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T-EXTENSION AS A METHOD OF CONSTRUCTION OF A GENERALIZED AGGREGATION OPERATOR

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Generalized aggregation operators are the tool for aggregation of fuzzy sets. The apparatus was introduced by Takači in [11]. T-extension is a construction method of a generalized aggregation operator and we study it in the paper. We observe the behavior of a T-extension with respect to different order relations and we investigate properties of the construction.

Keywords: aggregation operator, t-norm, T-extension

Classification: 03E72, 94D05

1. INTRODUCTION

This paper is a contribution to the theory of generalized aggregation operators (shortly gagops) introduced by Takači in [11]. The term *generalized* refers to the inputs of an aggregation operator (shortly agop), they are a special type of fuzzy sets. In the sequel we study a method of construction of a gagop by means of an arbitrary continuous t-norm, called T-extension. Another construction method and namely pointwise extension of an agop was previously studied in [6].

For more convenient reading the basic knowledge of the theory of fuzzy sets and the theory of agops is required. For the information on the first topic the reader can refer e. g. to [2, 5, 10]. Basic knowledge on the theory of agops is provided in the sequel, but for the deeper understanding sources [1, 3] can be advised.

The paper is organized as follows: Section 2 contains basic notions related to the theory of agops; Section 3 contains results on a continuous t-norm, these results play an important role in many proofs provided in the contribution. We recall the definition of a gagop in Section 4; Section 5 is the main part of the contribution and it is structured in the following way: first we recall the definition of a T-extension, after that we study possible sets of input of a T-extension, then we study T-extension w.r.t. different order relations, we conclude the section by some properties of a T-extension; and we conclude the paper by Section 6.

2. PRELIMINARIES

Aggregation of several input values into a single output value is an indispensable tool not only of mathematics or physics, but also when studying different problems in engineering, economics and other fields of science. The problem of aggregation is very broad in general, we generalize agops defined on the unit interval and having the finite number of input values. In the sequel we give the definition, examples and the main properties of agops which are needed for our work. For more information an interested reader can refer e. g. to [1, 3].

Definition 2.1. A mapping $A : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is an agop on the unit interval if for every $n \in \mathbb{N}$ the following conditions hold:

(A1) $A(0,\ldots,0) = 0$

(A2) $A(1,\ldots,1) = 1$

(A3) $(\forall i = \overline{1,n})\ (x_i \le y_i) \Rightarrow A(x_1,x_2,\ldots,x_n) \le A(y_1,y_2,\ldots,y_n)$.

Conditions (A1) and (A2) are called boundary conditions, and they ensure that aggregation of completely bad (good) results will give the completely bad (good) output. Condition (A3) resembles the monotonicity property of A.

In general, the number of the input values to be aggregated is unknown, and therefore an agop can be presented as a family $A = (A_{(n)})_{n\in\mathbb{N}}$, where $A_{(n)} = A|_{[0,1]^n}$. Operators $A_{(n)}$ and $A_{(m)}$ for different n and m need not be related.

A specific case is the aggregation of a singleton, i. e., the unary operator $A_{(1)} : [0,1] \to [0,1]$. Throughout the work we will follow the convention $A_{(1)}(x) = x, x \in [0,1]$.

Definition 2.2. An element $x \in [0;1]$ is called A-idempotent element whenever $A_{(n)}(x,\ldots,x) = x, \forall n \in \mathbb{N}$. A is called an idempotent agop if each $x \in [0;1]$ is an idempotent element of A.

0 and 1 are trivial A-idempotent elements for an arbitrary agop.

Definition 2.3. An agop A is called a symmetric agop if
$\forall n \in \mathbb{N}, \forall x_1,\ldots,x_n \in [0;1] : A(x_1,\ldots,x_n) = A(x_{\pi(1)},\ldots,x_{\pi(n)})$ for all permutations $\pi = (\pi(1),\ldots,\pi(n))$ of $(1,\ldots,n)$.

A weighted mean $W(x_1,\ldots,x_n) = \sum_{i=1}^{n} w_i^{(n)} x_i$, with $w_i^{(n)} \ge 0$ and such that $\sum_{i=1}^{n} w_i^{(n)} = 1$ for all $n \in \mathbb{N}$ is an example of nonsymmetric agop.

