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Applications of Mathematics, Vol. 56 (2011), No. 3, 315–339

Persistent URL: http://dml.cz/dmlcz/141490

56 (2011) APPLICATIONS OF MATHEMATICS No. 3, 315–339

AN ANALYTIC METHOD FOR THE INITIAL VALUE PROBLEM OF THE ELECTRIC FIELD SYSTEM IN VERTICAL INHOMOGENEOUS ANISOTROPIC MEDIA*

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(Received February 5, 2009)

Abstract. The time-dependent system of partial differential equations of the second order describing the electric wave propagation in vertically inhomogeneous electrically and magnetically biaxial anisotropic media is considered. A new analytical method for solving an initial value problem for this system is the main object of the paper. This method consists in the following: the initial value problem is written in terms of Fourier images with respect to lateral space variables, then the resulting problem is reduced to an operator integral equation. After that the operator integral equation is solved by the method of successive approximations. Finally, a solution of the original initial value problem is found by the inverse Fourier transform.

Keywords: equations of electromagnetic theory, hyperbolic system of second order partial differential equations, initial value problem, analytical method, Fourier transform

MSC 2010: 35Q60, 35L55, 35L15

1. Introduction

The study of wave propagation inside electrically and magnetically anisotropic materials constitutes an important interdisciplinary area of research with many cutting-edge scientific and technological applications. The dynamics of electric fields inside of electrically and magnetically anisotropic materials are described by the time-dependent system (see for example, [7], [5], [20])

$$\mathcal{E}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \mathrm{curl}_x(\mathcal{M}^{-1}\,\mathrm{curl}_x\,\mathbf{E}) = \mathbf{f}, \tag{1}$$

* This work was supported by Dokuz Eylul University of Turkey under the research grant 2006.KB.FEN.024.

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable, $t \in \mathbb{R}$ is the time variable, $\mathbf{E} = (E_1, E_2, E_3)$ is a vector function with components $E_k = E_k(x,t)$, $k = 1,2,3$, $\mathbf{f} = -\partial \mathbf{J}/\partial t$, $\mathbf{J} = (J_1, J_2, J_3)$ is the density of the electric current, $J_k = J_k(x,t)$, $k = 1,2,3$; $\mathcal{E} = (\varepsilon_{ij})_{3\times 3}$ is a symmetric positive definite matrix of the electric permittivity; $\mathcal{M} = (\mu_{ij})_{3\times 3}$ is a symmetric positive definite matrix of the magnetic permeability, $\mathcal{M}^{-1}$ is inverse to $\mathcal{M}$.

We note that for the isotropic medium ($\mathcal{E} = \varepsilon_0 I$ and $\mathcal{M} = \mu_0 I$, I is the identity matrix; ε_0, μ_0 are positive constants) the equation (1) under conditions $\mathbf{f} = 0$ and $\operatorname{div}_x(\varepsilon_0 \mathbf{E}) = 0$ can be written as

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta_x \mathbf{E}(x,t) = 0.$$

This equation is a classical wave equation and dozens of Initial Value Problems (IVPs) and Initial Boundary Value Problems (IBVPs) have been formulated and studied for it (see for example [8], [17]). Nowadays in view of the growing interest to the development of new anisotropic materials the analysis of electromagnetic fields in anisotropic media is an important issue and the study of IVPs and IBVPs for the equation (1) with arbitrary positive definite symmetric matrices $\mathcal{E}$ and $\mathcal{M}$ becomes actual. A special case of (1) is the system of crystal optics. In this case $\mathcal{E}$ is a symmetric positive definite matrix and $\mathcal{M} = \mu_0 I$. IVP for the system of crystal optics, where $\mathcal{E}$ is a diagonal matrix with different positive constant elements and $\mathcal{M}$ is the identity matrix, with smooth initial data and solving this problem has been studied by Courant and Hilbert [6] (see pp. 603–612). Burridge and Qian in [4] have used a plane wave approach to obtain an explicit formula for a fundamental solution of the same system of crystal optics. We note that the system of crystal optics is of great interest in applied mathematics and the different aspects of this system have been studied in [11], [12], [16]. Yakhno [18] has used matrix symbolic computations for constructing the time-dependent electromagnetic fields for the system (1), when $\mathcal{E}$ and $\mathcal{M}$ are symmetric positive definite matrices with constant elements.

The main object of our paper is IVP for the system (1) which consists in finding the vector function $\mathbf{E}$ satisfying (1) and the initial data

$$\mathbf{E}|_{t=0} = 0, \quad \frac{\partial \mathbf{E}}{\partial t}\Big|_{t=0} = 0. \tag{2}$$

The following notation and assumptions will be used throughout the paper: α, β, T are given positive numbers, $\alpha \leqslant \beta$, $c = \sqrt{\beta/\alpha}$; Δ is the triangle given by

$$\Delta = \{(x_3, t)\colon\ 0 \leqslant t \leqslant T,\ -c(T-t) \leqslant x_3 \leqslant c(T-t)\}; \tag{3}$$

elements of the diagonal matrices $\mathcal{E} = \operatorname{diag}(\varepsilon_{11}(x_3), \varepsilon_{22}(x_3), \varepsilon_{33}(x_3))$ and $\mathcal{M} = \operatorname{diag}(\mu_{11}(x_3), \mu_{22}(x_3), \mu_{33}(x_3))$ are twice continuously differentiable functions over $[-cT, cT]$ and have such positive values that $0 < \alpha \leqslant \varepsilon_{jj}(x_3) \leqslant \beta$, $0 < \alpha \leqslant m_{jj}(x_3) \equiv 1/\mu_{jj}(x_3) \leqslant \beta$, $j = 1, 2, 3$. Besides, we assume that there exists the Fourier transform with respect to x_1, x_2 of the components of the vector function $\mathbf{f}(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t))$ which appears in (1); the Fourier images of $f_j(x,t)$, denoted as $\tilde{f}_j(\nu, x_3, t)$, are such that $\tilde{f}_j(\nu, x_3, t) \in C(\mathbb{R}^2 \times \Delta)$, $j = 1, 2, 3$; $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$.

The assumptions we have made about $\mathcal{E}$ and $\mathcal{M}$ are related to biaxial vertically inhomogeneous anisotropic media where electric waves are propagated and the system (1) is a mathematical model of these waves. We note that various aspects of electromagnetic waves in homogeneous biaxial anisotropic materials were considered in the works [10], [13], [14].

The main result of the present paper is a new method for solving IVP (1), (2). This method consists of the following. First, IVP (1), (2) is written in terms of Fourier images with respect to lateral variables x_1, x_2. We denote this problem as FTIVP. Secondly, the resulting (FTIVP) is reduced to an operator integral equation. After that the operator integral equation is solved by the method of successive approximations. Finally, to find a solution of IVP (1), (2) we apply the inverse Fourier transform with respect to ν_1, ν_2 to the solution of the operator integral equation. At the same time a class of vector functions, where a unique solution of (1), (2) is constructed, is described.

The paper is organized as follows. In Section 2 IVP (1), (2) is written in terms of the Fourier transform with respect to the lateral variables. The reduction of the resulting problem (FTIVP) to an equivalent operator integral equation is given in Section 3. The properties of the operator integral equation are described in Section 4. Using these properties, a unique solution of the operator integral equation is constructed in Section 5. Finding a solution of the original IVP (1), (2) is given in Section 6.

2. Set-up of FTIVP

Let the components of the vector functions $\tilde{\mathbf{E}}(\nu, x_3, t) = (\tilde{E}_1(\nu, x_3, t), \tilde{E}_2(\nu, x_3, t), \tilde{E}_3(\nu, x_3, t))$ and $\tilde{\mathbf{f}}(\nu, x_3, t) = (\tilde{f}_1(\nu, x_3, t), \tilde{f}_2(\nu, x_3, t), \tilde{f}_3(\nu, x_3, t))$ be defined by

$$\tilde{E}_j(\nu, x_3, t) = \mathcal{F}_{x_1 x_2}[E_j](\nu, x_3, t), \quad \tilde{f}_j(\nu, x_3, t) = \mathcal{F}_{x_1 x_2}[f_j](\nu, x_3, t),$$
$$j = 1, 2, 3, \quad \nu = (\nu_1, \nu_2) \in \mathbb{R}^2,$$

where $\mathcal{F}_{x_1x_2}$ is the operator of the Fourier transform with respect to x_1, x_2, i.e.

$$\mathcal{F}_{x_1x_2}[\mathbf{E}](\nu, x_3, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{E}(x,t)\mathrm{e}^{\mathrm{i}(\nu_1x_1+\nu_2x_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2, \quad \mathrm{i}^2 = -1,$$

and $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ is the Fourier transform parameter.

Applying the operator $\mathcal{F}_{x_1x_2}$ to (1), (2) and using the properties of the Fourier transform, we can write the problem (1), (2) in terms of the Fourier image $\tilde{\mathbf{E}}(\nu, x_3, t)$ as follows:

$$\text{(4)} \quad \begin{aligned} \varepsilon_{jj}(x_3)\frac{\partial^2\tilde{E}_j}{\partial t^2} - \frac{\partial}{\partial x_3}\Big(m_{kk}(x_3)\frac{\partial \tilde{E}_j}{\partial x_3}\Big) = &- \nu_k^2 m_{33}(x_3)\tilde{E}_j + \nu_j\nu_k m_{33}(x_3)\tilde{E}_k \\ &+ (\mathrm{i}\nu_j)\frac{\partial}{\partial x_3}(m_{kk}(x_3)\tilde{E}_3) + \tilde{f}_j, \end{aligned}$$

$$\text{(5)} \quad \begin{aligned} \varepsilon_{33}(x_3)\frac{\partial^2\tilde{E}_3}{\partial t^2} + (\nu_1^2 m_{22}(x_3) + \nu_2^2 m_{11}(x_3))\tilde{E}_3 = &\,(\mathrm{i}\nu_1)m_{22}(x_3)\frac{\partial \tilde{E}_1}{\partial x_3} \\ &+ (\mathrm{i}\nu_2)m_{11}(x_3)\frac{\partial \tilde{E}_2}{\partial x_3} + \tilde{f}_3, \end{aligned}$$

$$\text{(6)} \qquad \tilde{\mathbf{E}}|_{t=0} = 0, \quad \frac{\partial \tilde{\mathbf{E}}}{\partial t}\Big|_{t=0} = 0,$$

where $j = 1, 2$, $j \neq k$, $k = 1, 2$.

