x;\mu)$ and $\theta(x;\mu)$ by

$$\begin{cases} u(x;\mu) = r(x;\mu)\sin_p\theta(x;\mu), \\ u'(x;\mu) = r(x;\mu)\cos_p\theta(x;\mu), \end{cases}$$

where $' = \mathrm{d}/\mathrm{d}x$. It can be shown that

$$\theta'(x;\mu) = |\cos_p\theta(x;\mu)|^p + \frac{\lambda a(x) f\big(r(x;\mu)\sin_p\theta(x;\mu)\big)\sin_p\theta(x;\mu)}{(p-1)[r(x;\mu)]^{p-1}} > 0$$

for $x \in [0,1]$, which implies that $\theta(x;\mu)$ is strictly increasing in $x \in [0,1]$ for each fixed $\mu > 0$. (See, for example, [3] or [4].) The initial condition (3.1) yields that $\theta(0;\mu) \equiv 0 \pmod{2\pi_p}$. For simplicity we take $\theta(0;\mu) = 0$. Lemma 3.1 implies that $\theta(x;\mu)$ is continuous in $(x;\mu) \in [0,1]\times(0,\infty)$. We easily see that $u(x;\mu)$ has exactly $k-1$ zeros in $(0,1)$ if and only if $(k-1)\pi_p < \theta(1;\mu) \leqslant k\pi_p$.

Lemma 3.2. (i) *Assume that $\lambda f(s)/(|s|^{p-2}s) > \lambda_k$ for $s \in \mathbb{R}\setminus\{0\}$ with some $k \in \mathbb{N}$. Then for each $\mu \neq 0$ the solution $u(x;\mu)$ has at least k zeros in $(0,1)$.*

(ii) *Assume that $\lambda f(s)/(|s|^{p-2}s) < \lambda_k$ for $s \in \mathbb{R}\setminus\{0\}$ with some $k \in \mathbb{N}$. Then for each $\mu \neq 0$ the solution $u(x;\mu)$ has at most $k-1$ zeros in $(0,1]$.*

P r o o f. (i) We observe that $u = u(x;\mu)$ satisfies the equation

$$(|u'|^{p-2}u')' + b(x;\lambda)|u|^{p-2}u = 0, \tag{3.2}$$

where

$$b(x;\lambda) = \lambda a(x)\frac{f(u(x;\mu))}{|u(x;\mu)|^{p-2}u(x;\mu)}. \tag{3.3}$$

Note that $f(s)/(|s|^{p-2}s)$ is continuous at $s = 0$ if $f_0 < \infty$. Then the function $b(x; \lambda)$ is continuous for $x \in [0, 1]$ if $f_0 < \infty$.

First, assume that $f_0 < \infty$. Then $b(x; \lambda)$ given by (3.3) is continuous for $x \in [0, 1]$, and satisfies

$$b(x; \lambda) \geqslant \lambda_k a(x), \; b(x; \lambda) \not\equiv \lambda_k a(x) \quad \text{for } 0 \leqslant x \leqslant 1.$$

By Lemma 2.2, the solution $u(x; \mu)$ has at least k zeros in $(0, 1)$.

Next, assume that $f_0 = \infty$. Let φ_k be an eigenfunction corresponding to λ_k, and let $\{x_j\}_{j=0}^{k}$ be zeros of φ_k satisfying (2.3). We will show that $u(x; \mu)$ has at least one zero in each interval (x_{i-1}, x_i) for $i = 1, 2, \ldots, k$, which implies that $u(x; \mu)$ has at least k zeros in $(0, 1)$. Assume to the contrary that $u(x; \mu)$ has no zero in (x_{i_0-1}, x_{i_0}) for some $i_0 \in \{0, 1, 2, \ldots, k\}$. Then $b(x; \lambda)$ given by (3.3) is continuous for $x \in (x_{i_0-1}, x_{i_0})$ and satisfies $b(x; \lambda) > \lambda_k a(x)$ for $x_{i_0-1} < x < x_{i_0}$. We observe that, due to $f_0 = \infty$, there exists $\tilde{\lambda} > \lambda_k$ such that