Definition 2.4. An agop A is associative if $\forall n,m \in \mathbb{N}, \forall x_1,\ldots,x_n,y_1,\ldots,y_m \in [0;1] : A(x_1,\ldots,x_n,y_1,\ldots,y_m) = A(A(x_1,\ldots,x_n), A(y_1,\ldots,y_n))$.

The associativity of an agop allows to aggregate first some subsystems of all inputs, and then the partial outputs. For practical purposes we can start with aggregation procedure before knowing all inputs to be aggregated.

Definition 2.5. An agop A is bisymmetric if $\forall n,m \in \mathbb{N}, \forall x_{11},\ldots,x_{mn} \in [0;1]$:
$A_{(mn)}(x_{11},\ldots,x_{mn}) = A_{(m)}(A_{(n)}(x_{11},\ldots,x_{1n}),\ldots,A_{(n)}(x_{m1},\ldots,x_{mn})) =$
$= A_{(n)}(A_{(m)}(x_{11},\ldots,x_{m1}),\ldots,A_{(m)}(x_{1n},\ldots,x_{mn}))$.

The bisymmetry allows to aggregate first rows and then partial outputs or first columns and then partial outputs if information is stored in the form of the matrix. Bisymmetry is implied by associativity and symmetry.

Definition 2.6. An element $e \in [0;1]$ is called a neutral element of A if $\forall n \in \mathbb{N}, \forall x_1, \ldots, x_n, \in [0;1]$ if $x_i = e$ for some $i \in \{1, \ldots, n\}$ then

$$A(x_1, \ldots, x_n) = A(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

So, the neutral element can be omitted from aggregation inputs without influencing the final output.

Typical examples are the following:

$$\Pi(x_1, \ldots, x_n) = \Pi_{i=1}^n x_i \text{ with } e = 1,$$

$$min(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n) \text{ with } e = 1,$$

$$max(x_1, \ldots, x_n) = \max(x_1, \ldots, x_n) \text{ with } e = 0.$$

Agops

$$M(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^n x_i,$$

$$\forall n \geq 2, (x_1, \ldots, x_n) \neq (1, \ldots, 1) : A_w(x_1, \ldots, x_n) = 0,$$

$$\forall n \geq 2, (x_1, \ldots, x_n) \neq (0, \ldots, 0) : A_s(x_1, \ldots, x_n) = 1,$$

$$G(x_1, \ldots, x_n) = (\Pi_{i=1}^n x_i)^{1/n}$$

do not have neutral elements.

The existence of the neutral element is not related to the previous properties such as continuity, symmetry, associativity or bisymmetry.

Definition 2.7. An element $a \in [0;1]$ is called an absorbing element of A if

$$\forall n \in \mathbb{N},\ \forall x_1, \ldots, x_n, \in [0;1] : a \in \{x_1, \ldots, x_n\} \Rightarrow A(x_1, \ldots, x_n) = a.$$

3. RESULTS FOR CONTINUOUS T-NORM

In this section we provide results related to continuous t-norms. Basics and important results on t-norms can be found e. g. in [4]. For the sake of brevity we skip proof of theorem 3.3, it follows from theorem 3.1, and it can be also found in the author's thesis [7].

Below provided results appear in literature with different combinations of t-norm and properties of fuzzy sets. Combination of continuous t-norm and upper semicontinuous fuzzy sets with bounded α-cuts $\forall \alpha > 0$ is important for us, thus we prove these results for this case.

Theorem 3.1. If $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous operation, T is a continuous t-norm and $P, Q \in F(\mathbb{R})$ are upper semicontinuous fuzzy sets with bounded α-cuts $\forall \alpha > 0$ then for all $z \in \mathbb{R}$, $z = x * y$ $\exists x_0, y_0 \in \mathbb{R}$ such that $z = x_0 * y_0$ and $(P * Q)_T(z) = T(P(x_0), Q(y_0))$.

P r o o f. According to the extension principle

$$(P * Q)_T(z) = \sup_{x*y=z} T(P(x), Q(y)).$$

The case when $(P*Q)_T(z) = 0$ is evident. Therefore we assume that $(P*Q)_T(z) = \alpha > 0$ and

$$T(P(x), Q(y)) < \alpha = \sup_{x,y \in \mathbb{R}: x*y=z} T(P(x), Q(y))$$

for all $x, y : x * y = z$.

According to the definition of supremum there exists a sequence (α_n): $\alpha_n \to \alpha$ from below and moreover we can construct sequences $(x_n), (y_n) : \forall n \; x_n * y_n = z$ and $T(P(x_n), Q(y_n)) \geq \alpha_n$.