3. Reduction of FTIVP to operator integral equation

The main aim of this section is to show that FTIVP is equivalent to a second kind operator integral equation of the Volterra type. This section is organized as follows. In Subsection 3.1 we obtain the equivalence of (4) under data (6) to some integral equalities for $\tilde{E}_j(\nu, x_3, t)$, $j = 1, 2$. The equivalence of (5) under data (6) to an integral equality for $\tilde{E}_3(\nu, x_3, t)$ is described in Subsection 3.2. The integral equality for $(\partial\tilde{E}_3/\partial t)(\nu, x_3, t)$ is presented in Subsection 3.2 also. Subsection 3.3 contains integral equalities for $(\partial\tilde{E}_j/\partial x_3)(\nu, x_3, t)$, $j = 1, 2$ in the forms which are necessary to get a closed system of integral equations for unknowns $\tilde{E}_j$, $\tilde{E}_3$, $\partial\tilde{E}_3/\partial t$, $\partial\tilde{E}_j/\partial x_3$, $j = 1, 2$. This system of integral equations is written in the form of a second kind operator integral equation of the Volterra type in Subsection 3.4.

3.1. Equivalence of (4), (6) to integral equalities

Now we show that for each $j = 1, 2$ the equation (4) is written in the terms of a new function $U_j(\nu, y_j, t)$. The resulting equation is a partial differential equation with constant coefficients in the principal part. We find the integral equality for $U_j(\nu, y_j, t)$ by inverting the principal part of the resulting differential equation. As a next step the integral equality is written in terms of $\tilde{E}_j(\nu, x_3, t)$.

Let us consider the transformation

$$y_j = \tau_j(x_3), \quad \tau_j(x_3) = \int_0^{x_3} c_j(\xi)\,\mathrm{d}\xi, \tag{7}$$

where

$$c_1^2(\xi) = \frac{\varepsilon_{11}(\xi)}{m_{22}(\xi)}, \quad c_2^2(\xi) = \frac{\varepsilon_{22}(\xi)}{m_{11}(\xi)}.$$

Lemma 1. *Under the assumptions mentioned in Section 1 the function $\tau_j(x_3)$, defined by (7) for each $j = 1, 2$, has the following properties:*

(a) *$\tau_j(x_3)$ is a monotonically increasing function from $[-cT, cT]$ into $[Y_j^-, Y_j^+]$, where $Y_j^- = \tau_j(-cT)$, $Y_j^+ = \tau_j(cT)$;*
(b) *$\tau_j(x_3)$ has a monotonically increasing inverse function $\tau_j^{-1}(y_j)$ from $[Y_j^-, Y_j^+]$ into $[-cT, cT]$;*
(c) *$\tau_j(0) = 0$, $\tau_j^{-1}(0) = 0$;*
(d) *$\tau_j(x_3) \in C^3[-cT, cT]$, $\tau_j^{-1}(y_j) \in C^3[Y_j^-, Y_j^+]$.*

P r o o f. Proof of Lemma 1 follows from calculus [2]. □

Let

$$W_j(\nu, y_j, t) = \tilde{E}_j(\nu, x_3, t)|_{x_3=\tau_j^{-1}(y_j)}. \tag{8}$$

Then we have

$$\frac{\partial \tilde{E}_j}{\partial x_3}(\nu, x_3, t)|_{x_3=\tau_j^{-1}(y_j)} = c_j(\tau_j^{-1}(y_j))\frac{\partial W_j}{\partial y_j}(\nu, y_j, t). \tag{9}$$

The equation (4) may be written in terms of y_j and $W_j(\nu, y_j, t)$ as

$$\begin{aligned} \frac{\partial^2 W_j}{\partial t^2} - \frac{\partial^2 W_j}{\partial y_j^2} = &- K_j(y_j)\frac{\partial W_j}{\partial y_j} - \nu_k^2 \frac{m_{33}(x_3)}{\varepsilon_{jj}(x_3)}\Big|_{x_3=\tau_j^{-1}(y_j)} W_j \\ &+ \nu_j \nu_k \frac{m_{33}(x_3)}{\varepsilon_{jj}(x_3)}\Big|_{x_3=\tau_j^{-1}(y_j)} \tilde{E}_k \\ &+ (\mathrm{i}\nu_j)\Big[\frac{1}{\varepsilon_{jj}(x_3)}\frac{\partial}{\partial x_3}(m_{kk}(x_3)\tilde{E}_3(\nu, x_3, t))\Big]_{x_3=\tau_j^{-1}(y_j)} \\ &+ \frac{\tilde{f}_j(\nu, x_3, t)}{\varepsilon_{jj}(x_3)}\Big|_{x_3=\tau_j^{-1}(y_j)}, \end{aligned} \tag{10}$$

where

$$
(11)\qquad K_j(y_j)=\frac{\mathrm{d}}{\mathrm{d}y_j}(\ln A_j(y_j)),\quad A_j(y_j)=\frac{1}{\sqrt{m_{kk}(x_3)\varepsilon_{jj}(x_3)}}\Big|_{x_3=\tau_j^{-1}(y_j)},
$$
$$
j=1,2,\ k\neq j,\ k=1,2.
$$

Let us introduce the function $U_j(\nu,y_j,t)$ by the equality

$$
(12)\qquad W_j(\nu,y_j,t)=S_j(y_j)U_j(\nu,y_j,t),
$$

where the function $S_j(y_j)$ is defined by

$$
(13)\qquad S_j(y_j)=\exp\Bigl(\frac{1}{2}\int_0^{y_j}K_j(\xi)\,\mathrm{d}\xi\Bigr).
$$

Substituting (12) into (10), we find

$$
\begin{aligned}
(14)\qquad &\frac{\partial^2 U_j}{\partial t^2}-\frac{\partial^2 U_j}{\partial y_j^2}\\
&=[q_j(y_j)-\nu_k^2M_{3j}(y_j)N_j(y_j)]U_j+\nu_j\nu_kM_{3j}(y_j)L_j(y_j)\tilde{E}_k(\nu,\tau_j^{-1}(y_j),t)\\
&\quad+(\mathrm{i}\nu_j)\frac{\partial}{\partial y_j}[C_j(y_j)L_j(y_j)M_{kj}(y_j)\tilde{E}_3(\nu,\tau_j^{-1}(y_j),t)]\\
&\quad-(\mathrm{i}\nu_j)M_{kj}(y_j)\frac{\partial}{\partial y_j}[C_j(y_j)L_j(y_j)]\tilde{E}_3(\nu,\tau_j^{-1}(y_j),t)+F_j(\nu,y_j,t),\\
&\qquad j=1,2,\ k\neq j,\ k=1,2.
\end{aligned}
$$

Here the following notation was used:

$$
\begin{aligned}
(15)\qquad &q_j(y_j)=\frac{1}{2}K_j'(y_j)-\frac{1}{4}K_j^2(y_j),\quad C_j(y_j)=c_j(x_3)|_{x_3=\tau_j^{-1}(y_j)},\\
&N_j(y_j)=\frac{1}{\varepsilon_{jj}(x_3)}\Big|_{x_3=\tau_j^{-1}(y_j)},\quad M_{lj}(y_j)=m_{ll}(x_3)|_{x_3=\tau_j^{-1}(y_j)},\quad l=1,2,3;\\
&L_j(y)=\frac{N_j(y)}{S_j(y)},\quad F_j(\nu,y_j,t)=\tilde{f}_j(\nu,\tau_j^{-1}(y_j),t)L_j(y_j),
\end{aligned}
$$

where $K_j(y_j)$, $S_j(y_j)$ are defined by (11), (13). Using the d'Alembert formula ([17], see also Appendix A), we can show that equation (14) with zero initial data is equivalent to the integral equation

$$(16)\ U_j(\nu, y_j, t) = \frac{1}{2}\int_0^t \int_{y_j-(t-\tau)}^{y_j+(t-\tau)} \Big\{[q_j(\xi) - \nu_k^2 M_{3j}(\xi)N_j(\xi)]U_j(\nu,\xi,\tau)$$
$$+ \nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu, \tau_j^{-1}(\xi), \tau)$$
$$- (\mathrm{i}\nu_j)M_{kj}(\xi)\frac{\partial}{\partial\xi}[C_j(\xi)L_j(\xi)]\tilde{E}_3(\nu,\tau_j^{-1}(\xi),\tau) + F_j(\nu,\xi,\tau)\Big\}\,\mathrm{d}\xi\,\mathrm{d}\tau$$
$$+ \frac{\mathrm{i}\nu_j}{2}\int_0^t \{C_j(\xi)L_j(\xi)M_{kj}(\xi)\tilde{E}_3(\nu,\tau_j^{-1}(\xi),\tau)\}\Big|_{\xi=y_j-(t-\tau)}^{\xi=y_j+(t-\tau)}\,\mathrm{d}\tau,$$
$$j=1,2,\ k\neq j,\ k=1,2.$$

Here and throughout the paper we have used the notation $\{f(\xi)\}\big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)} = f(y+(t-\tau)) - f(y-(t-\tau))$.

