Czechoslovak Mathematical Journal

Lech Pasicki

Uniformly convex spaces, bead spaces, and equivalence conditions

Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 2, 383–388

Persistent URL: `http://dml.cz/dmlcz/141541`

Czechoslovak Mathematical Journal, 61 (136) (2011), 383–388

UNIFORMLY CONVEX SPACES, BEAD SPACES, AND EQUIVALENCE CONDITIONS

Lech Pasicki, Kraków

(Received January 12, 2010)

Abstract. The notion of a metric bead space was introduced in the preceding paper (L. Pasicki: Bead spaces and fixed point theorems, Topology Appl., vol. 156 (2009), 1811–1816) and it was proved there that every bounded set in such a space (provided the space is complete) has a unique central point. The bead spaces themselves can be considered in particular as natural extensions of convex sets in uniformly convex spaces. It appears that normed bead spaces are identical with uniformly convex spaces. On the other hand the "metric" approach leads to new elementary conditions equivalent to the uniform convexity. The initial part of the paper contains the proof that discus spaces (they seem to have a richer structure) are identical with bead spaces.

Keywords: uniformly convex space, bead space, central point

MSC 2010: 46B20, 54E35

Let us recal the notion of a discus space [2, Def. 1].

Definition 1. A metric space (X, d) is a discus space if there exists a mapping $\varrho\colon [0,\infty) \times (0,\infty) \to [0,\infty)$ such that

(1) $\varrho(\beta, r) < \varrho(0, r) = r,\ \beta, r > 0,$

(2) $\varrho(\cdot, r)$ is nonincreasing, $r > 0$,

(3) $\varrho(\delta, \cdot)$ is upper semicontinuous, $\delta \geqslant 0$,

(4) for each $x, y \in X$, $r, \varepsilon > 0$ there exists a $z \in X$ such that $B(x, r) \cap B(y, r) \subset B(z, \varrho(d(x, y), r) + \varepsilon)$.

This definition is a complicated one. Therefore we have introduced the notion of a bead space [3, Def. 6]

Definition 2. A metric space (X, d) is a bead space if the following is satisfied:

(5) For every $r > 0, \beta > 0$ there exists a $\delta > 0$ such that
for each $x, y \in X$ with $d(x, y) \geqslant \beta$ there exists a $z \in X$
such that $B(x, r + \delta) \cap B(y, r + \delta) \subset B(z, r - \delta)$.

Def. 2 looks much simpler than Def. 1 and it was proved in [3, Lemma 8] that each discus space is a bead space. Let us prove

Lemma 3. *Each bead space is a discus space.*

P r o o f. Assume that (X, d) is a bead space. For $\beta > 0$, $r > 0$ let us adopt $\gamma(0, r) = 0$ and

(6) $$\gamma(\beta, r) = \sup\{\delta \in (0, r) \colon \text{ for each } x, y \in X \text{ with } d(x, y) \geqslant \beta \text{ there exists a } z \in X \text{ such that } B(x, r + \delta) \cap B(y, r + \delta) \subset B(z, r - \delta)\}.$$

Let us consider $\varrho(\beta, r) = r - \gamma(\beta, r)$. It is clear that $\varrho\colon [0, \infty) \times (0, \infty) \to [0, \infty)$ is a mapping. From (5) it follows that for $\beta > 0$ and $r > 0$ we have $\gamma(\beta, r) > 0$, and hence we obtain $\varrho(\beta, r) < \varrho(0, r) = r$, i.e. (1). Now let us prove that $\gamma(\cdot, r)$ is nondecreasing for $r > 0$. Once again by (5) if a δ is "good" for a $\beta > 0$ then it works for any $\beta_1 > \beta$. Therefore $\beta < \beta_1$ implies $\gamma(\beta, r) \leqslant \gamma(\beta_1, r)$, i.e. $\gamma(\cdot, r)$ is nondecreasing and $\varrho(\cdot, r)$ is nonincreasing (see (2)). Let us show that $\gamma(\beta, \cdot)$ is lower semicontinuous, $\beta \geqslant 0$. We have $\gamma(0, \cdot) = 0$ and it is sufficient to consider $\beta > 0$. Assume $\gamma(\beta, r_0) > \alpha > 0$. We ought to show that $\gamma(\beta, (r_0 - \varepsilon, r_0 + \varepsilon)) \subset (\alpha, \infty)$ for an $\varepsilon > 0$. The inequality $\gamma(\beta, r_0) > \alpha$ means that there exists a $\delta > \alpha$ such that $B(x, r_0 + \delta) \cap B(y, r_0 + \delta) \subset B(z, r_0 - \delta)$ for a $z \in X$. For $\varepsilon > 0$ with $r_0 + \delta - 2\varepsilon > 0$, $\delta - \varepsilon > 0$ we have