P, Q are upper semicontinuous fuzzy sets with bounded α-cuts $P^\alpha, Q^\alpha \; \forall \alpha > 0$, this implies that $\forall \alpha > 0$ α-cuts are closed and bounded intervals and as a result sequences $(x_n), (y_n)$ are bounded. It is a known fact that a bounded sequence has a convergent subsequence, therefore $\exists (x_{n_k}) \subseteq (x_n)$ which converges to some point x_0.

Further we consider (y_{n_k}) a subsequence of (y_n) with corresponding to (x_{n_k}) numbers. Again (y_{n_k}) is a bounded sequence in compact sets Q^{α_n} and we can extract $(y_{n_{k_l}})$:

$$(y_{n_{k_l}}) \subseteq (y_{n_k}) \text{ and } y_{n_{k_l}} \to y_0 \text{ when } l \to \infty.$$

We go back to (x_{n_k}) and extract subsequence $(x_{n_{k_l}})$ with corresponding to $(y_{n_{k_l}})$ numbers. $x_{n_{k_l}} \to x_0$ (as a subsequence of the convergent sequence). The continuity of $*$ and constructions of $(x_n), (y_n)$ allow us to state that $x_0 * y_0 = z$.

Further we assume that

$$P(x_{n_{k_l}}) = \beta_{n_{k_l}}$$

$$Q(y_{n_{k_l}}) = \gamma_{n_{k_l}}.$$

As $(\beta_{n_{k_l}}), (\gamma_{n_{k_l}})$ are bounded sequences, then we can extract convergent subsequences (similar reasoning like above allows us to extract subsequences with the same index numbers):

$$(\beta_m) \subseteq (\beta_{n_{k_l}}) \text{ and } \beta_m \to \beta \text{ when } m \to \infty$$

$$(\gamma_m) \subseteq (\gamma_{n_{k_l}}) \text{ and } \gamma_m \to \gamma \text{ when } m \to \infty.$$

By construction of $(\beta_m), (\gamma_m)$ we have:

$$T(\beta_m, \gamma_m) \geq \alpha_m.$$

By construction of the sequence (α_m) $\alpha_m \to \alpha$ from below.

By continuity of $*$:

$$x_0 * y_0 = z,$$

by continuity of T:

$$T(\beta, \gamma) \geq \alpha.$$

Since $P(x_m) \geq \beta_m, \forall m$ this implies that $P(x_0) \geq \beta_m, \forall m$ and as a result $P(x_0) \geq \beta$, and similarly $Q(y_0) \geq \gamma$.

Using monotonicity of T we can write:

$$T(P(x_0), Q(y_0)) \geq T(\beta, \gamma) \geq \alpha,$$

thus we have obtained that $T(P(x_0), Q(y_0)) \geq \alpha$, but α is the supremum, so only the equality is possible. $\square$

We use the following definition of fuzzy quantity in the paper:

Definition 3.2. A convex, upper semicontinuous fuzzy set $M : \mathbb{R} \to [0, 1]$ with bounded α-cuts for all $\alpha > 0$ is called a fuzzy quantity.

Theorem 3.3. If $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous operation, T is an arbitrary continuous t-norm and P, Q are fuzzy quantities then $(P * Q)_T$ is a fuzzy quantity.

4. GENERALIZED AGGREGATION

The problem of aggregation can be generalized if we use fuzzy subsets as input information. This approach is initiated in [11] by Takači. Other interesting, conceptually different approaches of generalization can be found in the literature, e. g. in [8, 9, 12] and others.

Let $F(X)$ be the set of all fuzzy subsets of the universe X, $\leq$ be some order relation defined on $F(X)$. Element $\tilde{0} \in F(X)$ is the minimal and $\tilde{1} \in F(X)$ is the maximal element w.r.t. $\leq$.

Definition 4.1. (Takači [11]) A mapping $\tilde{A} : \cup_{n \in \mathbb{N}} F(X)^n \to F(X)$ is called a generalized aggregation operator w.r.t. the order relation $\leq$, if for every $n \in \mathbb{N}$ the following conditions hold:

($\tilde{A}1$) $\tilde{A}(\tilde{0}, \ldots, \tilde{0}) = \tilde{0}$

($\tilde{A}2$) $\tilde{A}(\tilde{1}, \ldots, \tilde{1}) = \tilde{1}$

($\tilde{A}3$) $(\forall i = \overline{1, n})\ (P_i \leq Q_i) \Rightarrow \tilde{A}(P_1, \ldots, P_n) \leq \tilde{A}(Q_1, \ldots, Q_n)$,
where $P_1, \ldots, P_n, Q_1, \ldots, Q_n \in F(X)$.

