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EXTREME POINTS OF SUBORDINATION AND WEAK SUBORDINATION FAMILIES OF HARMONIC MAPPINGS

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Abstract. The aim of the paper is to discuss the extreme points of subordination and weak subordination families of harmonic mappings. Several necessary conditions and sufficient conditions for harmonic mappings to be extreme points of the corresponding families are established.

Keywords: planar harmonic mapping, extreme point, subordination, weak subordination, class N

MSC 2010: 30C65, 30C45, 30C20

1. Preliminaries

A complex-valued function f is said to be *harmonic* in a simply connected domain Ω of the complex plane $\mathbb{C}$ if and only if both $\operatorname{Re}\{f\}$ and $\operatorname{Im}\{f\}$ are real harmonic in Ω.

Every harmonic mapping f in Ω has the canonical decomposition

$$f = h + \overline{g}, \tag{1.1}$$

where both h and g are analytic in Ω and $g(z_0) = 0$ for some prescribed point $z_0 \in \Omega$ (cf. [6]). Moreover, a necessary and sufficient condition for f of the form (1.1) to be locally univalent and sense preserving is that its Jacobian

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2 = |h'|^2 - |g'|^2$$

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is positive. The class of all sense-preserving univalent harmonic mappings of the unit disk $\mathbb{D} = \{z\colon |z| < 1\}$ with $h(0) = g(0) = h'(0) - 1 = 0$ is denoted by $S_{\mathcal{H}}$, and $S^0_{\mathcal{H}} = \{f\colon f \in S_{\mathcal{H}} \text{ and } g'(0) = 0\}$.

Definition 1.1. Let X be a topological vector space over the field of complex numbers, and let D be a subset of X. A point $x \in D$ is called an *extreme point* of D if it has no representation of the form $x = ty + (1-t)z$ $(0 < t < 1)$ as a proper convex combination of two distinct points y and z in D.

We denote by ED the set of extreme points of D and by HD the *closed convex hull* of D, that is, the smallest closed convex set containing D (cf. [5, §9.3]).

Let B_0 denote the set of all functions φ analytic in $\mathbb{D}$ with $|\varphi(z)| < 1$ and $\varphi(0) = 0$.

Definition 1.2. We say that a harmonic mapping f is *subordinate* to F, denoted by $f \prec F$, if there is $\varphi \in B_0$ such that $f(z) = F(\varphi(z))$.

Suppose $f = h + \overline{g}$ and $F = H + \overline{G}$, where h, g, H and G are analytic in $\mathbb{D}$ with $h(0) = H(0)$ and $g(0) = G(0)$. If $f \prec F$, then, obviously, $h \prec H$ and $g \prec G$.

It is known that for analytic functions f and F, if F is univalent then $f(\mathbb{D}) \subset F(\mathbb{D})$ if and only if $f \prec F$ (cf. [5]). See [3], [5], [13] and [15] for more properties of subordinate analytic functions. But this important property is not valid for harmonic mappings. In [10], Muir introduced the following concept for harmonic mappings.

Definition 1.3. Suppose f and F are harmonic mappings in $\mathbb{D}$ with $f(0) = F(0) = 0$. We say that f is *weakly subordinate* to F if $f(\mathbb{D}) \subset F(\mathbb{D})$.

Extreme points of analytic functions play an important role in solving extremal problems. Many results have appeared in literature, see [1], [2], [7], [8], [9], [14] etc. But, up to now, there are no corresponding results for harmonic mappings. As the first aim of this paper, we discuss the extreme points of weak subordination families of harmonic mappings. Several necessary conditions are established. Our main results are Theorems 2.1 and 2.3. And then, we discuss the extreme points of subordination families of harmonic mappings. Several sufficient conditions are proved. Theorems 3.1 and 3.2 are the main results.

2. Extreme points of weak subordination families of harmonic mappings

We begin this section with two concepts.