$$\begin{aligned} B(x, r_0 - \varepsilon + (\delta - \varepsilon)) \cap B(y, r_0 - \varepsilon + (\delta - \varepsilon)) \subset B(x, r_0 + \delta) \\ \cap B(y, r_0 + \delta) \subset B(z, r_0 - \delta) = B(z, r_0 - \varepsilon - (\delta - \varepsilon)). \end{aligned}$$

Consequently, for $\delta - 2\varepsilon > 0$ and $\delta - \varepsilon > \alpha$, i.e. small $\varepsilon < \delta - \alpha$, we obtain $\gamma(\beta, (r_0 - \varepsilon, r_0]) \subset (\alpha, \infty)$. On the other hand, for $\delta - \varepsilon > \alpha$ we have

$$\begin{aligned} B(x, r_0 + \varepsilon + (\delta - \varepsilon)) \cap B(y, r_0 + \varepsilon + (\delta - \varepsilon)) \\ = B(x, r_0 + \delta) \cap B(y, r_0 + \delta) \subset B(z, r_0 - \delta) \\ = B(z, r_0 + \varepsilon - (\delta + \varepsilon)) \subset B(z, r_0 + \varepsilon - (\delta - \varepsilon)). \end{aligned}$$

and consequently $\gamma(\beta, [r_0, r_0 + \varepsilon)) \subset (\alpha, \infty)$. Finally, we have $\gamma(\beta, (r_0 - \varepsilon, r_0 + \varepsilon)) \subset (\alpha, \infty)$ and $\gamma(\beta, \cdot)$ is lower semicontinuous. Directly from (5) it follows that

$$B(x, r) \cap B(y, r) \subset B(x, r + \delta) \cap B(y, r + \delta) \subset B(z, r - \delta) \subset B(z, r - \delta + \varepsilon)$$

and consequently

$$B(x, r) \cap B(y, r) \subset B(z, \varrho(d(x, y), r) + \varepsilon),$$

i.e. (4) holds. □

In view of Lemma 3 and [3, Lemma 8] we have

Theorem 4. *Any metric space is a bead space if and only if it is a discus space.*

Remark 5. If (X, d) is a metric space then for any $r > 2\varepsilon > 0$ we have

$$B(x, r - 2\varepsilon) \subset \overline{B}(x, r - \varepsilon) \subset B(x, r).$$

Therefore one may use closed balls in Definitions 1, 2.

Condition (5) guarantees the uniqueness of the so called central points of sets. If the space under consideration is complete then central points exist (see [3, Lemma 10]).

Our definition of a central point for a bounded set A in a metric space X coincides with the old notion of the Chebyshev centre of A in X (see e.g. [1]).

Definition 6 ([3, Def. 1]). Let (X, d) be a metric space and $\emptyset \neq A \subset X$ a bounded set. An $x \in X$ is a central point for A if

$$\begin{aligned} r(A) &:= \inf\{t \in (0, \infty)\colon \text{ there exists a } z \in X \text{ with } A \subset B(z, t)\} \\ &= \inf\{t \in (0, \infty)\colon A \subset B(x, t)\}. \end{aligned} \tag{7}$$

The centre $c(A)$ for A is the set of all central points for A, and $r(A)$ is the radius of A.

We have [3, Prop. 2]

Proposition 7. *Let (X, d) be a metric space satisfying*

$$\begin{aligned} &\textit{for each } r > 0 \textit{ and } x, y \in X, x \neq y \textit{ there exist } \delta > 0, \\ &z \in X \textit{ such that } B(x, r) \cap B(y, r) \subset B(z, r - \delta). \end{aligned} \tag{8}$$

If $\emptyset \neq A \subset X$ is bounded then $c(A)$ contains at most one point.