$$b(x; \lambda) > \tilde{\lambda} a(x) \quad \text{for } x_{i_0-1} < x < x_{i_0},$$

even if $u(x_{i_0-1}; \mu) = 0$ or $u(x_{i_0}; \mu) = 0$. By Lemma 2.4, Eq. (2.4) has a nontrivial solution w such that $w(t_1) = w(t_2) = 0$ with $t_1, t_2 \in (x_{i_0-1}, x_{i_0})$. Lemma 2.1 implies that $u(x; \mu)$ has at least one zero in $(t_1, t_2) \subset (x_{i_0-1}, x_{i_0})$. This is a contradiction. Thus $u(x; \mu)$ has at least one zero in each interval (x_{i-1}, x_i) for $i = 1, 2, \ldots, k$, and hence $u(x; \mu)$ has at least k zeros in $(0, 1)$.

(ii) By the assumption, $f_0 < \infty$. Then the function $b(x; \lambda)$ given by (3.3) is continuous for $x \in [0, 1]$ and satisfies

$$b(x; \lambda) \leqslant \lambda_k a(x), \; b(x; \lambda) \not\equiv \lambda_k a(x) \quad \text{for } 0 \leqslant x \leqslant 1.$$

By Lemma 2.3, the solution $u(x; \mu)$ has at most $k - 1$ zeros in $(0, 1]$. □

Proof of Theorem 1.1. Assume that $\lambda \in (0, \lambda_k/f^*)$. In this case, we have

$$\lambda f(s)/(|s|^{p-2}s) < \lambda_k \quad \text{for } s \in \mathbb{R} \setminus \{0\}.$$

Then, by Lemma 3.2 (ii), the solution $u(x; \mu)$ has at most $k - 1$ zeros in $(0, 1]$ for every $\mu \neq 0$. This implies that $S_k^+ = S_k^- = \emptyset$. In the case $\lambda \in (\lambda_k/f_*, \infty)$ we obtain $S_k^+ = S_k^- = \emptyset$ by a similar argument with a slight modification.

Proof of Corollary 1.1. Note that $\lambda_k < \lambda_{k+1}$ for $k = 1, 2, \ldots$. Then Theorem 1.1 implies that, if $\lambda \in (\lambda_{k-1}/f_*, \infty)$, then $S_j^+ = S_j^- = \emptyset$ for each $j =$

$1, 2, \ldots, k-1$, and that, if $\lambda \in (0, \lambda_k/f^*)$, then $S_j^+ = S_j^- = \emptyset$ for each $j = k, k+1, \ldots$. By Lemma 3.1, the number of zeros of nontrivial solutions of (1.1)–(1.2) is finite. Hence (1.1)–(1.2) has no nontrivial solution.

4. Proof of theorems 1.2 and 1.3

We denote by $u(x;\mu,\lambda)$ the solution of the problem (1.1) and (3.1). As in Section 3, we define functions $r(x;\mu,\lambda)$ and $\theta(x;\mu,\lambda)$ by

$$\begin{cases} u(x;\mu,\lambda) = r(x;\mu,\lambda)\sin_p\theta(x;\mu,\lambda), \\ u'(x;\mu,\lambda) = r(x;\mu,\lambda)\cos_p\theta(x;\mu,\lambda) \end{cases}$$

with $\theta(0;\mu,\lambda) = 0$, where $' = \mathrm{d}/\mathrm{d}x$. We see that $\theta(x;\mu,\lambda)$ is continuous in $(x,\mu,\lambda) \in [0,1]\times\mathbb{R}\times(0,\infty)$ by Lemma 3.1, and that $\theta(x;\mu,\lambda)$ is strictly increasing in $x \in [0,1]$ for each fixed $\mu > 0$ and $\lambda > 0$. From $\theta(0;\mu,\lambda) = 0$ it follows that $u(x;\mu,\lambda)$ has exactly $k-1$ zeros in $(0,1)$ if and only if $(k-1)\pi_p < \theta(1;\mu,\lambda) \leqslant k\pi_p$.

By Lemmas 4.1–4.4 in [12] we obtain the following.