Definition 2.1. A function h analytic in $\mathbb{D}$ is said to belong to the *Hardy space* H^p $(0 < p < \infty)$ if the integral means

$$M_p(r,h) = \Big\{\frac{1}{2\pi}\int_0^{2\pi} |h(r\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta\Big\}^{1/p}$$

is bounded for each $r \in (0,1)$.

We use h^p to denote the set of all harmonic mappings f for which $M_p(r,f)$ $(0 < r < 1)$ are bounded.

If $\lim\limits_{r\to 1} M_p(r,h) < \infty$ or $\lim\limits_{r\to 1} M_p(r,f) < \infty$, then we always use $\|h\|_p$ or $\|f\|_p$ respectively to denote this limit although it does not give a norm when $0 < p < 1$.

By [6, Theorem 1 in §8.5], if $F = H + \overline{G} \in S_{\mathcal{H}}$, then $F \in h^p$ for all $p < 0.0004$. Hence the radial limit $\lim\limits_{r\to 1} F(r\mathrm{e}^{\mathrm{i}\theta}) = F(\mathrm{e}^{\mathrm{i}\theta})$ exists for almost all $\theta \in [0,2\pi]$ (cf. [6, §8.5]). If $F \in S_{\mathcal{H}}$ and $f \prec F$, then $\lim\limits_{r\to 1} f(r\mathrm{e}^{\mathrm{i}\theta}) = f(\mathrm{e}^{\mathrm{i}\theta})$ also exists for almost all $\theta \in [0,2\pi]$. This is because, under the assumptions, $f \in h^p$ for all $p < 0.0004$ (cf. [5, Theorem 6.1]).

Definition 2.2. A function h analytic in $\mathbb{D}$ is said to be of *class* N if the integral

$$\int_0^{2\pi} \log^+ |h(r\mathrm{e}^{\mathrm{i}\theta})| \,\mathrm{d}\theta$$

is bounded for each $r < 1$, where $\log^+ x = \max\{0, \log x\}$.

[4, Theorem 2.1] says that an analytic function belongs to the class N if and only if it is the quotient of two bounded analytic functions. It is clear that for each $p > 0$, $H^p \subset N$.

In [1], it was proved that for a univalent analytic function H, if H' belongs to the class N and $\varphi \in EB_0$, then $\int_0^{2\pi} \log d\big(H(\varphi(\mathrm{e}^{\mathrm{i}\theta})), \partial\mathcal{D}_H\big)\,\mathrm{d}\theta = -\infty$, where $\mathcal{D}_H = H(\mathbb{D})$ and $d\big(H(\varphi(\mathrm{e}^{\mathrm{i}\theta})), \partial\mathcal{D}_H\big)$ denotes the distance between $H(\varphi(\mathrm{e}^{\mathrm{i}\theta}))$ and the boundary $\partial\mathcal{D}_H$ of $\mathcal{D}_H$. It was conjectured that $\int_0^{2\pi} \log d\big(H(\varphi(\mathrm{e}^{\mathrm{i}\theta})), \partial\mathcal{D}_H\big)\,\mathrm{d}\theta = -\infty$ for any univalent function H and $\varphi \in EB_0$.

In [2], Abu-Muhanna and Hallenbeck discussed a weaker conjecture: If $H \circ \varphi \in Es(H)$, where $\varphi \in B_0$, $s(H) = \{h\colon h \prec H\}$ and H is univalent, then $\int_0^{2\pi} \log d\big(H(\varphi(\mathrm{e}^{\mathrm{i}\theta})),\, \partial\mathcal{D}_H\big)\,\mathrm{d}\theta = -\infty$. They showed that the answer to this weaker conjecture is affirmative under the assumption that H' belongs to the class N.

The main aim of this section is to discuss the corresponding weaker conjecture for harmonic mappings. Our result is as follows, which is a partial answer to this weaker conjecture.