Let us prove

Lemma 8. *Let (X, d) be a complete metric space satisfying*

$$\text{(9)} \quad \begin{array}{l} \textit{for every } r > 0,\ \beta > 0 \textit{ there exists a } \delta > 0 \textit{ such that} \\ \textit{for each } x, y \in X \textit{ with } d(x, y) \geqslant \beta \textit{ there exists a } z \in X \\ \textit{such that } B(x, r + \delta) \cap B(y, r + \delta) \subset B(z, r). \end{array}$$

If $\emptyset \neq A \subset X$ is bounded then $c(A)$ is nonempty.

Proof. Let $(r_n)_{n\in\mathbb{N}}$ decrease to $r = r(A)$ while $A \subset B(x_n, r_n)$. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence then it converges, say, to an $x \in X$. We have $A \subset B(x_n, r_n) \subset B(x, r + \delta)$ for arbitrary $\delta > 0$ and large $n \in \mathbb{N}$, which means $x \in c(A)$. Suppose $(x_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence, i.e. $d(x_n, x_k) \geqslant \beta > 0$ for infinitely many $k, n \in \mathbb{N}$. In view of (9) for $r_n \leqslant r_k < r + \delta$ we have

$$A \subset B(x_n, r_k) \cap B(x_k, r_k) \subset B(x_n, r + \delta) \cap B(x_k, r + \delta) \subset B(z, r)$$

for a $z \in X$, which means $z \in c(A)$. $\square$

Clearly, condition (9) follows from (5).

Let us consider normed spaces.

Lemma 9. *For any normed space $(X, \|\cdot\|)$ conditions (5), (9) and*

$$\text{(10)} \quad \begin{array}{l} \textit{for every } r > 0,\ \beta > 0 \textit{ there exists a } \delta > 0 \\ \textit{such that } x \in X \textit{ and } 2\|x\| \geqslant \beta \textit{ imply } B(-x, r) \cap B(x, r) \subset B(0, r - \delta) \end{array}$$

are equivalent.

Proof. It is sufficient to prove the equivalence for $y = -x$. The set $C = B(-x, r) \cap B(x, r)$ is symmetric and therefore $C \subset B(z, t)$ implies $C \subset B(-z, t) \cap B(z, t) \subset B(0, t)$. Thus we may put $z = (x + y)/2 = 0$ in the conditions (5), (9), (10). Clearly (5) implies (9), (10). Let us adopt $r = 1$ in (10). Then we obtain

$$\text{(11)} \quad \begin{array}{l} \text{for every } \beta > 0 \text{ there exists a } \delta > 0 \text{ such that } u, x \in X, \\ \|u + x\|, \|u - x\| < 1 \text{ and } 2\|x\| \geqslant \beta > 0 \text{ imply } \|u\| < 1 - \delta. \end{array}$$

Assume that $\|u + x\|$, $\|u - x\| < r + \varepsilon$ and $2\|x\| \geqslant (r + \varepsilon)\beta$ hold. Then we have $\|u + x\|/(r + \varepsilon)$, $\|u - x\|/(r + \varepsilon) < 1$ and $2\|x\|/(r + \varepsilon) \geqslant \beta$, and in view of (11) $\|u\|/(r + \varepsilon) < 1 - \delta$ holds, i.e. $\|u\| < (r + \varepsilon)(1 - \delta) = r(1 - \delta) + \varepsilon(1 - \delta) < r(1 - \delta/2)$ for small $\varepsilon > 0$. Consequently, for $y = -x$, $\delta_1 := \min\{\varepsilon, r\delta/2\}$ and $2\|x\| \geqslant \beta_1 = 3r\beta/2 > (r + \varepsilon)\beta$ we obtain $B(x, r + \delta_1) \cap B(y, r + \delta_1) \subset B(0, r - \delta_1)$, i.e. (5). Thus we have proved that conditions (5), (10) and (11) are equivalent. Let us consider (9). Then we have $\|u+x\|/(r-\varepsilon)$, $\|u-x\|/(r-\varepsilon) < 1+\delta$ and $2\|x\|/(r-\varepsilon) \geqslant \beta$, which imply $\|u\|/(r-\varepsilon) < 1$ (see (9)). This dependence can be written as $\|u+x\|$, $\|u-x\| < (r-\varepsilon)(1+\delta) = r + r\delta - \varepsilon(1+\delta)$ and $2\|x\| \geqslant r\beta > (r-\varepsilon)\beta$, which imply $\|u\| < r-\varepsilon$. Thus for $\varepsilon < r\delta - \varepsilon(1+\delta)$, i.e. $\varepsilon < r\delta/(2+\delta)$ we have $\|u+x\|$, $\|u-x\| < r+\varepsilon$ and $2\|x\| \geqslant r\beta$, which imply $\|u\| < r-\varepsilon$, i.e. (5) holds. □

