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# NATURAL EXTENSION OF A CONGRUENCE OF A LATTICE TO ITS LATTICE OF CONVEX SUBLATTICES

S. Parameshwara Bhatta and H. S. Ramananda*

Abstract. Let $L$ be a lattice. In this paper, corresponding to a given congruence relation $\Theta$ of $L$, a congruence relation $\Psi_\Theta$ on $CS(L)$ is defined and it is proved that

1. $CS(L/\Theta)$ is isomorphic to $CS(L)/\Psi_\Theta$;
2. $L/\Theta$ and $CS(L)/\Psi_\Theta$ are in the same equational class;
3. if $\Theta$ is representable in $L$, then so is $\Psi_\Theta$ in $CS(L)$.

## 1. Introduction

Let $L$ be a lattice and $CS(L)$ be the set of all convex sublattices of $L$. It is proved in [3] that, there exists a partial order on $CS(L)$ with respect to which $CS(L)$ is a lattice such that both $L$ and $CS(L)$ are in the same equational class. A natural question that arises is the following:

If $\Theta$ is a congruence relation of $L$, does there exists a natural extension $\Psi_\Theta$ of $\Theta$ to $CS(L)$ such that $L/\Theta$ and $CS(L)/\Psi_\Theta$ are in the same equational class?

This paper gives an affirmative answer to this question. Further, it is proved that, if $\Theta$ is representable in $L$, then so is $\Psi_\Theta$ in $CS(L)$.

## 2. Notation and definitions

Let $L$ be a lattice and $CS(L)$ be the set of all convex sublattices of $L$. Define an ordering $\leq$ on $CS(L)$ by, for $A,\ B \in CS(L)$, $A \leq B$ if and only if for each $a \in A$ there exists $b \in B$ such that $a \leq b$ and for each $b \in B$ there exists $a \in A$ such that $b \geq a$. Then $(CS(L); \leq)$ is a lattice called the *lattice of convex sublattices* of $L$ (see [3]), denoted by $CS(L)$ in this paper.

Let $L$ be a lattice and $A$ and $B$ be convex sublattices of $L$. Then in $CS(L)$,

$$A \wedge B := \{z \in L \mid a_1 \wedge b_1 \leq z \leq a_2 \wedge b_2 \text{ for some } a_1,\ a_2 \in A,\ b_1,\ b_2 \in B\}\,;$$
$$A \vee B := \{z \in L \mid a_1 \vee b_1 \leq z \leq a_2 \vee b_2 \text{ for some } a_1,\ a_2 \in A,\ b_1,\ b_2 \in B\}$$

(see [3]).

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Let $L$ be a lattice and $X$ be a sublattice of $L$. Then the convex sublattice generated by $X$ in $L$, denoted by $\langle X\rangle$, is given by

$$\langle X\rangle = \{z \in L|\ a_1 \leq z \leq a_2 \text{ for some } a_1,\ a_2 \in X\}$$

(see [1]).

Let $L$ be a lattice and $\Theta$ be a congruence relation of $L$. Then $L/\Theta$ denotes the quotient lattice of $L$ modulo $\Theta$ and for $a \in L$, $a/\Theta$ denotes the congruence class containing $a$ (see [2]).

A congruence relation $\Theta$ of a lattice $L$ is said to be *representable* if there is a sublattice $L_1$ of $L$ such that the map $f$: $L_1 \to L/\Theta$ defined by $f(a) = a/\Theta$ is an isomorphism (see [1]).

## 3. Extending a congruence relation of $L$ to $CS(L)$

The following lemma is often used in the paper.

**Lemma 3.1.** *Let $L$ be a lattice, $\Theta$ be a congruence relation of $L$ and $A$ be a convex sublattice of $L$. Suppose that the elements $x_1$, $x$, $x_2$ of $L$ satisfy the following conditions:*

(1) $x_1 \leq x \leq x_2$;
(2) $x_1 \equiv a_1(\Theta)$ *for some* $a_1 \in A$;
(3) $x_2 \equiv a_2(\Theta)$ *for some* $a_2 \in A$.