A gagop can be presented as a family $\tilde{A} = (\tilde{A}_{(n)})_{n \in \mathbb{N}}$, usually we consider an arbitrary n-ary restriction of this family.

We use convention $\tilde{A}_{(1)}(P(x)) = P(x)\ \forall P(x) \in F(X)$.

We use the concept of definition 4.1, but we vary properties of relation $\leq$ (is should not necessarily be a partial ordering with properties of reflexivity, antisymmetry and transitivity); we also consider different modification of $\tilde{0}$ and $\tilde{1}$; and we change sets of input and output values.

5. T-EXTENSION OF AN AGOP A

Definition 4.1 provides the concept a gagop. We need to specify a construction method to be able investigate it in more details. Different methods are summarized in [11]. We recall definition of T-extension.

Let T be an arbitrary t-norm, A be an arbitrary agop defined on the subset of X, then:

Definition 5.1. (Takači [11]) A mapping $\tilde{A} : \cup_{n\in\mathbb{N}} F(X)^n \to F(X)$ defined in the following way

$$\begin{aligned}\tilde{A}(P_1,\dots,P_n)(x) = \sup\{T(P_1(x_1),\dots,P_n(x_n))|\\ (x_1,\dots,x_n)\in X^n : A(x_1,\dots,x_n)=x\},\end{aligned} \tag{1}$$

is called a T-extension of an agop A.

We put the following restriction on T-extension studied in the paper:

1. number of input values is finite
2. t-norm T is continuous
3. agop A is continuous agop defined on the unit interval
4. input values are upper semicontinuous fuzzy sets with bounded α-cuts for every $\alpha > 0$ defined on the unit interval and taking values from the unit interval.

Formulating results further in the paper we by default assume that conditions 1. – 3. are fulfilled. If it is not required we specify it separately.

In the paper $F^*([0,1]) = \{P|P : [0,1] \to [0,1]\}$ denotes the set of upper semicontinuous fuzzy sets with bounded α-cuts for every $\alpha > 0$.

We also use sets $FQ([0,1])$, $FI([0,1])$, $FN([0,1])$ (correspondingly the set of all fuzzy quantities, fuzzy intervals and fuzzy numbers defined on the unit interval) in the role of the set of input and output values. When we restrict the set of inputs we require that output values belong to the same set.

We give definitions of fuzzy interval and fuzzy number used in the paper (definition of fuzzy quantity is specified before, see definition 3.2):

Definition 5.2. A fuzzy quantity P is called a fuzzy interval if $\exists I = [a,b] \subseteq (-\infty,+\infty) : P(x) = 1 \Leftrightarrow x \in I$. Interval I is called the core of P.

Definition 5.3. Fuzzy quantity P is called a fuzzy number if $\exists! x \in \mathbb{R} : P(x) = 1$. Point x is called the core of P.

5.1. The set of inputs of a T-extension

Provided restrictions $1, 2$ and 3 we ensure that aggregated result belongs to $F^*([0,1])$ when input values are taken from the same set.

Consider $FQ([0,1])$ in the role of the set of inputs of $\tilde{A}$.

As a corollary from theorem 3.3 we get the following result:

Corollary 5.4. If $\tilde{A}$ is a T-extension of an agop A given by (1) and $P_1,\dots,P_n \in FQ([0,1])$ then

$$\tilde{A}(P_1,\dots,P_n) \in FQ([0,1]).$$

The proposition formulated below indicates case when the result of aggregation of fuzzy intervals is a fuzzy interval (as a corollary we get the same result for fuzzy numbers).

Proposition 5.5. If $\tilde{A}$ is a T-extension of an agop A given by (1), $P_1, \ldots, P_n \in FI([0,1])$ and $I_1, \ldots, I_n$ are their corresponding cores then $\tilde{A}(P_1, \ldots, P_n) \in FI([0,1])$ and its core is $I = \{A(x_1, \ldots, x_n) | (x_1, \ldots, x_n) : x_i \in I_i, i = 1, \ldots, n\}$.