Theorem 2.1. *Suppose* $F = H + \overline{G} \in S^0_{\mathcal{H}}$ *and* H' *belongs to the class* N. *Let* $\mathscr{F}$ *be the set of all harmonic mappings* f *which are weakly subordinate to* F. *If* $f = F \circ \varphi$ *is an extreme point of* $\mathscr{F}$, *where* $\varphi \in B_0$, *then*

$$\int_0^{2\pi} \log d\big(F(\varphi(\mathrm{e}^{\mathrm{i}\theta})), \partial\mathcal{D}_F\big)\, \mathrm{d}\theta = -\infty. \tag{2.1}$$

The following lemmas are crucial for the proof of Theorem 2.1.

Lemma 2.1. *Let* $F = H + \overline{G} \in S^0_{\mathcal{H}}$. *Then*

$$\frac{1}{16}\Big(\frac{1-r}{1+r}\Big)^{\alpha} \leqslant d(F(z), \partial\mathcal{D}_F) \leqslant a\Big(\frac{1+r}{1-r}\Big)^{\alpha},$$

where $|z| = r$, $a = 2\pi\sqrt{6}/9$ *and* $\alpha = \sup\{\frac{1}{2}|h''(0)|\colon\ f = h + \overline{g} \in S_{\mathcal{H}}\}$.

P r o o f. Obviously, for any $f = h + \overline{g} \in S_{\mathcal{H}}$ there exists a function $f_0 \in S^0_{\mathcal{H}}$ such that $f = f_0 + \overline{b_1(g) f_0}$, where $b_1(g) = g'(0)$. By [6, Theorem 2 in §6.2],

$$\begin{aligned} d(0, \partial\mathcal{D}_f) = \liminf_{|z|\to 1} |f(z)| &\geqslant \liminf_{|z|\to 1} \big(1 - |b_1(g)|\big)|f_0(z)| \\ &\geqslant \liminf_{|z|\to 1} \big(1 - |b_1(g)|\big)\frac{|z|}{4(1+|z|)^2} = \frac{1 - |b_1(g)|}{16}. \end{aligned}$$

On the other hand, by [6, Theorem 1 in §6.2], we see that each function in $S_{\mathcal{H}}$ omits some point on the circle $\{z\colon\ |z| = a\}$, so

$$d(0, \partial\mathcal{D}_f) = \liminf_{|z|\to 1} |f(z)| \leqslant a.$$

Thus

$$\frac{1 - |b_1(g)|}{16} \leqslant d(0, \partial\mathcal{D}_f) \leqslant a. \tag{2.2}$$

For each $F \in S^0_{\mathcal{H}}$ and a fixed $\zeta \in \mathbb{D}$, let

$$F_1(z) = \frac{F((z+\zeta)/(1+\overline{\zeta}z)) - F(\zeta)}{(1-|\zeta|^2)H'(\zeta)}.$$

Then it is easy to verify that $F_1 = H_1 + \overline{G_1} \in S_{\mathcal{H}}$. So

$$d(F(\zeta), \partial\mathcal{D}_F) = d(0, \partial\mathcal{D}_{F_1})(1 - |\zeta|^2)|H'(\zeta)|.$$

It follows from (2.2) that

$$\frac{1 - |b_1(G_1)|}{16}(1 - |\zeta|^2)|H'(\zeta)| \leqslant d(F(\zeta), \partial\mathcal{D}_F) \leqslant a(1 - |\zeta|^2)|H'(\zeta)|. \tag{2.3}$$

Obviously, $b_1(G_1) = G'(\zeta)/H'(\zeta)$. This shows

$$\frac{1}{16}(1 - |\zeta|^2)\big(|H'(\zeta)| - |G'(\zeta)|\big) \leqslant d(F(\zeta), \partial\mathcal{D}_F) \leqslant a(1 - |\zeta|^2)|H'(\zeta)|. \tag{2.4}$$