*Then there exists $y \in A$ such that $x \equiv y(\Theta)$.*

**Proof.** From (1) and (2), we get

$$x = x \vee x_1 \equiv x \vee a_1(\Theta) \tag{3.1}$$

and from (1) and (3), we get

$$x = x \wedge x_2 \equiv x \wedge a_2(\Theta). \tag{3.2}$$

Take $y = (a_1 \wedge a_2) \vee (a_2 \wedge x)$. Then

$$a_1 \wedge a_2 \leq y \leq a_2 \tag{3.3}$$

and

$$a_2 \wedge x \leq y \leq a_1 \vee x\,. \tag{3.4}$$

Now from (3.1), (3.2) and (3.4), $x \equiv y(\Theta)$ and from (3.3), $y \in A$. □

In the following lemma a congruence relation on $CS(L)$ corresponding to a congruence relation of a lattice $L$ is constructed. Note that, in [4], a similar congruence relation is defined on $I(L)$ of a trellis $L$, and it is used for proving some results.

**Lemma 3.2.** *Let $L$ be a lattice and $\Theta$ be a congruence relation of $L$. Then the binary relation $\Psi$ on $CS(L)$ defined by "$X \equiv Y(\Psi)$ if and only if for each $x \in X$ there exists $y \in Y$ such that $x \equiv y(\Theta)$ and for each $y \in Y$ there exists $x \in X$ such that $x \equiv y(\Theta)$", is a congruence relation on $CS(L)$.*

**Proof.** Clearly $\Psi$ is an equivalence relation on $CS(L)$. To show that $\Psi$ satisfies the substitution property, consider $A$, $B$, $C \in CS(L)$ with $A \equiv C(\Psi)$. It is enough to prove that

$$A \wedge B \equiv C \wedge B(\Psi);$$

$$A \vee B \equiv C \vee B(\Psi).$$

Let $x \in A \wedge B$. Then, by the definition of $A \wedge B$ in $CS(L)$, there exist $a_1$, $a_2$ $\in A$ and $b_1$, $b_2 \in B$ such that $a_1 \wedge b_1 \leq x \leq a_2 \wedge b_2$. Since $a_1 \in A$ and $A \equiv C(\Psi)$, there exists $c_1 \in C$ such that $a_1 \equiv c_1(\Theta)$. But then $a_1 \wedge b_1 \equiv c_1 \wedge b_1(\Theta)$. Similarly, $a_2 \wedge b_2 \equiv c_2 \wedge b_2(\Theta)$ for some $c_2 \in C$. Note that $c_1 \wedge b_1$ and $c_2 \wedge b_2 \in C \wedge B$. Applying Lemma 3.1 for $a_1 \wedge b_1$, $x$, $a_2 \wedge b_2$ in $L$, noting that $C \wedge B \in CS(L)$, there exists $y \in C \wedge B$ such that $x \equiv y(\Theta)$.

Similarly, for each $x \in C \wedge B$ there exists $y \in A \wedge B$ such that $x \equiv y(\Theta)$. Hence $A \wedge B \equiv C \wedge B(\Psi)$.

By the dual argument it follows that $A \vee B \equiv C \vee B(\Psi)$. □

**Definition 3.3.** For a given congruence relation $\Theta$ on $L$, the congruence relation on $CS(L)$ defined in Lemma 3.2 is denoted by $\Psi_\Theta$.

One can easily verify the following lemma.

**Lemma 3.4** ([3])**.** *$L/\Theta$ is a suborder of $CS(L)$ for any $\Theta \in \operatorname{Con} L$.*

**Theorem 3.5.** *Let $L$ be a lattice and $\Theta$ be a congruence relation of $L$. Then $CS(L/\Theta)$ is isomorphic to $CS(L)/\Psi_\Theta$.*

**Proof.** Define a map $f \colon CS(L/\Theta) \to CS(L)/\Psi_\Theta$ by

$$f(X) = (\cup X)/\Psi_\Theta\,.$$

It is easy to see that $\cup X$ is a convex sublattice of $L$ and hence the map $f$ is well-defined.

To prove $f$ is one to one, suppose that $(\cup X)/\Psi_\Theta = (\cup Y)/\Psi_\Theta$. We assert that $\cup X = \cup Y$ which eventually proves $X = Y$. Let $x \in \cup X$. Since $(\cup X) \equiv (\cup Y)(\Psi_\Theta)$, there is a $y \in \cup Y$ such that $x \equiv y(\Theta)$. Now $x/\Theta = y/\Theta \in Y$ so that $x \in \cup Y$. Hence $\cup X \subseteq \cup Y$. Similarly it follows that $\cup Y \subseteq \cup X$. Thus $f$ is one to one.