Proof. The class of fuzzy intervals is a subclass of fuzzy quantities, therefore according to the results of corollary 5.4 $\tilde{A}(P_1, \ldots, P_n)$ will be a fuzzy quantity at least, when $P_1, \ldots, P_n \in FI([0,1])$. This implies, that if $\tilde{A}(P_1, \ldots, P_n)$ has a core, then it is in the form of continuous interval, otherwise $\tilde{A}(P_1, \ldots, P_n)$ will not be convex.

Now we show that $\tilde{A}(P_1, \ldots, P_n)(x)$=1 iff $x \in I$.

Assume $x \in I$ this implies that $\exists (x_1^*, \ldots, x_n^*) : x_i^* \in I_i, i = 1, \ldots, n$ and $A(x_1^*, \ldots \ldots, x_n^*) = x$.

Consider

$$\begin{aligned}\tilde{A}(P_1, \ldots, P_n)(x) &= \sup\{T(P_1(x_1), \ldots, P_n(x_n)) | x_i \in [0,1] : A(x_1, \ldots, x_n) = x\} \\ &\geq T(P_1(x_1^*), \ldots, P_n(x_n^*)) = 1\end{aligned}$$

and hence $\tilde{A}(P_1, \ldots, P_n)(x) = 1$.

Assume $\tilde{A}(P_1, \ldots, P_n)(x) = 1$, according to theorem 3.1
$\exists x_1^*, \ldots, x_n^* : A(x_1^*, \ldots, x_n^*) = x$ and $\tilde{A}(P_1, \ldots, P_n)(x) = T(P_1(x_1^*), \ldots, P_n(x_n^*))$.
Since $\tilde{A}(P_1, \ldots, P_n)(x) = 1$ by neutrality of t-norm versus 1 we have $P_i(x_i^*) = 1$, $i = 1, \ldots, n$, thus $x_i^* \in I_i$ and by definition of I we have $x \in I$. □

5.2. T-extension w.r.t. vertical order relations

We introduce the class of order relations (together with the minimal and the maximal element) and explore behavior of T-extension w.r.t. it, namely we study properties $(\tilde{A}1), (\tilde{A}2)$ and $(\tilde{A}3)$ from definition 4.1.

We introduce order relation $\subseteq_{F1}^{\alpha}$.

Definition 5.6. Let $\alpha \in [0,1]$, $P, Q \in F([0,1])$

$$P \subseteq_{F1}^{\alpha} Q \Leftrightarrow (\forall x \in [0,1])(P(x) \geq \alpha \Rightarrow P(x) \leq Q(x)).$$

Let denote $=_{F1}^{\alpha}$ the following relation:

$$P =_{F1}^{\alpha} Q \Leftrightarrow P \subseteq_{F1}^{\alpha} Q \text{ and } Q \subseteq_{F1}^{\alpha} P.$$

Relation $=_{F1}^{\alpha}$ is equivalence relation on the set $F([0,1])$ and $\subseteq_{F1}^{\alpha}$ is reflexive, anti-symmetric and transitive order relation w.r.t. it.

The maximal element w.r.t. $\subseteq_{F1}^{\alpha}$ is defined in the following way:

$$\tilde{1}(x) = 1, \forall x \in [0,1]. \tag{2}$$

Let

$$\Theta = \{\tilde{0}(x) | \tilde{0}(x) \leq \alpha, \forall x \in [0,1]\}.$$

Θ is the class of minimal elements.

We consider all elements of Θ to be equivalent and we say that boundary condition $(\tilde{A}1)$ in definition 4.1 amounts to

$$\tilde{A}_{(n)}(\tilde{0}_1, \dots, \tilde{0}_n) \in \Theta \tag{3}$$

for all $n \in \mathbb{N}$ and for all $\tilde{0}_1, \dots, \tilde{0}_n \in \Theta$.

Now we formulate result, which shows that T-extension is a gagop (in the sense of definition 4.1) w.r.t. $\subseteq^{\alpha}_{F1}$.

Theorem 5.7. Let $\alpha \in [0,1]$. An arbitrary T-extension $\tilde{A} : \cup_{n\in\mathbb{N}} F^*([0,1])^n \to F^*([0,1])$ of an arbitrary agop A given by (1) is a gagop w.r.t. the order relation $\subseteq^{\alpha}_{F1}$.