From the proof of [6, Theorem in §6.4] we know that

$$\frac{(1-r)^{\alpha-1}}{(1+r)^{\alpha+1}} \leqslant |H'(\zeta)| - |G'(\zeta)| \tag{2.5}$$

and

$$|H'(\zeta)| \leqslant \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}, \tag{2.6}$$

where $|\zeta| = r$. By (2.4), (2.5) and (2.6) we complete the proof. □

Lemma 2.2. *Let $F \in S^0_{\mathcal{H}}$ and let $\mathscr{F}$ be the set of all harmonic mappings which are weakly subordinate to F. If $f = F \circ \varphi$ is an extreme point of $\mathscr{F}$, where $\varphi \in B_0$, then φ is an extreme point of B_0.*

P r o o f. Suppose, on the contrary, that φ is not an extreme point of B_0. Then, by a discussions similar to the proof of [2, Theorem 1], we have

$$\log\big(1 - |\varphi(z)|\big) \geqslant \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\log\big(1 - |\varphi(\mathrm{e}^{\mathrm{i}\theta})|\big)\,\mathrm{d}\theta, \tag{2.7}$$

where $P_z(\theta) = \operatorname{Re}((\mathrm{e}^{\mathrm{i}\theta} + z)/(\mathrm{e}^{\mathrm{i}\theta} - z))$ and $\theta \in [0, 2\pi)$.

Hence

$$\exp\left(\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\log\big(1 - |\varphi(\mathrm{e}^{\mathrm{i}\theta})|\big)^{\alpha}\,\mathrm{d}\theta\right) \leqslant \big(1 - |\varphi(z)|\big)^{\alpha}, \tag{2.8}$$

where α is the same as in Lemma 2.1.

Let

$$f_1(z) = h_1(z) + \overline{h_1(z)},$$

where

$$h_1(z)=\frac{z}{2^{\alpha+5}}\exp\biggl(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\bigl(1-|\varphi(e^{i\theta})|\bigr)^{\alpha}\,d\theta\biggr).$$

Obviously, $f_1(0)=0$ and f_1 is a non-constant harmonic mapping in $\mathbb{D}$. We also know that

$$|f_1(z)|<\frac{1}{2^{\alpha+4}}\exp\biggl(\frac{1}{2\pi}\int_0^{2\pi}P_z(\theta)\log\bigl(1-|\varphi(e^{i\theta})|\bigr)^{\alpha}\,d\theta\biggr),$$

which, together with (2.8), implies that

$$|f_1(z)|<\frac{1}{2^{\alpha+4}}\bigl(1-|\varphi(z)|\bigr)^{\alpha}. \tag{2.9}$$

Lemma 2.1 yields that

$$\frac{1}{2^{\alpha+4}}\bigl(1-|\varphi(z)|\bigr)^{\alpha}\leqslant d\bigl(F(\varphi(z)),\partial\mathcal{D}_F\bigr)=d(f(z),\partial\mathcal{D}_F).$$

Hence

$$|f_1(z)|<d(f(z),\partial\mathcal{D}_F)$$

for all z in $\mathbb{D}$. We conclude that $f(z)\pm f_1(z)\in\mathcal{D}_F$ for any z in $\mathbb{D}$, so $f\pm f_1\in\mathscr{F}$, which contradicts the assumption $f\in E\mathscr{F}$. Hence $\varphi\in EB_0$ and the proof is complete. □

We remark that Lemma 2.2 is a generalization of [2, Theorem 1] to the case of harmonic mappings, and [2, Theorem 1] is an affirmative answer to a conjecture raised by Abu-Muhanna in [1].

The following lemma is an analog of [1, Theorem 1] for harmonic mappings.