By virtue of (8) and (12) equation (16) may be written as follows:

$$(17)\ \tilde{E}_j(\nu, x_3, t)$$
$$= \frac{S_j(\tau_j(x_3))}{2}\int_0^t \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \Big\{[q_j(\xi) - \nu_k^2 M_{3j}(\xi)N_j(\xi)]$$
$$\times \frac{\tilde{E}_j(\nu,\tau_j^{-1}(\xi),\tau)}{S_j(\xi)} + \nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu,\tau_j^{-1}(\xi),\tau)$$
$$- (\mathrm{i}\nu_j)M_{kj}(\xi)\frac{\partial}{\partial\xi}[C_j(\xi)L_j(\xi)]\tilde{E}_3(\nu,\tau_j^{-1}(\xi),\tau) + F_j(\nu,\xi,\tau)\Big\}\,\mathrm{d}\xi\,\mathrm{d}\tau$$
$$+ \frac{\mathrm{i}\nu_j}{2}S_j(\tau_j(x_3))\int_0^t \{C_j(\xi)L_j(\xi)M_{kj}(\xi)\tilde{E}_3(\nu,\tau_j^{-1}(\xi),\tau)\}\Big|_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)}\,\mathrm{d}\tau,$$
$$j=1,2,\ k\neq j,\ k=1,2.$$

3.2. Integral equalities for $\tilde{E}_3$, $\partial\tilde{E}_3/\partial t$

Let us view (5) as the inhomogeneous ordinary differential equation whose coefficients depend on the parameters ν_1, ν_2, and x_3. Let the right-hand side of (5) be the inhomogeneous term. Then integrating the equation (5) with respect to t with zero initial data, we find the integral equality for $\tilde{E}_3(\nu, x_3, t)$:

$$(18)\qquad \tilde{E}_3(\nu,x_3,t) = \frac{1}{\varepsilon_{33}(x_3)}\int_0^t \Big[\mathrm{i}\nu_1 m_{22}(x_3)\frac{\partial\tilde{E}_1}{\partial x_3}(\nu,x_3,\tau)$$
$$+ \mathrm{i}\nu_2 m_{11}(x_3)\frac{\partial\tilde{E}_2}{\partial x_3}(\nu,x_3,\tau) + \tilde{f}_3(\nu,x_3,\tau)\Big]$$
$$\times \frac{\sin(d(\nu,x_3)(t-\tau))}{d(\nu,x_3)}\,\mathrm{d}\tau,$$

where

$$d(\nu, x_3) = \sqrt{\frac{\nu_1^2 m_{22}(x_3) + \nu_2^2 m_{11}(x_3)}{\varepsilon_{33}(x_3)}}. \tag{19}$$

Differentiating (18) with respect to t, we find the integral equality for $\frac{\partial \tilde{E}_3}{\partial t}(\nu, x_3, t)$:

$$\begin{aligned} \frac{\partial \tilde{E}_3}{\partial t}&(\nu, x_3, t) \\ &= \frac{1}{\varepsilon_{33}(x_3)} \int_0^t \Big[\mathrm{i}\nu_1 m_{22}(x_3) \frac{\partial \tilde{E}_1}{\partial x_3}(\nu, x_3, \tau) \\ &\quad + \mathrm{i}\nu_2 m_{11}(x_3) \frac{\partial \tilde{E}_2}{\partial x_3}(\nu, x_3, \tau) + \tilde{f}_3(\nu, x_3, \tau)\Big] \cos(d(\nu, x_3)(t-\tau))\,\mathrm{d}\tau. \end{aligned} \tag{20}$$

3.3. Integral equalities for $\partial \tilde{E}_j / \partial x_3$, $j = 1, 2$

In this subsection we will obtain integral equalities for $(\partial \tilde{E}_j/\partial x_3)(\nu, x_3, t)$, $j = 1, 2$, in the form containing functions $\tilde{E}_j(\nu, x_3, t)$, $(\partial \tilde{E}_j/\partial x_3)(\nu, x_3, t)$, $j = 1, 2$, $\tilde{E}_3(\nu, x_3, t)$, $(\partial \tilde{E}_3/\partial t)(\nu, x_3, t)$. A starting point here is the equation (17) which can be written in the form

$$\begin{aligned} \tilde{E}_j&(\nu, x_3, t) \\ &= \frac{S_j(\tau_j(x_3))}{2} \int_0^t \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \Big\{ [q_j(\xi) - \nu_k^2 M_{3j}(\xi) N_j(\xi)] \frac{\tilde{E}_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} \\ &\quad + \nu_j \nu_k M_{3j}(\xi) L_j(\xi) \tilde{E}_k(\nu, \tau_j^{-1}(\xi), \tau) \\ &\quad - (\mathrm{i}\nu_j) M_{kj}(\xi) \frac{\partial}{\partial \xi}[C_j(\xi) L_j(\xi)] \tilde{E}_3(\nu, \tau_j^{-1}(\xi), \tau) + F_j(\nu, \xi, \tau) \Big\}\,\mathrm{d}\xi\,\mathrm{d}\tau \\ &\quad + \frac{\mathrm{i}\nu_j}{2} S_j(\tau_j(x_3)) \\ &\quad \times \Big\{ \int_{\tau_j(x_3)}^{\tau_j(x_3)+t} C_j(\eta) L_j(\eta) M_{kj}(\eta) \tilde{E}_3(\nu, \tau_j^{-1}(\eta), \tau_j(x_3) + (t-\eta))\,\mathrm{d}\eta \\ &\quad - \int_{\tau_j(x_3)-t}^{\tau_j(x_3)} C_j(\mu) L_j(\mu) M_{kj}(\mu) \tilde{E}_3(\nu, \tau_j^{-1}(\mu), -\tau_j(x_3) + (t+\mu))\,\mathrm{d}\mu \Big\}, \\ &\qquad\qquad j = 1, 2,\ k \neq j,\ k = 1, 2. \end{aligned} \tag{21}$$

Differentiating (21) with respect to x_3 we find

$$
\begin{aligned}
(22)\quad & \frac{\partial \tilde{E}_j}{\partial x_3}(\nu, x_3, t) \\
&= \frac{c_j(\tau_j(x_3))S_j'(\tau_j(x_3))}{2}\biggl\{\int_0^t \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \Bigl\{[q_j(\xi) - \nu_k^2 M_{3j}(\xi)N_j(\xi)] \\
&\quad \times \frac{\tilde{E}_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} + \nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu, \tau_j^{-1}(\xi), \tau) \\
&\quad - (\mathrm{i}\nu_j)M_{kj}(\xi)\frac{\partial}{\partial \xi}[C_j(\xi)L_j(\xi)]\tilde{E}_3(\nu, \tau_j^{-1}(\xi), \tau) + F_j(\nu, \xi, \tau)\Bigr\}\,\mathrm{d}\xi\,\mathrm{d}\tau \\
&\quad + (\mathrm{i}\nu_j)\biggl[\int_{\tau_j(x_3)}^{\tau_j(x_3)+t} C_j(\eta)L_j(\eta)M_{kj}(\eta)\tilde{E}_3(\nu, \tau_j^{-1}(\eta), \tau_j(x_3) + (t-\eta))\,\mathrm{d}\eta \\
&\quad - \int_{\tau_j(x_3)-t}^{\tau_j(x_3)} C_j(\mu)L_j(\mu)M_{kj}(\mu)\tilde{E}_3(\nu, \tau_j^{-1}(\mu), -\tau_j(x_3) + (t+\mu))\,\mathrm{d}\mu\biggr]\biggr\} \\
&\quad + \frac{S_j(\tau_j(x_3))}{2}\int_0^t \Bigl\{[q_j(\xi) - \nu_k^2 M_{3j}(\xi)N_j(\xi)]\frac{\tilde{E}_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} \\
&\quad + \nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu, \tau_j^{-1}(\xi), \tau) - (\mathrm{i}\nu_j)M_{kj}(\xi)\frac{\partial}{\partial \xi}[C_j(\xi)L_j(\xi)] \\
&\quad \times \tilde{E}_3(\nu, \tau_j^{-1}(\xi), \tau) + F_j(\nu, \xi, \tau)\Bigr\}_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)}\,\mathrm{d}\tau + \frac{\mathrm{i}\nu_j c_j(x_3)}{2}S_j(\tau_j(x_3)) \\
&\quad \times \biggl\{\int_{\tau_j(x_3)}^{\tau_j(x_3)+t} C_j(\eta)L_j(\eta)M_{kj}(\eta)\frac{\partial \tilde{E}_3}{\partial t}(\nu, \tau_j^{-1}(\eta), \tau_j(x_3) + (t-\eta))\,\mathrm{d}\eta \\
&\quad + \int_{\tau_j(x_3)-t}^{\tau_j(x_3)} C_j(\mu)L_j(\mu)M_{kj}(\mu)\frac{\partial \tilde{E}_3}{\partial t}(\nu, \tau_j^{-1}(\mu), -\tau_j(x_3) + (t+\mu))\,\mathrm{d}\mu\biggr\} \\
&\quad - (\mathrm{i}\nu_j)c_j(x_3)S_j(\tau_j(x_3))C_j(\tau_j(x_3))L_j(\tau_j(x_3))M_{kj}(\tau_j(x_3))\tilde{E}_3(\nu, x_3, t), \\
&\qquad\qquad\qquad\qquad\qquad\qquad j = 1,2,\ k \neq j,\ k = 1,2.
\end{aligned}
$$