To prove $f$ is onto, we need some preliminary considerations.

Let $A \in CS(L)$ and $S = \bigcup\{B \in CS(L) \mid B \equiv A(\Psi_\Theta)\}$.

**Claim 1**: $S$ is a convex sublattice of $L$.

Let $x$, $y \in S$. Then $x \in A_1 \equiv A(\Psi_\Theta)$ and $y \in A_2 \equiv A(\Psi_\Theta)$ for some $A_1$, $A_2 \in CS(L)$. Now $A_1 \underset{CS(L)}{\wedge} A_2 \equiv A_1 \underset{CS(L)}{\vee} A_2 \equiv A(\Psi_\Theta)$. Note that $x \wedge y \in A_1 \underset{CS(L)}{\wedge} A_2$ and $x \vee y \in A_1 \underset{CS(L)}{\vee} A_2$. Hence $x \wedge y$ and $x \vee y \in S$.

Let $a \leq x \leq b$ in $L$ and $a, b \in S$. Then $a \in A_1 \equiv A(\Psi_\Theta)$ and $b \in A_2 \equiv A(\Psi_\Theta)$ for some $A_1$, $A_2 \in CS(L)$. We can assume w.l.g that $A_1 \underset{CS(L)}{\leq} A_2$. Let $C$= $[A_1) \cap (A_2]$, where $[A_1)$ is the filter of $L$ generated by $A_1$ and $(A_2]$ is the ideal of $L$ generated by $A_2$. Then $C$ is a convex sublattice of $L$. Also $A_1 \underset{CS(L)}{\leq} C \underset{CS(L)}{\leq} A_2$ so that $A_1 \equiv C \equiv A_2(\Psi_\Theta)$. Thus $x \in C \subseteq S$. Claim 1 holds.

**Claim 2**: $S \equiv A(\Psi_\Theta)$.
Let $x \in A$. Since $A \subseteq S$, clearly $x \in S$ and $x \equiv x(\Theta)$. On the other hand, let $y \in S$. Then $y \in B \equiv A(\Psi_\Theta)$ for some $B$ in $CS(L)$, i.e. there exists $x \in A$ such that $y \equiv x(\Theta)$. Claim 2 holds.

Now set

$$X := \{x/\Theta \in L/\Theta |\ x \in S\}.$$

We shall prove that $X$ is a convex sublattice of $L/\Theta$. Let $a/\Theta$, $b/\Theta \in X$. Then $a/\Theta = x/\Theta$ and $b/\Theta = y/\Theta$ for some $x$, $y \in S$ . Now, since $S$ is a sublattice of $L$, $x \wedge y$ and $x \vee y \in S$. Therefore $x \wedge y/\Theta = x/\Theta \wedge y/\Theta = a/\Theta \wedge b/\Theta \in X$ and $x \vee y/\Theta = x/\Theta \vee y/\Theta = a/\Theta \vee b/\Theta \in X$.

Let $a/\Theta \underset{L/\Theta}{\leq} c/\Theta \underset{L/\Theta}{\leq} b/\Theta$ and $a/\Theta$, $b/\Theta \in X$. We can assume w.l.g that $a$, $b \in S$. Using Lemma 3.4, there exist $x \in c/\Theta$ and $b_1 \in b/\Theta$ such that $a \leq x \leq b_1$. Applying Lemma 3.1 for $a \leq x \leq b_1$ in $L$ and $S \in CS(L)$, there exists $y \in S$ such that $x \equiv y(\Theta)$, i.e., $y/\Theta = x/\Theta = c/\Theta \in X$. Hence $X$ is a convex sublattice of $L/\Theta$.

It is easy to see that $\cup X \equiv S(\Psi_\Theta)$. Now $X \in CS(L/\Theta)$ and from claim 2, $\cup X \equiv S \equiv A(\Psi_\Theta)$, so that $f$ is onto.

To prove that $f$ is order preserving, let $X \underset{CS(L/\Theta)}{\leq} Y$. Consider any $x \in \cup X$. Then $x/\Theta \in X \underset{CS(L/\Theta)}{\leq} Y$ and hence there exists $y/\Theta \in Y$ such that $x/\Theta \underset{L/\Theta}{\leq} y/\Theta$. Now $x/\Theta \vee y/\Theta = (x \vee y)/\Theta = y/\Theta \in Y$. Hence $x \vee y \in \cup Y$ and also $x \leq x \vee y$. Similarly for each $y \in \cup Y$ we can find $x \in \cup X$ such that $x \leq y$. Thus $\cup X \underset{CS(L)}{\leq} \cup Y$. Therefore $(\cup X)/\Psi_\Theta \underset{CS(L)/\Psi_\Theta}{\leq} (\cup Y)/\Psi_\Theta$, proving $f$ is order preserving.