P r o o f. If we consider an arbitrary vector $(x_1, \dots, x_n) \in [0,1]^n$ then applying the restriction from above of an arbitrary t-norm by T_M we obtain:

$$T(\tilde{0}_1(x_1), \dots, \tilde{0}_n(x_n)) \leq T_M(\tilde{0}_1(x_1), \dots, \tilde{0}_n(x_n)). \tag{4}$$

According to the definition of $\tilde{0}_i$, $\forall x_i \in [0,1]$ $\tilde{0}_i(x_i) \leq \alpha$ thus the same is true for the minimum, i. e.:

$$T_M(\tilde{0}_1(x_1), \dots, \tilde{0}_1(x_n)) \leq \alpha. \tag{5}$$

Evidently using formulas (4), (5) for an arbitrary $x \in [0,1]$ we obtain modified $(\tilde{A}1)$ (formula (3)).

According to the definition of $\tilde{1}$ (formula (2)) for an arbitrary vector $(x_1, \dots, x_n) \in [0,1]^n$ the following holds:

$$T(\tilde{1}(x_1), \dots, \tilde{1}(x_n)) = T(1, \dots, 1) = 1 \text{ for an arbitrary } x \in [0,1]$$

and thus $(\tilde{A}2)$ is straightforward.

Consider the proof of $(\tilde{A}3)$. We take an arbitrary $x \in [0,1] : \tilde{A}(P_1, \dots, P_n)(x) \geq \alpha$ and consider $\tilde{A}(P_1, \dots, P_n)(x)$ and $\tilde{A}(Q_1, \dots, Q_n)(x)$:
according to theorem 3.1 $\exists$ $(x_1^*, \dots, x_n^*)$:

$$A(x_1^*, \dots, x_n^*) = x$$

and

$$\tilde{A}(P_1, \dots, P_n)(x) = T(P_1(x_1^*), \dots, P_n(x_n^*)). \tag{6}$$

Similarly $\exists$ $(x_1', \dots, x_n')$:

$$A(x_1', \dots, x_n') = x$$

and

$$\tilde{A}(Q_1, \dots, Q_n)(x) = T(Q_1(x_1'), \dots, Q_n(x_n')).$$

We remind that

$$\tilde{A}(P_1, \dots, P_n)(x) \geq \alpha \tag{7}$$

and using $\forall i$ $P_i \subseteq^{\alpha}_{F_1} Q_i$, i. e.:

$$(P_i(x_i^*) \geq \alpha) \Rightarrow (P_i(x_i^*) \leq Q_i(x_i^*))$$

and formula (6) we can write

$$\alpha \leq T(P_1(x_1^*), \ldots, P_n(x_n^*)) \leq P_i(x_i^*) \leq Q_i(x_i^*),\ \forall i.$$

Using the monotonicity of t-norm we can continue in the following way:

$$T(P_1(x_1^*), \ldots, P_n(x_n^*)) \leq T(Q_1(x_1^*), \ldots, Q_n(x_n^*)).$$

But according to the definition of vector $(x_1', \ldots, x_n')$

$$T(Q_1(x_1^*), \ldots, Q_n(x_n^*)) \leq T(Q_1(x_1'), \ldots, Q_n(x_n'))$$

and thus

$$\tilde{A}(P_1, \ldots, P_n)(x) \leq \tilde{A}(Q_1, \ldots, Q_n)(x). \tag{8}$$

Point x was chosen according to the formula (7) and we obtained inequality (8), thus we have shown that $(\tilde{A}3)$ holds. □

Order of fuzzy sets by inclusion is a special case of $\subseteq_{F_1}^{\alpha}$ when $\alpha = 0$, thus the above result holds also for this order relation.

5.3. T-extension w.r.t. horizontal order relations

In this subsection we study the behavior of a T-extension w.r.t. another class of order relations.

Definition 5.8. Let $\alpha \in (0,1]$, $P, Q \in F([0,1])$

$$P \subseteq_{F_2}^{\alpha} Q \Leftrightarrow \overline{P}^{\alpha} \leq \underline{Q}^{\alpha},$$

where

$$\begin{aligned} P^{\alpha} &= \{x : P(x) \geq \alpha\}, & \min P^{\alpha} &= \underline{P}^{\alpha}, & \max P^{\alpha} &= \overline{P}^{\alpha}, \\ Q^{\alpha} &= \{x : Q(x) \geq \alpha\}, & \min Q^{\alpha} &= \underline{Q}^{\alpha}, & \max Q^{\alpha} &= \overline{Q}^{\alpha}. \end{aligned}$$