Using (18), the equation (22) can be written as

$$
\begin{aligned}
(23)\quad & \frac{\partial \tilde{E}_j}{\partial x_3}(\nu, x_3, t) \\
&= \frac{c_j(\tau_j(x_3))S_j'(\tau_j(x_3))}{2}\biggl\{\int_0^t \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \Bigl\{[q_j(\xi) - \nu_k^2 M_{3j}(\xi)N_j(\xi)] \\
&\quad \times \frac{\tilde{E}_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} + \nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu, \tau_j^{-1}(\xi), \tau) \\
&\quad - (\mathrm{i}\nu_j)M_{kj}(\xi)\frac{\partial}{\partial \xi}[C_j(\xi)L_j(\xi)]\tilde{E}_3(\nu, \tau_j^{-1}(\xi), \tau) + F_j(\nu, \xi, \tau)\Bigr\}\,\mathrm{d}\xi\,\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ (\mathrm{i}\nu_j) \int_0^t \{C_j(z)L_j(z)M_{kj}(z)\tilde{E}_3(\nu, \tau_j^{-1}(z), \tau)\}\Big|_{z=\tau_j(x_3)-(t-\tau)}^{z=\tau_j(x_3)+(t-\tau)} \,\mathrm{d}\tau \Big\} \\
&+ \frac{S_j(\tau_j(x_3))}{2} \int_0^t \Big\{ [q_j(\xi) - \nu_k^2 M_{3j}(\xi) N_j(\xi)] \frac{\tilde{E}_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} \\
&+ \nu_j \nu_k M_{3j}(\xi) L_j(\xi) \tilde{E}_k(\nu, \tau_j^{-1}(\xi), \tau) - (\mathrm{i}\nu_j) M_{kj}(\xi) \frac{\partial}{\partial \xi}[C_j(\xi)L_j(\xi)] \\
&\times \tilde{E}_3(\nu, \tau_j^{-1}(\xi), \tau) + F_j(\nu, \xi, \tau) \Big\} \Big|_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)} \,\mathrm{d}\tau \\
&+ \frac{\mathrm{i}\nu_j c_j(x_3)}{2} S_j(\tau_j(x_3)) \\
&\times \Big\{ \int_0^t \Big\{ C_j(z)L_j(z)M_{kj}(z) \frac{\partial \tilde{E}_3}{\partial t}(\nu, \tau_j^{-1}(z), \tau) \Big\} \Big|_{z=\tau_j(x_3)+(t-\tau)} \\
&+ \Big\{ C_j(z)L_j(z)M_{kj}(z) \frac{\partial \tilde{E}_3}{\partial t}(\nu, \tau_j^{-1}(z), \tau) \Big\}_{z=\tau_j(x_3)-(t-\tau)} \,\mathrm{d}\tau \Big\} \\
&- \frac{(\mathrm{i}\nu_j)c_j(x_3)}{\varepsilon_{33}(x_3)} S_j(\tau_j(x_3)) C_j(\tau_j(x_3)) L_j(\tau_j(x_3)) M_{kj}(\tau_j(x_3)) \\
&\times \int_0^t \Big[\mathrm{i}\nu_1 m_{22}(x_3) \frac{\partial \tilde{E}_1}{\partial x_3}(\nu, x_3, \tau) + \mathrm{i}\nu_2 m_{11}(x_3) \frac{\partial \tilde{E}_2}{\partial x_3}(\nu, x_3, \tau) \\
&+ \tilde{f}_3(\nu, x_3, \tau) \Big] \frac{\sin\big(d(\nu, x_3)(t-\tau)\big)}{d(\nu, x_3)} \,\mathrm{d}\tau, \\
&\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2,\ k \neq j,\ k = 1, 2.
\end{aligned}
$$

3.4. An operator integral equation

Equations (17), (18), (20), (23) represent a system of integral equations with respect to the unknowns $\tilde{E}_j$, $\tilde{E}_3$, $\partial \tilde{E}_3/\partial t$, $\partial \tilde{E}_j/\partial x_3$, $j = 1, 2$. Reasonings of Subsections 3.1–3.3 show that this system is equivalent to (4), (5) under condition (6). The system (17)–(20), (23) can be written in the form of the operator integral equation

$$
\mathbf{V}(\nu, x_3, t) = \mathbf{G}(\nu, x_3, t) + \int_0^t (\mathbf{K}\mathbf{V})(\nu, x_3, t, \tau) \,\mathrm{d}\tau, \tag{24}
$$

where $\mathbf{V} = (V_1, V_2, V_3, V_4, V_5, V_6)$ is the unknown vector-function whose components are

$$
\begin{gathered}
V_1 = \tilde{E}_1, \quad V_2 = \tilde{E}_2, \quad V_3 = \tilde{E}_3, \\
V_4 = \frac{\partial \tilde{E}_3}{\partial t}, \quad V_5 = \frac{\partial \tilde{E}_1}{\partial x_3}, \quad V_6 = \frac{\partial \tilde{E}_2}{\partial x_3};
\end{gathered} \tag{25}
$$

$\mathbf{G} = (G_1, G_2, G_3, G_4, G_5, G_6)$ is the given vector-function whose components are defined by

$$G_j(\nu, x_3, t) = \frac{S_j(\tau_j(x_3))}{2} \int_0^t \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} F_j(\nu, \xi, \tau)\, \mathrm{d}\xi\, \mathrm{d}\tau, \quad j = 1, 2, \tag{26}$$

$$G_3(\nu, x_3, t) = \frac{1}{\varepsilon_{33}(x_3)} \int_0^t \tilde{f}_3(\nu, x_3, \tau) \frac{\sin(d(\nu, x_3)(t-\tau))}{d(\nu, x_3)}\, \mathrm{d}\tau, \tag{27}$$

$$G_4(\nu, x_3, t) = \frac{1}{\varepsilon_{33}(x_3)} \int_0^t \tilde{f}_3(\nu, x_3, \tau) \cos(d(\nu, x_3)(t-\tau))\, \mathrm{d}\tau, \tag{28}$$

$$\begin{aligned} G_{4+j}(\nu, x_3, t) &= \frac{c_j(\tau_j(x_3)) S_j'(\tau_j(x_3))}{2} \int_0^t \bigg\{ \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} F_j(\nu, \xi, \tau)\, \mathrm{d}\xi \\ &\quad + (\mathrm{i}\nu_j)\{F_j(\nu, \xi, \tau)\}\Big|_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)} \bigg\} \mathrm{d}\tau \\ &\quad - (\mathrm{i}\nu_j) \frac{c_j(x_3)}{\varepsilon_{33}(x_3)} S_j(\tau_j(x_3)) L_j(\tau_j(x_3)) M_{kj}(\tau_j(x_3)) \\ &\quad \times \int_0^t \tilde{f}_3(\nu, x_3, \tau) \frac{\sin(d(\nu, x_3)(t-\tau))}{d(\nu, x_3)}\, \mathrm{d}\tau, \quad j = 1, 2. \end{aligned} \tag{29}$$

The components of the vector-operator $\mathbf{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5, \mathcal{K}_6)$ are defined by

$$\begin{aligned} &(\mathcal{K}_j \mathbf{V})(\nu, x_3, t, \tau) \\ &= \frac{S_j(\tau_j(x_3))}{2} \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \bigg\{ [q_j(\xi) - \nu_k^2 M_{3j}(\xi) N_j(\xi)] \frac{V_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} \\ &\quad + \nu_j \nu_k M_{3j}(\xi) L_j(\xi) V_k(\nu, \tau_j^{-1}(\xi), \tau) \\ &\quad - (\mathrm{i}\nu_j) M_{kj}(\xi) \frac{\partial}{\partial \xi}[C_j(\xi) L_j(\xi)] V_3(\nu, \tau_j^{-1}(\xi), \tau) \bigg\} \mathrm{d}\xi \\ &\quad + \frac{\mathrm{i}\nu_j}{2} S_j(\tau_j(x_3)) \{C_j(\xi) L_j(\xi) M_{kj}(\xi) V_3(\nu, \tau_j^{-1}(\xi), \tau)\}\Big|_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)}, \\ &\qquad j = 1, 2,\ k \neq j,\ k = 1, 2; \end{aligned} \tag{30}$$

$$\begin{aligned} (\mathcal{K}_3 \mathbf{V})(\nu, x_3, t, \tau) &= \frac{1}{\varepsilon_{33}(x_3)} [\mathrm{i}\nu_1 m_{22}(x_3) V_5(\nu, x_3, \tau) \\ &\quad + \mathrm{i}\nu_2 m_{11}(x_3) V_6(\nu, x_3, \tau)] \frac{\sin(d(\nu, x_3)(t-\tau))}{d(\nu, x_3)}, \end{aligned} \tag{31}$$

$$\begin{aligned} (\mathcal{K}_4 \mathbf{V})(\nu, x_3, t, \tau) &= \frac{1}{\varepsilon_{33}(x_3)} [i\nu_1 m_{22}(x_3) V_5(\nu, x_3, \tau) \\ &\quad + \mathrm{i}\nu_2 m_{11}(x_3) V_6(\nu, x_3, \tau)] \cos(d(\nu, x_3)(t-\tau)), \end{aligned} \tag{32}$$