Lemma 2.3. *Suppose $F=H+\overline{G}\in S^0_{\mathcal{H}}$, where H' belongs to the class N. Then for any $f=F\circ\varphi$ ($\varphi\in B_0$),*
(1) $\int_0^{2\pi}\log^+ d\bigl(f(e^{i\theta}),\partial\mathcal{D}_F\bigr)\,d\theta$ *is convergent;*
(2) $\int_0^{2\pi}\log d\bigl(f(e^{i\theta}),\partial\mathcal{D}_F\bigr)\,d\theta=-\infty$ *if and only if* $\int_0^{2\pi}\log\bigl(1-|\varphi(e^{i\theta})|\bigr)\,d\theta=-\infty$, *where* $\log^+ x=\max\{0,\log x\}$.

P r o o f. The proof of this lemma follows from Lemma 2.1 and a reasoning similar to the proof of [1, Theorem 1]. □

The proof of Theorem 2.1.

P r o o f. Lemma 2.2 implies that $\varphi\in EB_0$ and [4, §7.6] shows that $\int_0^{2\pi}\log\bigl(1-|\varphi(e^{i\theta})|\bigr)\,d\theta=-\infty$. Hence the proof follows from the assumption that H' belongs to the class N and Lemma 2.3. □

We conjecture that (2.1) holds without the hypothesis that "H' belongs to the class N". The following theorem is an indirect evidence for this conjecture, which is also a generalization of [2, Theorem 3].

Theorem 2.2. *Suppose that F and $\mathscr{F}$ are the same as in Lemma 2.2 and that $f = F \circ \varphi \in E\mathscr{F}$. Then*

$$\inf_r \int_0^{2\pi} \log d\big(F(\varphi(r\mathrm{e}^{\mathrm{i}\theta})), \partial\mathcal{D}_F\big)\,\mathrm{d}\theta = -\infty.$$

P r o o f. As in the proof of [2, Theorem 3], we know that

$$\inf_r \int_0^{2\pi} \log\big(1 - |\varphi(r\mathrm{e}^{\mathrm{i}\theta})|^2\big)\,\mathrm{d}\theta = \int_0^{2\pi} \log\big(1 - |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2\big)\,\mathrm{d}\theta. \tag{2.10}$$

Since $f = F \circ \varphi \in E\mathscr{F}$, Lemma 2.2 implies that $\varphi \in EB_0$. Since $H'(z) \neq 0$, we know that both $\log|H'(z)|$ and $\log|H'(\varphi(z))|$ are harmonic, and so

$$\int_0^{2\pi} \log|H'(\varphi(r\mathrm{e}^{\mathrm{i}\theta}))|\,\mathrm{d}\theta = 2\pi\log|H'(\varphi(0))| = 2\pi\log|H'(0)| = 0.$$

Therefore, by (2.4),

$$\int_0^{2\pi} \log d\big(F(\varphi(r\mathrm{e}^{\mathrm{i}\theta})), \partial\mathcal{D}_F\big)\,\mathrm{d}\theta \leqslant \int_0^{2\pi} \log\big(1 - |\varphi(r\mathrm{e}^{\mathrm{i}\theta})|^2\big)\,\mathrm{d}\theta + 2\pi\log a. \tag{2.11}$$

It follows from (2.10), (2.11) and the fact that $\int_0^{2\pi} \log\big(1 - |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2\big)\,\mathrm{d}\theta = -\infty$ when $\varphi \in EB_0$ that

$$\inf_r \int_0^{2\pi} \log d\big(F(\varphi(r\mathrm{e}^{\mathrm{i}\theta})), \partial\mathcal{D}_F\big)\,\mathrm{d}\theta = -\infty.$$

□

The next two results are sufficient conditions for $\int_0^{2\pi} \log d\big(F(\varphi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\vartheta}), \partial\mathcal{D}_F\big)\,\mathrm{d}\theta$ to be $-\infty$.