Lemma 4.1. *Let $k \in \mathbb{N}$.*

(i) *Assume that $\lambda f_0 < \lambda_k$. Then there exists $\mu_* > 0$ such that, for each $\mu \in (0,\mu_*]$, the solution $u(x;\mu,\lambda)$ has at most $k-1$ zeros in $(0,1)$.*

(ii) *Assume that $\lambda f_0 > \lambda_k$. Then there exists $\mu_* > 0$ such that, for each $\mu \in (0,\mu_*]$, the solution $u(x;\mu,\lambda)$ has at least k zeros in $(0,1)$.*

(iii) *Assume that $\lambda f_\infty > \lambda_k$. Then there exists $\mu^* > 0$ such that, for each $\mu \geqslant \mu^*$, the solution $u(x;\mu,\lambda)$ has at least k zeros in $(0,1)$.*

(iv) *Assume that $\lambda f_\infty < \lambda_k$. Then there exists $\mu^* > 0$ such that, for each $\mu \geqslant \mu^*$, the solution $u(x;\mu,\lambda)$ has at most $k-1$ zeros in $(0,1)$.*

We will prove Theorem 1.2 only, since Theorem 1.3 can be shown by an argument similar to the proof of Theorem 1.2 with a slight modification.

Proof of Theorem 1.2. (i) We observe that $u = u(x;\mu,\lambda)$ satisfies (3.2) with

$$b(x;\lambda) = \lambda a(x)\frac{f(u(x;\mu,\lambda))}{|u(x;\mu,\lambda)|^{p-2}u(x;\mu,\lambda)} \quad \text{for } 0 \leqslant x \leqslant 1. \tag{4.1}$$

If $f_0 < \infty$, then the function $b(x;\lambda)$ is continuous for $x \in [0,1]$.

Let $\mu > 0$. Due to $f_0 = f_\infty = f^* \in (0,\infty)$, the function $b(x;\lambda)$ given by (4.1) satisfies

$$b(x;\lambda_k/f^*) \leqslant \lambda_k a(x) \quad \text{for } x \in [0,1].$$

By Lemma 2.3, the solution $u(x;\mu,\lambda)$ has at most $k-1$ zeros on $(0,1)$, that is, $\theta(1;\mu,\lambda_k/f^*) \leqslant k\pi_p$. Assume that $\theta(1;\mu,\lambda_k/f^*) = k\pi_p$ with some $\mu > 0$. Then, by Lemma 2.3, we obtain

$$b(x;\lambda_k/f^*) \equiv \lambda_k a(x) \quad \text{for } x \in [0,1],$$

which implies that

$$\frac{f(s)}{|s|^{p-2}s} \equiv f^* \quad \text{for } 0 < s \leqslant \max_{x\in[0,1]} u(x;\mu,\lambda).$$

This contradicts (1.6). Thus we obtain $\theta(1;\mu,\lambda_k/f^*) < k\pi_p$ for any $\mu > 0$. This implies that $S_k^+ = \emptyset$ if $\lambda = \lambda_k/f^*$. By a similar argument, we obtain $\theta(1;\mu,\lambda_k/f^*) < k\pi_p$ for any $\mu < 0$, and hence $S_k^- = \emptyset$ if $\lambda = \lambda_k/f^*$.

(ii) Put $\mu_0 > 0$. By (i) we have $\theta(1;\mu_0,\lambda_k/f^*) < k\pi_p$. By the continuity of $\theta(1;\mu_0,\lambda)$ with respect to $\lambda > 0$ there exists $\delta_k^+ > \lambda_k/f^*$ such that $\theta(1;\mu_0,\lambda) < k\pi_p$ for $\lambda \in (\lambda_k/f^*, \delta_k^+)$. Let $\lambda \in (\lambda_k/f^*, \delta_k^+)$. Then we have $\lambda f_0 = \lambda f_\infty > \lambda_k$. By Lemmas 4.1 (ii), (iii) there are μ_*, $\mu^* > 0$ such that, if either $\mu \in (0,\mu_*]$ or $\mu \in [\mu^*,\infty)$, the solution $u(x;\mu,\lambda)$ has at least k zeros in $(0,1)$. This implies that