$$
\begin{aligned}
(33)\qquad &(\mathcal{K}_{4+j}\mathbf{V})(\nu, x_3, t, \tau)\\
&= \frac{c_j(\tau_j(x_3))S'_j(\tau_j(x_3))}{2}\Big\{\int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)}\Big\{[q_j(\xi)-\nu_k^2 M_{3j}(\xi)N_j(\xi)]\\
&\quad\times\frac{V_j(\nu,\tau_j^{-1}(\xi),\tau)}{S_j(\xi)}+\nu_j\nu_k M_{3j}(\xi)L_j(\xi)V_k(\nu,\tau_j^{-1}(\xi),\tau)\\
&\quad-(\mathrm{i}\nu_j)M_{kj}(\xi)\frac{\partial}{\partial\xi}[C_j(\xi)L_j(\xi)]V_3(\nu,\tau_j^{-1}(\xi),\tau)\Big\}\,\mathrm{d}\xi\\
&\quad+(\mathrm{i}\nu_j)\{C_j(z)L_j(z)M_{kj}(z)V_3(\nu,\tau_j^{-1}(z),\tau)\}\Big|_{z=\tau_j(x_3)-(t-\tau)}^{z=\tau_j(x_3)+(t-\tau)}\Big\}\\
&\quad+\frac{S_j(\tau_j(x_3))}{2}\Big\{[q_j(\xi)-\nu_k^2M_{3j}(\xi)N_j(\xi)]\frac{V_j(\nu,\tau_j^{-1}(\xi),\tau)}{S_j(\xi)}\\
&\quad+\nu_j\nu_kM_{3j}(\xi)L_j(\xi)V_k(\nu,\tau_j^{-1}(\xi),\tau)\\
&\quad-(\mathrm{i}\nu_j)M_{kj}(\xi)\frac{\partial}{\partial\xi}[C_j(\xi)L_j(\xi)]V_3(\nu,\tau_j^{-1}(\xi),\tau)\Big\}\Big|_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)}\\
&\quad+\frac{\mathrm{i}\nu_jc_j(x_3)}{2}S_j(\tau_j(x_3))\\
&\quad\times\{\{C_j(z)L_j(z)M_{kj}(z)V_4(\nu,\tau_j^{-1}(z),\tau)\}|_{z=\tau_j(x_3)+(t-\tau)}\\
&\quad+\{C_j(z)L_j(z)M_{kj}(z)V_4(\nu,\tau_j^{-1}(z),\tau)\}|_{z=\tau_j(x_3)-(t-\tau)}\}\\
&\quad-\frac{(\mathrm{i}\nu_j)c_j(x_3)}{\varepsilon_{33}(x_3)}S_j(\tau_j(x_3))C_j(\tau_j(x_3))L_j(\tau_j(x_3))M_{kj}(\tau_j(x_3))\\
&\quad\times[\mathrm{i}\nu_1m_{22}(x_3)V_5(\nu,x_3,\tau)\\
&\quad+\mathrm{i}\nu_2m_{11}(x_3)V_6(\nu,x_3,\tau)]\frac{\sin(d(\nu,x_3)(t-\tau))}{d(\nu,x_3)},\\
&\qquad\qquad\qquad j=1,2,\ k\neq j,\ k=1,2.
\end{aligned}
$$

4. Properties of the operator integral equation (24)

To establish the existence and uniqueness of the solution for (24) we have used specific properties of the inhomogeneous term and the kernel of (24). In this section these properties are established by the following two propositions.

Proposition 1. *Let the components of* $\mathbf{G}=(G_1,G_2,\dots,G_6)$ *be defined by* (26)–(29). *Then under the assumptions mentioned in Section* 1 *these components are continuous functions for* $(x_3,t)\in\Delta$, $\nu\in\mathbb{R}^2$.

Proposition 2. *Let the components of the vector operator* $\mathbf{K}=(\mathcal{K}_1,\mathcal{K}_2,\dots,\mathcal{K}_6)$ *be defined by* (30)–(33). *Then under the assumptions mentioned in Section* 1

(i) *the expressions*

$$\int_0^t (\mathcal{K}_m \mathbf{V})(\nu, x_3, t, \tau)\,\mathrm{d}\tau, \quad m = 1, 2, \ldots, 6,$$

are continuous functions for $(x_3, t) \in \Delta$, $\nu \in \mathbb{R}^2$ *and any vector function* $\mathbf{V}(\nu, x_3, t) = (V_1(\nu, x_3, t), V_2(\nu, x_3, t), \ldots, V_6(\nu, x_3, t))$ *with continuous components for* $(x_3, t) \in \Delta$, $\nu \in \mathbb{R}^2$*;*

(ii) *for any positive number* Ω *the following inequalities are satisfied:*

$$\left|\int_0^t (\mathcal{K}_m \mathbf{V})(\nu, x_3, t, \tau)\,\mathrm{d}\tau\right| \leqslant B \int_0^t \|\mathbf{V}\|(\nu, \tau)\,\mathrm{d}\tau, \quad m = 1, 2, \ldots, 6, \tag{34}$$

where $(x_3, t) \in \Delta$, $|\nu| \leqslant \Omega$, B *is a positive number depending on* α, β, T, Ω*;*

$$\|\mathbf{V}\|(\nu, \tau) = \max_{m=1,2,\ldots,6} \max_{\xi \in [-c(T-\tau), c(T-\tau)]} |V_m(\nu, \xi, \tau)|. \tag{35}$$

P r o o f of Proposition 1. Let numbers α, β, T, c, the set Δ, and functions $\varepsilon_{jj}(x_3)$, $m_{jj}(x_3)$ satisfy the assumptions of Section 1, numbers Y_j^-, Y_j^+ and functions τ_j, τ_j^{-1} have the properties from Lemma 1. Using the formulae (11), (13), (15), we find that the functions A_j, S_j, C_j, N_j, L_j, M_{lj} are twice continuously differentiable on $[Y_j^-, Y_j^+]$; the functions $\mathcal{K}_j$ are once continuously differentiable on $[Y_j^-, Y_j^+]$, and q_j, F_j are continuous on $[Y_j^-, Y_j^+]$. The function $d(\nu, x_3)$ defined by (19) is twice continuously differentiable with respect to $x_3 \in [-cT, cT]$ for any $\nu \in \mathbb{R}^2$ and $\sin(d(\nu, x_3)(t - \tau))/d(\nu, x_3)$ is bounded and twice continuously differentiable with respect to $(x_3, t) \in \Delta$ for any $\nu \in \mathbb{R}^2$, $0 \leqslant \tau \leqslant t$. Using properties τ_j described in Lemma 1, we find that $G_m(\nu, x_3, t)$, $m = 1, 2, \ldots, 6$, are continuous functions for $(x_3, t) \in \Delta$ and $\nu \in \mathbb{R}^2$. Proposition 1 is proved. $\square$

P r o o f of Proposition 2 (i). Using the reasoning made in the proof of Proposition 1 and formulae (30)–(33), we find that

$$\int_0^t (\mathcal{K}_m \mathbf{V})(\nu, x_3, t, \tau)\,\mathrm{d}\tau, \quad m = 1, 2, \ldots, 6,$$

are continuous functions with respect to $(x_3, t) \in \Delta$ for any $\nu \in \mathbb{R}^2$ and any vector function $\mathbf{V} = (V_1, V_2, \ldots, V_6)$ with continuous components $V_j(\nu, x_3, t)$ for $(x_3, t) \in \Delta$ and $\nu \in \mathbb{R}^2$. Hence, Proposition 2 (i) is proved. $\square$

To prove Proposition 2 (ii) we need the following lemma.

Lemma 2. *Let c, T be the numbers and Δ the triangle defined by* (3)*, let τ_j be the function defined by* (7)*, τ_j^{-1} the inverse function to τ_j; Y_j^-, Y_j^+ the numbers defined in Lemma* 1*. Then for any $(x_3,t)\in\Delta$, $\tau\in[0,t]$ and $\xi\in[\tau_j(x_3)-(t-\tau),\tau_j(x_3)+(t-\tau)]$, the following relations are satisfied:*

$$\tau_j^{-1}(\xi)\in[-c(T-\tau),c(T-\tau)],\quad \xi\in[Y_j^-,Y_j^+].$$

P r o o f of Lemma 2. Let $y=\tau_j(x_3)$. Using Lemma 1, we obtain

$$y=\int_0^{\tau_j^{-1}(y)}\sqrt{\frac{\varepsilon_{jj}(z)}{m_{jj}(z)}}\,\mathrm{d}z. \tag{36}$$

Differentiating both sides of (36) with respect to y, we get

$$\frac{\mathrm{d}\tau_j^{-1}(y)}{\mathrm{d}y}=\sqrt{\frac{\varepsilon_{jj}(\tau_j^{-1}(y))}{m_{jj}(\tau_j^{-1}(y))}}. \tag{37}$$

Integrating (37) from 0 to y and using $\tau_j^{-1}(0)=0$, we find

$$\tau_j^{-1}(y)=\int_0^{y}\sqrt{\frac{\varepsilon_{jj}(\tau_j^{-1}(z))}{m_{jj}(\tau_j^{-1}(z))}}\,\mathrm{d}z. \tag{38}$$

By (38), we come to

$$\begin{aligned}\tau_j^{-1}(\tau_j(x_3)-(t-\tau))&=\int_0^{\tau_j(x_3)-(t-\tau)}\sqrt{\frac{\varepsilon_{jj}(\tau_j^{-1}(z))}{m_{jj}(\tau_j^{-1}(z))}}\,\mathrm{d}z\\&=x_3-\int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)}\sqrt{\frac{\varepsilon_{jj}(\tau_j^{-1}(z))}{m_{jj}(\tau_j^{-1}(z))}}\,\mathrm{d}z.\end{aligned} \tag{39}$$

We have from (39)

$$\tau_j^{-1}(\tau_j(x_3)-(t-\tau))\geqslant x_3-c(t-\tau). \tag{40}$$

Using $(x_3,t)\in\Delta$, we have

$$x_3-c(t-\tau)\geqslant -c(T-t)-c(t-\tau)=-c(T-t)$$

and therefore,

$$\tau_j^{-1}(\tau_j(x_3) - (t-\tau)) \geqslant -c(T-\tau). \tag{41}$$

Similarly we find

$$\tau_j^{-1}(\tau_j(x_3) + (t-\tau)) \leqslant c(T-\tau). \tag{42}$$

As $\tau_j^{-1}(\xi)$ is monotonically increasing (see Lemma 1), using (41), (42), we obtain

$$-c(T-\tau) \leqslant \tau_j^{-1}(\xi) \leqslant c(T-\tau) \tag{43}$$

for any $\xi \in [\tau_j(x_3) - (t-\tau), \tau_j(x_3) + (t-\tau)]$. It follows from (43) that

$$-cT \leqslant \tau_j^{-1}(\xi) \leqslant cT$$

and therefore, as τ_j is monotonically increasing, we conclude that

$$Y_j^- = \tau_j(-cT) \leqslant \xi \leqslant \tau_j(cT) = Y_j^+ \tag{44}$$

for any $\xi \in [\tau_j(x_3) - (t-\tau), \tau_j(x_3) + (t-\tau)]$. Lemma 2 is proved. $\square$