Let denote $=_{F2}^{\alpha}$ the following relation:

$$P =_{F2}^{\alpha} Q \Leftrightarrow P \subseteq_{F2}^{\alpha} Q \text{ and } Q \subseteq_{F2}^{\alpha} P.$$

Relation $\subseteq_{F2}^{\alpha}$ is antisymmetric and transitive order relation w.r.t. $=_{F2}^{\alpha}$.
The classes

$$\Theta = \{\tilde{0}(x) | \tilde{0}(x) = 1,\ \text{if } x = 0 \text{ and } \tilde{0}(x) < \alpha \text{ if } x \in (0,1]\},$$

$$\Sigma = \{\tilde{1}(x) | \tilde{1}(x) = 1,\ \text{if } x = 1 \text{ and } \tilde{1}(x) < \alpha \text{ if } x \in [0,1)\}$$

will be called correspondingly the class of minimal and maximal elements.

We consider that elements of the class of minimal elements Θ (respectively from the class of maximal elements Σ) are equivalent. We say that the boundary condition $(\tilde{A}1)$ w.r.t. $\subseteq_{F_2}^{\alpha}$ is satisfied if (3) holds. The boundary condition $(\tilde{A}2)$ in definition 4.1 amounts to

$$\tilde{A}_{(n)}(\tilde{1}_1, \ldots, \tilde{1}_n) \in \Sigma$$

for all $n \in \mathbb{N}$ and for all $\tilde{1}_1, \ldots, \tilde{1}_n \in \Sigma$.

Theorem 5.9. Let $\alpha \in [0,1]$. An arbitrary T-extension $\tilde{A} : \cup_{n\in\mathbb{N}} F^*([0,1])^n \to F^*([0,1])$ of an arbitrary agop A given by (1) is a gagop w.r.t. the order relation $\subseteq^{\alpha}_{F2}$.

P r o o f. First we show that the modified border condition ($\tilde{A}1$) holds.

We consider $\tilde{A}_{(n)}(\tilde{0}_1, \dots, \tilde{0}_n)(x)$ in an arbitrary point $x \in [0,1]$ and for arbitrary $n \in \mathbb{N}$. Two different cases $x = 0$ and $x \neq 0$ will be considered separately.

1st case $x = 0$:
since $A(0, \dots, 0) = 0$ by ($A1$) of definition 2.1, we have that

$$1 \geq A_{(n)}(\tilde{0}_1, \dots, \tilde{0}_n)(0) \geq T(\tilde{0}_1(0), \dots, \tilde{0}_n(0)) = T(1, \dots, 1) = 1,$$

hence

$$A_{(n)}(\tilde{0}_1, \dots, \tilde{0}_n)(0) = 1.$$

2nd case $x \neq 0$:
according to theorem 3.1 and definition of T-extension $\exists$ $(x'_1, \dots, x'_n)$: $A(x'_1, \dots, x'_n) = x$ and

$$\tilde{A}_{(n)}(\tilde{0}_1, \dots, \tilde{0}_n)(x) = T(\tilde{0}_1(x'_1), \dots, \tilde{0}_n(x'_n)). \tag{9}$$

$(x'_1, \dots, x'_n) \neq (0, \dots, 0)$ otherwise $A(x'_1, \dots, x'_n) = A(0, \dots, 0) = 0$. Thus among $x'_i, i = 1, \dots, n$ there exists at least one x'_k such that $x'_k \neq 0$ and according to definition 5.8 $\tilde{0}_k(x'_k) < \alpha$. Evidently using t-norm neutrality versus 1 formula (9) can be continued in the following way:

$$\tilde{A}_{(n)}(\tilde{0}_1, \dots, \tilde{0}_n)(x) = T(\tilde{0}_1(x'_1), \dots, \tilde{0}_n(x'_n)) \leq \tilde{0}_k(x'_k) < \alpha.$$

Thus we have obtained that

$$\tilde{A}_{(n)}(\tilde{0}_1, \dots, \tilde{0}_n)(x) = \begin{cases} 1, & \text{if } x = 0 \\ \alpha_x < \alpha, & \text{otherwise} \end{cases}$$

and this means that modified ($\tilde{A}1$) holds. Similarly we show that modified ($\tilde{A}2$) holds.
In order to prove the monotonicity ($\tilde{A}3$) we should show the following implication:

$$(\forall i = 1, \dots, n, P_i \subseteq^{\alpha}_{F_2} Q_i) \Rightarrow (\tilde{A}(P_1, \dots, P_n) \subseteq^{\alpha}_{F_2} \tilde{A}(Q_1, \dots, Q_n)). \tag{10}$$