Lemma 10. *A normed space* $(X, \|\cdot\|)$ *is uniformly convex if and only if* (11) *is satisfied.*

Proof. From (11) for $u = (y+z)/2$, $x = (y-z)/2$ we obtain

(12) for every $\beta > 0$ there exists a $\delta > 0$ such that $y, z \in X$,

$\|y\|, \|z\| < 1$ and $\|y-z\| \geqslant \beta > 0$ imply $\|(y+z)/2\| < 1-\delta$,

i.e. X is uniformly convex (see Remark 5 if you prefer "$\leqslant$" in place of "$<$"). Now let us assume (12). Then for $y = u+x$, $z = u-x$ we obtain (11). □

Lemmas 9, 10 yield

Theorem 11. *A normed space* $(X, \|\cdot\|)$ *is uniformly convex if and only if one of conditions* (5), (9), (10), (11) *(or* (12)*) are satisfied.*

Remark 12. It can be noted that for normed spaces "$\geqslant \beta$" in conditions (5), (9), (10), (11) and (12) can be replaced by "$= \beta$". This observation is obvious for condition (10) (or (9)) and we apply the equivalences.

From Th. 11 and Remarks 5, 12 we obtain

Theorem 13. *A normed space* $(X, \|\cdot\|)$ *is uniformly convex if and only if for each* $\beta > 0$ *there exists a* $\delta > 0$ *such that for each* $u, x \in X$ *any of the following conditions is satisfied:*

(i) $\|x-u\|$, $\|x+u\| \leqslant 1+\delta$, $2\|x\| = \beta$ *imply* $\|u\| \leqslant 1-\delta$,

(ii) $\|x-u\|$, $\|x+u\| \leqslant 1+\delta$, $2\|x\| = \beta$ *imply* $\|u\| \leqslant 1$,

(iii) $\|x-u\|$, $\|x+u\| \leqslant 1$, $2\|x\| = \beta$ *imply* $\|u\| \leqslant 1-\delta$,

(iv) $\|u\|$, $\|x\| \leqslant 1+\delta$, $\|x-u\| = \beta$ *imply* $\|(x+u)/2\| \leqslant 1-\delta$,

(v) $\|u\|$, $\|x\| \leqslant 1+\delta$, $\|x-u\| = \beta$ *imply* $\|(x+u)/2\| \leqslant 1$,

(vi) $\|u\|$, $\|x\| \leqslant 1$, $\|x-u\| = \beta$ *imply* $\|(x+u)/2\| \leqslant 1-\delta$.

Theorems 4, 11 imply

Theorem 14. *For any normed space* $(X, \|\cdot\|)$ *the following conditions are equivalent:*

(i) X *is a bead space,*

(ii) X *is a discus space,*

(iii) X *is a uniformly convex space.*

Remark 15 (cf. [2, Ex. 3]). Each convex subset X of a uniformly convex space $(Z, \|\cdot\|)$ is a bead space with $z = (x+y)/2 \in X$ in condition (5) (see the initial part of the proof of Lemma 9).

References

[1] *T. C. Lim*: On asymptotic centers and fixed points of nonexpansive mappings. Canad. J. Math. *32* (1980), 421–430.

[2] *L. Pasicki*: A basic fixed point theorem. Bull. Polish Acad. Sci. Math. *54* (2006), 85–88.

[3] *L. Pasicki*: Bead spaces and fixed point theorems. Topology Appl. *156* (2009), 1811–1816.

Author's address: Lech Pasicki, Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Kraków, Poland, e-mail: pasicki@agh.edu.pl.