P r o o f of Proposition 2 (ii). Let the number Q be defined by

$$Q = \max_{j=1,2} \max_{y \in [Y_j^-, Y_j^+]} \Big\{ |q_j(y)|, |L_j(y)|, |N_j(y)|, |S_j(y)|, |C_j(y)|, \\ \Big|\frac{\partial}{\partial y}(C_j(y)L_j(y))\Big|, \max_{l=1,2,3} |M_{lj}(y)| \Big\}.$$

Using Lemma 2, we deduce that for any $(x_3, t) \in \Delta$ and $\tau \in [0, t]$, $\xi \in [\tau_j(x_3) - (t-\tau), \tau_j(x_3) + (t-\tau)]$ the following inequalities are satisfied:

$$|q_j(\xi)| \leqslant Q, \ |L_j(\xi)| \leqslant Q, \ |N_j(\xi)| \leqslant Q, \ |S_j(\xi)| \leqslant Q, \ |C_j(\xi)| \leqslant Q, \\ \Big|\frac{\partial}{\partial \xi}(C_j(\xi)L_j(\xi))\Big| \leqslant Q, \ \max_{l=1,2,3} |M_{lj}(\xi)| \leqslant Q, \ \|V_m(\nu, \tau_j^{-1}(\xi), \tau)\| \leqslant \|\mathbf{V}\|(\nu, \tau), \\ j = 1, 2, \ m = 1, 2, \ldots, 6.$$

From the above inequalities and the equation (30) we find the relation

$$|(\mathcal{K}_j \mathbf{V})(\nu, x_3, t, \tau)| \leqslant \frac{Q}{2} \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \{(Q + |\nu|^2 Q^2)|V_j(\nu, \tau_j^{-1}(\xi), \tau)| \\ + |\nu|^2 Q^2 |V_k(\nu, \tau_j^{-1}(\xi), \tau)| + |\nu| Q^2 |V_3(\nu, \tau_j^{-1}(\xi), \tau)|\} \, \mathrm{d}\xi \\ + |\nu| Q^4 \|\mathbf{V}\|(\nu, \tau), \quad j = 1, 2, \ k = 1, 2, \ k \neq j.$$

Let Ω be any positive number. Then the last relation implies

$$|(\mathcal{K}_j\mathbf{V})(\nu,x_3,t,\tau)| \leqslant B_j(T,\Omega)\|\mathbf{V}\|(\nu,\tau),$$

where $(x_3,t)\in\Delta$, $|\nu|\leqslant\Omega$,

$$B_j(T,\Omega) = \max_{(x_3,t)\in\Delta}\{Q^2T(1+2\Omega^2Q+\Omega Q)+\Omega Q^4\}, \quad j=1,2.$$

We find from the equation (31) that

$$\Big|(\mathcal{K}_3\mathbf{V})(\nu,x_3,t,\tau)\Big| \leqslant B_3(T,\Omega)\|\mathbf{V}\|(\nu,\tau),$$

where $(x_3,t)\in\Delta$, $|\nu|\leqslant\Omega$, $B_3(T,\Omega)=2c^2|\Omega|T$.

Using a similar reasoning, we can define constants $B_m(T,\Omega)$ for $m=4,5,6$ such that

$$(\mathcal{K}_m\mathbf{V})(\nu,x_3,t,\tau)| \leqslant B_m(T,\Omega)\|\mathbf{V}\|(\nu,\tau),$$

where $(x_3,t)\in\Delta$, $|\nu|\leqslant\Omega$.

Choosing

$$B = \max_{m=1,2,\dots,6} B_m(T,\Omega),$$

we complete the proof of Proposition 2 (ii). □

5. Solving operator integral equation (24)

Let α, β, T be positive numbers, $\alpha\leqslant\beta$, $c=\sqrt{\beta/\alpha}$, let Δ be defined by (3) and let all assumptions stated in Section 1 be satisfied. In this section we solve the integral equation (24) by the method of successive approximations and then show that this solution is unique in the class of vector functions with continuous components for $(\nu,x_3,t)\in\mathbb{R}^2\times\Delta$.

5.1. Successive approximations and convergence

Let Ω be an arbitrary positive number. Let us consider the integral equation (24) for $(x_3,t)\in\Delta$, $|\nu|\leqslant\Omega$. To find a solution of this equation we apply the successive approximations

$$\begin{aligned} &\mathbf{V}^{(0)}(\nu,x_3,t) = \mathbf{G}(\nu,x_3,t),\\ &\mathbf{V}^{(n)}(\nu,x_3,t) = \int_0^t (\mathbf{K}\mathbf{V}^{(n-1)})(\nu,x_3,t,\tau)\,\mathrm{d}\tau, \quad n=1,2\dots. \end{aligned} \tag{45}$$

Our goal is to show that for $(x_3, t) \in \Delta$, $|\nu| \leqslant \Omega$, the series

$$\sum_{n=0}^{\infty} \mathbf{V}^{(n)}(\nu, x_3, t) = \Big(\sum_{n=1}^{\infty} V_1^{(n)}(\nu, x_3, t), \dots, \sum_{n=1}^{\infty} V_6^{(n)}(\nu, x_3, t)\Big)$$

is uniformly convergent to a vector function

$$\mathbf{V}(\nu, x_3, t) = (V_1(\nu, x_3, t), V_2(\nu, x_3, t), \dots, V_6(\nu, x_3, t))$$

with continuous components and this vector function is a solution of (24).

Indeed, we find from (45) and Propositions 1, 2 of Section 4 that for $(x_3, t) \in \Delta$, $|\nu| \leqslant \Omega$, the vector functions $\mathbf{V}^{(n)}(\nu, x_3, t)$, $n = 0, 1, 2 \dots$, have continuous components and

$$|V_j^{(n)}(\nu, x_3, t)| \leqslant B \int_0^t \|\mathbf{V}^{(n-1)}\|(\nu, \tau)\, \mathrm{d}\tau, \tag{46}$$

where $\|\cdot\|(\nu, \tau)$ and B are defined in Proposition 2.

It follows from (46) that

$$|V_m^{(n)}(\nu, x_3, t)| \leqslant \frac{(BT)^n}{n!} \max_{|\nu| \leqslant \Omega} \|\mathbf{G}\|(\nu, T), \quad m = 1, 2, \dots, 6,\ n = 0, 1, 2 \dots. \tag{47}$$

The uniform convergence of $\sum\limits_{n=0}^{\infty} V_m^{(n)}(\nu, x_3, t)$ to a continuous function $V_m(\nu, x_3, t)$ follows from the inequality (47) and the first Weierstrass theorem ([2], p. 425). Let us show that the vector function $\mathbf{V}(\nu, x_3, t)$ is a solution of (24).

Summing the equation (45) with respect to n from 1 to N, we have

$$\sum_{n=1}^{N} \mathbf{V}^{(n)}(\nu, x_3, t) = \sum_{n=0}^{N-1} \int_0^t (\mathbf{K}\mathbf{V}^{(n)})(\nu, x_3, t, \tau)\, \mathrm{d}\tau, \tag{48}$$

where

$$\sum_{n=1}^{N} \mathbf{V}^{(n)}(\nu, x_3, t) = \Big(\sum_{n=1}^{N} V_1^{(n)}(\nu, x_3, t), \dots, \sum_{n=1}^{N} V_6^{(n)}(\nu, x_3, t)\Big).$$

Adding the vector function $\mathbf{G}(\nu, x_3, t)$ to both sides of (48), we arrive at

$$\sum_{n=0}^{N} \mathbf{V}^{(n)}(\nu, x_3, t) = \mathbf{G}(\nu, x_3, t) + \int_0^t \sum_{n=0}^{N-1} (\mathbf{K}\mathbf{V}^{(n)})(\nu, x_3, t, \tau)\, \mathrm{d}\tau. \tag{49}$$

Letting N tend to infinity and using the second Weierstrass theorem ([2], p. 426), we find that the vector function $\mathbf{V}(\nu, x_3, t)$ satisfies (24) for $(x_3, t) \in \Delta$, $|\nu| \leqslant \Omega$. Since

Ω is an arbitrary positive number, we conclude that the vector function $\mathbf{V}(\nu, x_3, t)$ with continuous components is a solution of (24) for $(x_3, t) \in \Delta$, $\nu \in \mathbb{R}^2$.

5.2. Uniqueness of the solution

We prove here that the solution $\mathbf{V}(\nu, x_3, t)$ of (24) is unique in the class of vector functions with components $V_m \in C(\mathbb{R}^2 \times \Delta)$, $m = 1, 2, \ldots, 6$. Indeed, let Ω be an arbitrary positive number, let $\mathbf{V}(\nu, x_3, t)$ and $\mathbf{V}^*(\nu, x_3, t)$ be two solutions of (24) with continuous components for $(x_3, t) \in \Delta$, $|\nu| \leqslant \Omega$. Setting $\hat{\mathbf{V}}(\nu, x_3, t) = \mathbf{V}(\nu, x_3, t) - \mathbf{V}^*(\nu, x_3, t)$, we find from (24)

$$\hat{\mathbf{V}}(\nu, x_3, t) = \int_0^t (\mathbf{K}\hat{\mathbf{V}})(\nu, x_3, t, \tau)\,\mathrm{d}\tau. \tag{50}$$

Using Proposition 2, we obtain from (50)

$$\|\hat{\mathbf{V}}\|(\nu, t) \leqslant B \int_0^t \|\hat{\mathbf{V}}\|(\nu, \tau)\,\mathrm{d}\tau, \tag{51}$$

where $|\nu| \leqslant \Omega$, $t \in [0, T]$; $\|\cdot\|(\nu, t)$ and B are defined in the statement of Proposition 2.