Theorem 2.3. *Let $F = H + \overline{G} \in S^0_{\mathcal{H}}$, $\varphi \in EB_0$ and $|\varphi(\mathrm{e}^{\mathrm{i}\theta})| < 1$ for almost all θ. If H is univalent in $\mathbb{D}$, then*

$$\int_0^{2\pi} \log d\big(F(\varphi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\vartheta}), \partial\mathcal{D}_F\big)\,\mathrm{d}\theta = -\infty \tag{2.12}$$

for almost all ϑ.

P r o o f. Since $F \in S^0_{\mathcal{H}}$ and $|\varphi(\mathrm{e}^{\mathrm{i}\theta})| < 1$ for almost all θ, we infer from (2.4) that

$$d\big(F(\varphi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\vartheta}), \partial\mathcal{D}_F\big) \leqslant a\big(1 - |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2\big)|H'\big(\varphi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\vartheta}\big)|$$

for almost all θ, all ϑ and $a = 2\pi\sqrt{6}/9$. It follows that

$$\int_0^{2\pi} \log d\big(F(\varphi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\vartheta}), \partial\mathcal{D}_F\big)\,\mathrm{d}\theta \leqslant 2\pi \log a + \int_0^{2\pi} \log\big(1 - |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2\big)^{1/2}\,\mathrm{d}\theta + \int_0^{2\pi} \log\big(1 - |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2\big)^{1/2} |H'(\varphi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\theta.$$

Since H is univalent, it follows from [2, Theorem 5] that

$$\int_0^{2\pi} \log\big(1 - |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2\big)^{1/2} |H'(\varphi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\theta < +\infty \tag{2.13}$$

for almost all ϑ. Since $\varphi \in EB_0$, we know from [4, §7.6] that $\int_0^{2\pi} \log\big(1 - |\varphi(\mathrm{e}^{\mathrm{i}\theta})|\big)\,\mathrm{d}\theta = -\infty$. So (2.12) follows. □

By Lemma 2.2 and Theorem 2.3 the following result is obvious.

Corollary 2.1. *Let $F = H + \overline{G} \in S^0_{\mathcal{H}}$ and $\varphi \in B_0$, where $|\varphi(\mathrm{e}^{\mathrm{i}\theta})| < 1$ for almost all θ. Suppose that $\mathscr{F}$ is the same as in Lemma 2.2. If H is univalent and $F \circ \varphi \in E\mathscr{F}$, then*

$$\int_0^{2\pi} \log d\big(F(\varphi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\vartheta}), \partial\mathcal{D}_F\big)\,\mathrm{d}\theta = -\infty$$

for almost all ϑ.

In order to state the next result, we introduce a new concept.

Definition 2.3. An *inner function* is an analytic function h in $\mathbb{D}$ with $|h(z)| \leqslant 1$ and $|h(\mathrm{e}^{\mathrm{i}\theta})| = 1$ for almost all θ (cf. [4]).

As a generalization of Theorem 2.3, we have

Theorem 2.4. *Suppose that $F = H + \overline{G} \in S^0_{\mathcal{H}}$, H is univalent, $\varphi \in EB_0$ and $|\varphi(\mathrm{e}^{\mathrm{i}\theta})| < 1$ for almost all θ. If ψ is an inner function with $\psi(0) \neq 1$, then*

$$\int_0^{2\pi} \log d\big(F(\varphi(\mathrm{e}^{\mathrm{i}\theta})\psi(\mathrm{e}^{\mathrm{i}\vartheta})), \partial\mathcal{D}_F\big)\,\mathrm{d}\theta = -\infty$$

for almost all ϑ.

The proof is similar to that of [2, Theorem 7]. We omit it here.

3. Extreme points of closed convex hulls of subordination families of harmonic mappings

In [8], the authors proved two results concerning the extreme points of the family of functions subordinate to an analytic function. Specifically, two classes of extreme points were determined. The main aim of this section is to generalize these results to the case of harmonic mappings.