$$\theta(1;\mu,\lambda) > k\pi_p \quad \text{for } \mu \in (0,\mu_*] \cup [\mu^*,\infty),$$

and that $\mu_0 \in (\mu_*,\mu^*)$. Since $\theta(1;\mu,\lambda)$ is continuous in $\mu \in (0,\infty)$, there exist μ_1 and μ_2 such that

$$0 < \mu_1 < \mu_0 < \mu_2 \quad \text{and} \quad \theta(1;\mu_1,\lambda) = \theta(1;\mu_2,\lambda) = k\pi_p,$$

which means $u(x;\mu_1,\lambda)$, $u(x;\mu_2,\lambda) \in S_k^+$.

By an argument similar to the above, there exists a sequence $\delta_k^- > \lambda_k/f^*$ such that, if $\lambda \in (\lambda_k/f^*, \delta_k^-)$, then (1.1)–(1.2) has two solutions v_1 and v_2 which have exactly $k-1$ zeros in $(0,1)$ and satisfy $v_1'(0) < 0$ and $v_2'(0) < 0$. This implies that v_1, $v_2 \in S_k^-$.

Finally, put $\delta_k = \min\{\delta_k^+,\ \delta_k^-\}$. If $\lambda \in (\lambda_k/f^*, \delta_k)$, then (1.1)–(1.2) has at least four solutions u_k^+, v_k^+, u_k^-, v_k^- which satisfy u_k^+, $v_k^+ \in S_k^+$ and u_k^-, $v_k^- \in S_k^-$.

References

[1] *R. P. Agarwal, H. Lü, D. O'Regan*: Eigenvalues and the one-dimensional p-Laplacian. J. Math. Anal. Appl. *266* (2002), 383–400.

[2] *M. del Pino, M. Elgueta, R. Manasevich*: A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$. J. Differential Equations *80* (1989), 1–13.

[3] *O. Došlý*: Half-linear differential equations. Handbook of Differential Equations, Volume 1: Ordinary Differential Equations. Vol. 1, 161–358, North-Holland, Amsterdam, Elsevier, 2004.

[4] *O. Došlý, P. Řehák*: Half-Linear Differential Equations. North-Holland, Amsterdam, Elsevier, 2005.

[5] *Á. Elbert*: A half-linear second order differential equation. Qualitative theory of differential equations. Colloq. Math. Soc. János Bolyai. *30* (1981), 153–180.

[6] *D. D. Hai, K. Schmitt, R. Shivaji*: Positive solutions of quasilinear boundary value problems. J. Math. Anal. Appl. *217* (1998), 672–686.

[7] *X. He, W. Ge*: Twin positive solutions for the one-dimensional p-Laplacian boundary value problems. Nonlinear Anal. *56* (2004), 975–984.

[8] *T. Kusano, M. Naito*: Sturm-Liouville eigenvalue problems for half-linear ordinary differential equations. Rocky Mountain J. Math. *31* (2001), 1039–1054.

[9] *L. Kong, J. Wang*: Multiple positive solutions for the one-dimensional p-Laplacian. Nonlinear Anal. *42* (2000), 1327–1333.

[10] *H. J. Li, C. C. Yeh*: Sturmian comparison theorem for half-linear second-order differential equations. Proc. Roy. Soc. Edinburgh Sect. *A 125* (1995), 1193–1204.

[11] *R. Manasevich, F. Zanolin*: Time-mappings and multiplicity of solutions for the one-dimensional p-Laplacian. Nonlinear Anal. *21* (1993), 269–291.

[12] *Y. Naito, S. Tanaka*: Sharp conditions for the existence of sign-changing solutions to equations involving the one-dimensional p-Laplacian. Nonlinear Anal., Theory Methods Appl. *69* (2008), 3070–3083.

[13] *J. Sánchez*: Multiple positive solutions of singular eigenvalue type problems involving the one-dimensional p-Laplacian. J. Math. Anal. Appl. *292* (2004), 401–414.

[14] *J. Wang*: The existence of positive solutions for the one-dimensional p-Laplacian. Proc. Amer. Math. Soc. *125* (1997), 2275–2283.

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