Applying Gronwall's lemma [9] to (51), we get

$$\|\hat{\mathbf{V}}\|(\nu, t) = 0, \quad t \in [0, T],\ |\nu| \leqslant \Omega. \tag{52}$$

Using the continuity of $\hat{\mathbf{V}}(\nu, x_3, t)$, we conclude that

$$\hat{\mathbf{V}}(\nu, x_3, t) \equiv 0, \quad (x_3, t) \in \Delta,\ |\nu| \leqslant \Omega.$$

Since Ω is an arbitrary positive number, we obtain that $\mathbf{V}(\nu, x_3, t) \equiv \mathbf{V}^*(\nu, x_3, t)$ for $(x_3, t) \in \Delta$, $\nu \in \mathbb{R}^2$. The uniqueness of the solution of (24) is proved.

6. Constructing a solution of initial value problem (1), (2)

In this section we show that a solution of (1), (2) may be found by the inverse Fourier transform of the first three components of $\mathbf{V}(\nu, x_3, t)$, where $\mathbf{V}(\nu, x_3, t)$ is the generalized solution of (24) found in Section 5. We describe the class of vector functions where the solution of (1), (2) is unique.

We will use the following notions and notation. For the exponent $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_j \in \{0, 1, 2, \ldots\}$ and $|\alpha| = \alpha_1 + \alpha_2$, the partial derivatives of higher order

$$\frac{\partial^{|\alpha|}}{\partial\nu_1^{\alpha_1}\partial\nu_2^{\alpha_2}}\tilde{f}_k(\nu, x_3, t), \quad \frac{\partial^{|\alpha|}}{\partial\nu_1^{\alpha_1}\partial\nu_2^{\alpha_2}}V_l(\nu, x_3, t), \quad k = 1, 2, 3,\ l = 1, 2, \ldots, 6,$$

will be denoted by

$$D_\nu^\alpha \tilde{f}_k(\nu, x_3, t), \quad D_\nu^\alpha V_l(\nu, x_3, t).$$

For vector functions $\mathbf{V} = (V_1, V_2, \ldots, V_6)$, $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ and each α we define $D_\nu^\alpha \mathbf{V}$ and $D_\nu^\alpha \tilde{\mathbf{f}}$ by

$$D_\nu^\alpha \mathbf{V} = (D_\nu^\alpha V_1, D_\nu^\alpha V_2, \ldots, D_\nu^\alpha V_6), \quad D_\nu^\alpha \tilde{\mathbf{f}} = (D_\nu^\alpha \tilde{f}_1, D_\nu^\alpha \tilde{f}_2, D_\nu^\alpha \tilde{f}_3).$$

We denote by $C(\mathbb{R}^2)$ the class consisting of all continuous functions that are defined on $\mathbb{R}^2$. Then for $m = 0, 1, 2, \ldots$ we define $C^m(\mathbb{R}^2)$ by $C^0(\mathbb{R}^2) = C(\mathbb{R}^2)$ and otherwise by

$$C^m(\mathbb{R}^2) = \{\varphi(\nu) \in C(\mathbb{R}^2)\colon \text{ for all } |\alpha| \leqslant m,\ D_\nu^\alpha \varphi(\nu) \in C(\mathbb{R}^2)\},$$
$$C^\infty(\mathbb{R}^2) = \bigcap_{m=1}^{\infty} C^m(\mathbb{R}^2).$$

Further, $C_c(\mathbb{R}^2)$ is the class of all functions from $C(\mathbb{R}^2)$ with compact supports; $\mathcal{L}_2(\mathbb{R}^2)$ is the class of all square integrable functions over $\mathbb{R}^2$; $\|\varphi\|_2$ is defined for each $\varphi(\nu) \in \mathcal{L}_2(\mathbb{R}^2)$ by

$$\|\varphi\|_2^2 = \int_{\mathbb{R}^2} |\varphi(\nu)|^2 \, \mathrm{d}\nu.$$

The Paley-Wiener space $PW(\mathbb{R}^2)$ is the space consisting of all functions $\varphi(x_1, x_2) \in C^\infty(\mathbb{R}^2)$ satisfying (see Appendix B)

(a) $\big(1 + \sqrt{x_1^2 + x_2^2}\big)^m \Delta^n \varphi(x_1, x_2) \in \mathcal{L}_2(\mathbb{R}^2)$ for all $m, n \in \{0, 1, 2 \ldots\}$,

(b) $R_\varphi^\Delta = \lim\limits_{n\to\infty} \|\Delta^n \varphi(x_1, x_2)\|_2^{1/2n} < \infty$,

where $\Delta^n = (\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2)^n$.

The class $C(\Delta; C_c(\mathbb{R}^2))$ consists of all continuous mappings of $(x_3, t) \in \Delta$ into the class $C(\mathbb{R}^2)$ of functions $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$; $C(\Delta; PW(\mathbb{R}^2))$ is the class of all continuous mappings of Δ into $PW(\mathbb{R}^2)$.

In this section we suppose that the assumptions of Section 1 hold and $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ is the Fourier transform with respect to x_1, x_2 of the inhomogeneous term $\mathbf{f}$ in (1) such that for each α

$$D_\nu^\alpha \tilde{f}_k \in C(\mathbb{R}^2 \times \Delta) \cap C(\Delta; C_c(\mathbb{R}^2)), \quad k = 1, 2, 3.$$

Let us consider the problem (4)–(6) (FTIVP). It was shown in Section 3 that this problem is equivalent to the operator integral equation (24). In Section 5 using the successive approximations a solution of (24) was constructed. Using this solution, we find a unique vector function $\tilde{\mathbf{E}}(\nu, x_3, t) = (\tilde{E}_1(\nu, x_3, t), \tilde{E}_2(\nu, x_3, t), \tilde{E}_3(\nu, x_3, t))$

such that $\tilde{E}_l, (\partial/\partial x_3)\tilde{E}_j, (\partial/\partial t)\tilde{E}_3 \in C(\mathbb{R}^2 \times \Delta)$, $l = 1,2,3$, $j = 1,2$; and this vector function $\tilde{\mathbf{E}}(\nu, x_3, t)$ will be a generalized solution of FTIVP (4)–(6).

To complete the reasoning of this section let us show that the inverse Fourier transform $\mathcal{F}_\nu^{-1}$ with respect to $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ is applicable to $\tilde{\mathbf{E}}(\nu, x_3, t)$, and $\mathbf{E}(x,t) = \mathcal{F}_\nu^{-1}[\tilde{\mathbf{E}}]$ is the unique generalized solution of (1), (2).

Using the Proposition 1 and the assumptions of this section, we see $G_m(\nu, x_3, t)$, $m = 1, 2, \ldots, 6$, defined by (26)–(29) satisfy the conditions

$$D_\nu^\alpha G_m(\nu, x_3, t) \in C(\mathbb{R}^2 \times \Delta) \cap C(\Delta; C_c(\mathbb{R}^2)) \tag{53}$$

for any $\alpha = (\alpha_1, \alpha_2)$, $\alpha_j \in \{0, 1, 2, \ldots\}$, $j = 1, 2$.

Applying D_ν^α to (24), we obtain

$$D_\nu^\alpha \mathbf{V}(\nu, x_3, t) = D_\nu^\alpha \mathbf{G}(\nu, x_3, t) + \int_0^t (\mathbf{K} D_\nu^\alpha \mathbf{V})(\nu, x_3, t, \tau)\,\mathrm{d}\tau, \\ \nu \in \mathbb{R}^2,\ (x_3, t) \in \Delta. \tag{54}$$

Equation (54) has the same form as (24). The solution $\mathbf{V}(\nu, y, t)$ of (24), which is found by the method of successive approximations described in Section 5, has the following property:

$$D_\nu^\alpha \mathbf{V}(\nu, x_3, t) \in C(\mathbb{R}^2 \times \Delta) \quad \text{for any } \alpha = (\alpha_1, \alpha_2),\ \alpha_j \in \{0, 1, 2, \ldots\},\ j = 1, 2.$$

Let us consider an arbitrary positive number Ω and an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2)$ with components from $\{0, 1, 2, \ldots\}$. Using (54) and (34), we obtain the inequality

$$\|D_\nu^\alpha \mathbf{V}\|(\nu, t) \leqslant \|D_\nu^\alpha \mathbf{G}\|(\nu, t) + B \int_0^t \|D_\nu^\alpha \mathbf{V}\|(\nu, \tau)\,\mathrm{d}\tau, \tag{55}$$

where $|\nu| \leqslant \Omega$, $t \in [0, T]$; B and $\|\cdot\|(\nu, t)$ are defined in the statement of Proposition 2 (see formula (35)).

Applying Gronwall's lemma [9] to the inequality (55), we find

$$\|D_\nu^\alpha \mathbf{V}\|(\nu, t) \leqslant \|D_\nu^\alpha \mathbf{G}\|(\nu, t)\mathrm{e}^{BT}, \quad |\nu| \leqslant \Omega,\ t \in [0, T]. \tag{56}$$

It follows from (53), (56) that the solution of (54) satisfies $D_\nu^\alpha \mathbf{V}(\nu, x_3, t) \in C(\Delta; C_c(\mathbb{R}^2))$ for any α. Hence, the components of the generalized solution

$$\tilde{\mathbf{E}}(\nu, x_3, t) = (\tilde{E}_1(\nu, x_3, t), \tilde{E}_2(\nu, x_3, t), \tilde{E}_3(\nu, x_3, t))$$

of (4)–(6) satisfy the conditions

$$\tilde{E}_l, \frac{\partial}{\partial x_3}\tilde{E}_j, \frac{\partial}{\partial t}\tilde{E}_3 \in C(\mathbb{R}^2 \times \Delta) \cap C(\Delta; C_c^\infty(\mathbb{R}^2)), \quad l = 1,2,3,\ j = 1,2.$$

Applying the inverse Fourier transform with respect to ν_1, ν_2 to (4)–(6) and using the real version of the Paley-Wiener theorem [1] (see also Appendix B) we find that $\mathbf{E}(x,t) = \mathcal{F}_\nu^{-1}[\tilde{\mathbf{E}}]$ is the unique generalized solution of (1), (2) such that $E_l(x,t)$, $(\partial/\partial x_3)E_j(x,t)$, $(\partial/\partial t)E_3(x,t)$ belong to the class $C(\mathbb{R}^2 \times \Delta) \cap C(\Delta; PW(\mathbb{R}^2))$, $l = 1,2,3$, $j = 1,2$.