It remains to prove that $f^{-1}$ is order preserving. First we observe the following fact.

**Claim 3**: Let $X \in CS(L/\Theta)$ and $S = \cup\{A \in CS(L) | A \equiv \cup X(\Psi_\Theta)\}$. Then $S = \cup X$. Since $\cup X \in CS(L)$ and $\cup X \equiv \cup X(\Psi_\Theta)$, $\cup X \subseteq S$. On the other hand, if $x \in S$, then $x \in A \equiv \cup X(\Psi_\Theta)$, for some $A \in CS(L)$. Now there exists $y \in \cup X$ such that $x \equiv y(\Theta)$. But then, $x/\Theta = y/\Theta \in X$. Hence $x \in \cup X$. Claim 3 holds.

Let $(\cup X)/\Psi_\Theta \underset{CS(L)/\Psi_\Theta}{\leq} (\cup Y)/\Psi_\Theta$. We prove that $\cup X \underset{CS(L)}{\leq} \cup Y$ which leads to $X \underset{CS(L/\Theta)}{\leq} Y$. Using Claim 3, it can be assumed that $\cup X = S_1$ and $\cup Y = S_2$ where $S_1$ and $S_2$ are as defined in Claim 3. It remains to show that $S_1 \underset{CS(L)}{\leq} S_2$.

Let $x \in S_1$. Then $x \in A \equiv \cup X(\Psi_\Theta)$, for some $A \in CS(L)$.
Since $S_1/\Psi_\Theta \underset{CS(L)/\Psi_\Theta}{\leq} S_2/\Psi_\Theta$ and $A \in S_1/\Psi_\Theta$ ; by Lemma 3.4, there exists $B \in S_2/\Psi_\Theta$ such that $A \underset{CS(L)}{\leq} B$. Since $x \in A \underset{CS(L)}{\leq} B$, there exists $y \in B$ such that $x \leq y$. Clearly $B \subseteq S_2$, so that $y \in S_2$. Similarly one can prove that for each $x \in S_2$ there exists $y \in S_1$ such that $y \leq x$. Thus $S_1 \underset{CS(L)}{\leq} S_2$. □

With the aid of Theorem 3.5, we obtain the following result.

**Corollary 3.6.** *Let $L$ be a lattice and $\Theta$ be a congruence relation of $L$. Then $L/\Theta$ and $CS(L)/\Psi_\Theta$ are in the same equational class.*

**Proof.** It is known that for a lattice $L$, $L/\Theta$ and $CS(L/\Theta)$ are in the same equational class ( [3]). Now by Theorem 3.5, $CS(L)/\Psi_\Theta$ is also in the same equational class. □

Next theorem shows that, the map $\Theta \to \Psi_\Theta$, preserves representability. But it requires a lemma.

In the following lemma a sublattice of $CS(L)$ corresponding to a sublattice of $L$ is constructed.

**Lemma 3.7.** *Let $L_1$ be a sublattice of $L$. Let*

$$Cvx(L_1) := \{\langle X\rangle \in CS(L) |\, X \in CS(L_1)\}.$$

*Then $Cvx(L_1)$ is a sublattice of $CS(L)$.*

**Proof.** The result follows by noting that, for $\langle X\rangle$, $\langle Y\rangle \in Cvx(L_1)$,

$$\langle X\rangle \underset{CS(L)}{\wedge} \langle Y\rangle = \left\langle X \underset{CS(L_1)}{\wedge} Y\right\rangle$$

and

$$\langle X\rangle \underset{CS(L)}{\vee} \langle Y\rangle = \left\langle X \underset{CS(L_1)}{\vee} Y\right\rangle.$$

□

**Theorem 3.8.** *If $\Theta$ is a representable congruence relation of $L$, then so is $\Psi_\Theta$ of $CS(L)$.*

**Proof.** Let $\Theta$ be a representable congruence relation of $L$. Then there exists a sublattice $L_1$ of $L$ such that the map $L_1 \to L/\Theta$, $a \mapsto a/\Theta$, defines an isomorphism. Let $Cvx(L_1)$ be the sublattice of $CS(L)$ as defined in Lemma 3.7.