Theorem 3.1. *Let $F = H + \overline{G}$ be harmonic in $\mathbb{D}$ with $H(0) = G(0) = 0$ and let $s(F)$ be the family of functions subordinate to F. Then the functions $f(z) = F(xz)$ $(|x| = 1)$ belong to $EHs(F)$.*

P r o o f. Suppose, on the contrary, that $f(z) = F(xz)$ does not belong to $EHs(F)$ for some x with $|x| = 1$. Then there exist f_1 and $f_2 \in Hs(F)$ such that $f_1 \neq f_2$ and

$$f(z) = F(xz) = tf_1(z) + (1-t)f_2(z),$$

where $0 < t < 1$. Using $H(0) = G(0) = 0$, we get

$$H(xz) = th_1(z) + (1-t)h_2(z)$$

and

$$G(xz) = tg_1(z) + (1-t)g_2(z),$$

where $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$. Hence either $H(xz)$ does not belong to $EHs(H)$ or $G(xz)$ does not belong to $EHs(G)$, which contradicts [8, Theorem 6].

Hence $f(z) = F(xz)$ $(|x| = 1)$ belongs to $EHs(F)$. □

Theorem 3.2. *Let F be harmonic in $\mathbb{D}$ and let $s(F)$ be the family of functions subordinate to F. Suppose $F \in h^p$, where $2 \leqslant p < \infty$. If φ is an inner function with $\varphi(0) = 0$ and $|\varphi(z)| < 1$, then $f = F \circ \varphi \in EHs(F)$.*

The following lemma plays an important role in the proof of Theorem 3.2.

Lemma 3.1. *Let $f = h + \overline{g}$ and $F = H + \overline{G}$ be two harmonic mappings with $h(0) = H(0)$, $g(0) = G(0)$ and $F \in h^2$. Then $f \prec F$ and $\|f\|_2 = \|F\|_2$ if and only if there is an inner function φ with $\varphi(0) = 0$ and $|\varphi(z)| < 1$ such that $f = F \circ \varphi$.*

Before the proof of Lemma 3.1, we introduce a result due to Ryff which is from [11].

Lemma 3.2 (11, [Theorem 3]). *In order that* $h \prec H$ *and* $\|h\|_p = \|H\|_p$ ($H \in H^p$, $0 < p < \infty$) *it is necessary and sufficient that* $h = H \circ \varphi$, *where* φ *is an inner function with* $\varphi(0) = 0$.

The proof of Lemma 3.1.

P r o o f. Since $\|f\|_2 = \|h\|_2 + \|g\|_2$ and $\|F\|_2 = \|H\|_2 + \|G\|_2$, we know that $\|f\|_2 = \|F\|_2$ if and only if $\|h\|_2 + \|g\|_2 = \|H\|_2 + \|G\|_2$. It follows from $f \prec F$, $h(0) = H(0)$ and $g(0) = G(0)$ that $h \prec H$ and $g \prec G$. By [5, Theorem 6.3],

$$\|h\|_2 \leqslant \|H\|_2, \ \|g\|_2 \leqslant \|G\|_2.$$

Hence

$$\|h\|_2 = \|H\|_2, \ \|g\|_2 = \|G\|_2.$$

The proof easily follows from Lemma 3.2 and a reasoning similar to its proof in [11]. □

The proof of Theorem 3.2.

P r o o f. Since $F \in h^p$ and $p \geqslant 2$, we see that $F \in h^2$. From the assumptions it is obvious that $f \prec F$ and hence $f \in h^2$ by [12, Theorem 2.4]. The remaining part of the proof easily follows from Lemma 3.1 and a reasoning similar to the proof of [8, Theorem 7]. □

The next result is an analog of [14, Lemma 2] for harmonic mappings.

Theorem 3.3. *Suppose that* $F \in S_{\mathcal{H}}$ *and* $f \in s(F)$. *If* f *does not belong to* $EHs(F)$ *and* $f_1 \prec f$, *then* f_1 *does not belong to* $EHs(F)$.

The proof is similar to that of [14, Lemma 2]. Here we omit it.

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