Conclusion

The initial value problem (1), (2) is a mathematical model of the time dependent electric field in vertically inhomogeneous biaxial anisotropic media. In the present paper a new method for solving this problem has been obtained. This method consists of the following. First, IVP (1), (2) is rewritten in the form of Fourier images with respect to the lateral variables x_1, x_2. Secondly, the resulting problem (FTIVP) is reduced to an operator integral equation. After that the operator integral equation is solved by the method of successive approximations. Finally, to find a solution of IVP (1), (2) we apply the inverse Fourier transform with respect to ν_1, ν_2 to the solution of the operator integral equation.

We note that if symmetric positive definite matrices $\mathcal{E}$ and $\mathcal{M}$ have constant elements then an explicit formula for the solution of IVP (1), (2) has been derived by symbolic computations in MATLAB [19], [20]. Using this formula, the simulation of electric fields has been derived in different homogeneous anisotropic materials [20]. Unfortunately, the explicit formulae for solutions cannot be constructed in the case when elements of matrices $\mathcal{E}$ and $\mathcal{M}$ are functions of one or several variables. In the case when $\mathcal{E}$ and $\mathcal{M}$ depend on one variable x_3 our method allows us to determine electric fields in inhomogeneous biaxial anisotropic media.

Appendix A

Generalized Cauchy problem for the wave equation

Let us consider IVP for the wave equation with two independent variables

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right) w(y,t) = f(y,t), \quad y \in \mathbb{R},\ t > 0, \tag{57}$$

$$w(y,t)|_{t=0+} = 0, \quad \frac{\partial w}{\partial t}(y,t)|_{t=0+} = 0. \tag{58}$$

If $f(y,t) \in C^1(\mathbb{R}\times[0,\infty))$ then there exists a unique solution $w(y,t) \in C^2(\mathbb{R}\times[0,\infty))$ which can be given by the D'Alembert formula (see, for example [17], p. 176).

Let now assume that $f(y,t) \in C(\mathbb{R}\times[0,\infty))$. In this case the problem (57), (58) will be interpreted as the generalized Cauchy problem ([17], pp. 171–178). According to the theorem from ([17], p. 174), there exists an inverse operator $(\partial^2/\partial t^2 - \partial^2/\partial y^2)^{-1}$ such that the function $w(y,t)$, defined by

$$(59)\quad w(y,t) = \Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\Big)^{-1} f(y,t) \equiv \iint_{\mathbb{R}^2} \theta((t-\tau) - |y-\xi|)\theta(\tau) f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau,$$

is the unique generalized solution of the generalized Cauchy problem (57), (58) for any $f(y,t) \in C(\mathbb{R}\times[0,\infty))$. This means that the equality (59) is equivalent to (57), (58), where (57) is understood as the equality of generalized functions [17].

R e m a r k 1. We note that (59) may be written as the d'Alembert formula

$$(60)\qquad w(y,t) = \frac{\theta(t)}{2}\int_0^t \int_{y-(t-\tau)}^{y+(t-\tau)} f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau,$$

or

$$(61)\qquad w(y,t) = \frac{\theta(t)}{2}\int_{y-t}^{y+t} \int_0^{t-|\xi-y|} f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau, \quad y\in\mathbb{R},\ t\in\mathbb{R}.$$

It follows from (60) that for $y \in \mathbb{R}$, $t > 0$ the derivatives $\partial w/\partial t$, $\partial w/\partial y$ can be found by

$$\begin{aligned}
\frac{\partial w}{\partial t}(y,t) &= \frac{1}{2}\int_0^t [f(y+(t-\tau),\tau) - f(y-(t-\tau),\tau)]\,\mathrm{d}\tau,\\
\frac{\partial w}{\partial y}(y,t) &= \frac{1}{2}\int_0^t [f(y+(t-\tau),\tau) - f(y-(t-\tau),\tau)]\,\mathrm{d}\tau.
\end{aligned}$$

This means that the generalized solution $w(y,t)$ of (57), (58) belongs to $C^1(\mathbb{R}\times[0,\infty))$ for any $f(y,t) \in C(\mathbb{R}\times[0,\infty))$.

R e m a r k 2. Let us consider IVP for the wave equation with two independent variables

$$(62)\quad \Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\Big) w(y,t) = \mathcal{F}\Big(y,t,w(y,t),\frac{\partial w}{\partial t}(y,t),\frac{\partial w}{\partial y}(y,t)\Big), \quad y\in\mathbb{R},\ t>0,$$

$$(63)\qquad w(y,t)|_{t=0+} = 0, \quad \frac{\partial w}{\partial t}(y,t)|_{t=0+} = 0,$$

where

$$\mathcal{F}\Big(y,t,w,\frac{\partial w}{\partial t},\frac{\partial w}{\partial y}\Big)=p_2(y,t)\frac{\partial w}{\partial t}+p_1(y,t)\frac{\partial w}{\partial y}+p_0(y,t)w+f(y,t),$$

$p_k(y,t)$, $f(y,t)\in C^1(\mathbb{R}\times[0,\infty))$, $k=0,1,2$, are given functions. Using the reasoning as above, we find that the generalized Cauchy problem (62), (63) is equivalent to the equation

$$\tag{64} w(y,t)=\Big(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}\Big)^{-1}\mathcal{F}\Big(y,t,w(y,t),\frac{\partial w}{\partial t}(y,t),\frac{\partial w}{\partial y}(y,t)\Big)$$
$$\equiv\int\int_{\mathbb{R}^2}\theta((t-\tau)-|y-\xi|)\theta(\tau)\mathcal{F}\Big(y,t,w(\xi,\tau),\frac{\partial w}{\partial t}(\xi,\tau),\frac{\partial w}{\partial y}(\xi,\tau)\Big)\,\mathrm{d}\xi\,\mathrm{d}\tau.$$

The equation (64) may be written in the form

$$\tag{65} w(y,t)=\frac{\theta(t)}{2}\int_0^t\int_{y-(t-\tau)}^{y+(t-\tau)}\mathcal{F}\Big(y,t,w(\xi,\tau),\frac{\partial w}{\partial t}(\xi,\tau),\frac{\partial w}{\partial y}(\xi,\tau)\Big)\,\mathrm{d}\xi\,\mathrm{d}\tau.$$

Appendix B

Paley-Wiener space and the real version of Paley-Wiener theorem

The result here has been taken from the paper [1] (see also [3], [15]).

As well known (see for example, [1], [3], [15]) the classical Fourier transform $\mathcal{F}$ is an isomorphism of the Schwartz space $S(\mathbb{R}^k)$ onto itself. The space $C_c^\infty(\mathbb{R}^k)$ of smooth functions with compact support is dense in $S(\mathbb{R}^k)$, and the classical Paley-Wiener theorem characterizes the image of $C_c^\infty(\mathbb{R}^k)$ under $\mathcal{F}$ as a rapidly decreasing function having a holomorphic extension to C^k of exponential type. In this appendix we will define the Paley-Wiener space and consider the real version of the Paley-Wiener theorem following the nice work [1].

Definition. We define the Paley-Wiener space $PW(\mathbb{R}^k)$ as the space of all functions $\varphi(x)\in C^\infty(\mathbb{R}^k)$ satisfying

(a) $(1+|x|)^m\Delta^n\varphi(x)\in\mathcal{L}_2(\mathbb{R}^2)$ for all $m,n\in\{0,1,2\ldots\}$,

(b) $R_\varphi^\Delta=\lim\limits_{n\to\infty}\|\Delta^n\varphi(x)\|_2^{1/2n}<\infty$,

where $\mathcal{L}_2(\mathbb{R}^2)$ is the space of square integrable functions with the norm $\|\varphi\|_2=\big(\int_{\mathbb{R}^2}|\varphi(x)|^2\,\mathrm{d}x\big)^{1/2}$ for any $\varphi(x)\in\mathcal{L}_2(\mathbb{R}^2)$; $\Delta=\partial^2/\partial x_1^2+\ldots+\partial^2/\partial x_k^2$ denotes the Laplacian on $\mathbb{R}^k$. Further, $PW_B(\mathbb{R}^k)=\{\varphi(x)\in PW(\mathbb{R}^k)\colon\ R_\varphi^\Delta=B\}$ for $B\geqslant 0$.

Theorem. *The inverse Fourier transform $\mathcal{F}^{-1}$ is a bijection on $C_c^\infty(\mathbb{R}^k)$ onto $PW(\mathbb{R}^k)$, mapping $C_B^\infty(\mathbb{R}^k)$ onto $PW_B(\mathbb{R}^k)$.*

Here $C_B^\infty(\mathbb{R}^k)$ is defined as

$$C_B^\infty(\mathbb{R}^k) = \{\varphi(x) \in C_B^\infty(\mathbb{R}^k)\colon\ R_\varphi = B\},$$

where $R_\varphi = \sup\limits_{x \in \operatorname{supp} \varphi} |x|$ is the radius of the support of $\varphi(x)$.

We note that the work [1] contains the proof of this theorem.

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