Define a map $f\colon Cvx(L_1) \to CS(L)/\Psi_\Theta$ by

$$f(\langle X\rangle) = \langle X\rangle /\Psi_\Theta\,,$$

where $X \in CS(L_1)$. We shall prove that $f$ is an isomorphism.

Clearly $f$ is well defined and a homomorphism.

Let $\langle X\rangle \equiv \langle Y\rangle\, (\Psi_\Theta)$. We claim that $X = Y$, which proves that $f$ is one to one. Let $x \in X$. Then there exists $y \in \langle Y\rangle$ such that $x \equiv y(\Theta)$. Since $y \in \langle Y\rangle$, there exist $y_1$, $y_2 \in Y$ such that $y_1 \le y \le y_2$. Then

$$y_1 = y \wedge y_1 \equiv x \wedge y_1(\Theta) \tag{3.5}$$

and

$$y_2 = y \vee y_2 \equiv x \vee y_2(\Theta)\,. \tag{3.6}$$

Since $x$, $y_1$, $y_2 \in L_1$ and $L_1$ has only one element in each congruence class, (3.5) and (3.6) give $y_1 \le x \le y_2$. Now $x \in Y$ by the convexity of $Y$ in $L_1$. Therefore $X \subseteq Y$. Similarly, by interchanging $X$ and $Y$, we get $Y \subseteq X$.

To prove that $f$ is onto, let $A \in CS(L)$. Set

$$X := \{x \in L_1 |\, A \cap (x/\Theta) \ne \emptyset\}\,.$$

Then $X$ is nonempty. In fact, $A$ is nonempty therefore there exists an element $a \in A$ and

$$A = A \cap L = A \cap (\bigcup_{x \in L_1} x/\Theta) = \bigcup_{x \in L_1} \big(A \cap (x/\Theta)\big)$$

so that $a \in A \cap (x/\Theta)$ for some $x \in L_1$. But then $x \in X$.

We prove that $X$ is a convex sublattice of $L_1$. Let $a$, $b \in X$. Since $L_1$ is a sublattice of $L$, $a \wedge b$ and $a \vee b \in L_1$. Further, since $A \cap (a/\Theta) \neq \emptyset$ and $A \cap (b/\Theta) \neq \emptyset$, take $x \in A \cap (a/\Theta)$ and $y \in A \cap (b/\Theta)$. Then $x \wedge y \in A \cap ((a \wedge b)/\Theta)$ and $x \vee y \in A \cap ((a \vee b)/\Theta)$, proving $A \cap ((a \wedge b)/\Theta) \neq \emptyset$ and $A \cap ((a \vee b)/\Theta) \neq \emptyset$. Thus $a \wedge b$ and $a \vee b \in X$.

Let $x_1$, $x_2 \in X$ and $x_1 \underset{L_1}{\leq} x \underset{L_1}{\leq} x_2$. Since $A \cap (x_1/\Theta) \neq \emptyset$ and $A \cap (x_2/\Theta) \neq \emptyset$, take $a \in A \cap (x_1/\Theta)$ and $b \in A \cap (x_2/\Theta)$. By Lemma 3.1, there exists $y \in A$ such that $x \equiv y(\Theta)$. Therefore $y \in A \cap (x/\Theta)$, so that $A \cap (x/\Theta) \neq \emptyset$. Thus $x \in X$. Hence $X$ is a convex sublattice of $L_1$.

Now we prove that $\langle X \rangle \equiv A(\Psi_\Theta)$.

Let $x \in \langle X \rangle$. Then there exist $x_1$, $x_2 \in X$ such that $x_1 \leq x \leq x_2$. Since $A \cap (x_1/\Theta) \neq \emptyset$ and $A \cap (x_2/\Theta) \neq \emptyset$, take $b_1 \in A \cap (x_1/\Theta)$ and $b_2 \in A \cap (x_2/\Theta)$. Then again by Lemma 3.1, there is a $y \in A$ such that $x \equiv y(\Theta)$.

On the other hand, if $x \in A$, then $x \in A \cap (y/\Theta)$ for some $y \in L_1$. Clearly $y \in X$ and $y \equiv x(\Theta)$ holds. $\square$

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