We denote A^{α}_P α-cut of $\tilde{A}(P_1, \dots, P_n)$, i.e.

$$A^{\alpha}_P = \{x : \tilde{A}(P_1, \dots, P_n)(x) \geq \alpha\}.$$

We take an arbitrary $x \in A^{\alpha}_P$ according to the definition of T-extension and theorem 3.1 $\exists$ $(x^*_1, \dots, x^*_n)$: $A(x^*_1, \dots, x^*_n) = x$ and

$$\tilde{A}(P_1, \dots, P_n)(x) = T(P_1(x^*_1), \dots, P_n(x^*_n)) \geq \alpha. \tag{11}$$

Formula (11) gives us the following result:

$$P_i(x_i^*) \geq \alpha \ \forall i = 1, \ldots, n$$

and this means, that x_i^* belongs to the α-cut of P_i for every $i \in \{1, \ldots, n\}$.

Similarly A_Q^α denotes α-cut of $\tilde{A}(Q_1, \ldots, Q_n)$:

$$A_Q^\alpha = \{y : \tilde{A}(Q_1, \ldots, Q_n)(y) \geq \alpha\}$$

and for arbitrary $y \in A_Q^\alpha$ $\exists$ $(y_1^*, \ldots, y_n^*)$: $A(y_1^*, \ldots, y_n^*) = y$ and

$$\tilde{A}(Q_1, \ldots, Q_n)(y) = T(Q_1(y_1^*), \ldots, Q_n(y_n^*)) \geq \alpha.$$

The same reasoning like above leads us to the following result:

$$y_i^* \in \{y : Q_i(y) \geq \alpha\}.$$

If we translate the left part of the implication (10) into language of α-cuts we get:

$$\overline{P}_i^\alpha \leq \underline{Q}_i^\alpha, \tag{12}$$

where $\overline{P}_i^\alpha = \max_x\{x : P_i(x) \geq \alpha\}$ and $\underline{Q}_i^\alpha = \min_y\{y : Q_i(y) \geq \alpha\}$. Given that, and the fact that x_i^* belongs to the α-cut of P_i for every $i \in \{1, \ldots, n\}$ we refer to formula (12) and get the following result:

$$x_i^* \leq y_i^* \ \forall i = 1, \ldots, n.$$

Applying the monotonicity of agop A we get:

$$A(x_1^*, \ldots, x_n^*) \leq A(y_1^*, \ldots, y_n^*),$$

thus for an arbitrary $x \in A_P^\alpha$ and an arbitrary $y \in A_Q^\alpha$ we get inequality

$$x \leq y$$

and as a result:

$$\max A_P^\alpha \leq \min A_Q^\alpha.$$

□

5.4. Symmetry, associativity and bisymmetry of a T-extension

This and the subsequent subsections are devoted to the properties of a T-extension.

In the sequel we show that it is easy to obtain symmetric, associative or bisymmetric $\tilde{A}$, and the corresponding properties are implied by the same properties of A. More precisely the same property of a t-norm is essential, but any t-norm is symmetric and associative (and as a result bisymmetric) by definition, therefore additional conditions for the t-norm are not required.

Result on associativity of a gagop is provided with proof, all other results are provided without proof as it can be performed in the similar manner (other proofs can be found in [7]).

Definition 5.10. [Symmetry] A gagop $\tilde{A} : \cup_{n\in\mathbb{N}} F(X)^n \to F(X)$ is called a symmetric gagop if

$$\forall n \in \mathbb{N}, \forall P_1, \ldots, P_n \in F(X) : \tilde{A}(P_1, \ldots, P_n) = \tilde{A}(P_{\pi(1)}, \ldots, P_{\pi(n)})$$

for all permutations $\pi = (\pi(1), \ldots, \pi(n))$ of $(1, \ldots, n)$

Definition 5.11. [Associativity] A gagop $\tilde{A} : \cup_{n\in\mathbb{N}} F(X)^n \to F(X)$ is associative if

$$\forall n, m \in \mathbb{N}, \forall P_1, \ldots, P_n, Q_1, \ldots, Q_m \in F(X) :$$
$$\tilde{A}(P_1, \ldots, P_n, Q_1, \ldots, Q_m) = \tilde{A}(\tilde{A}(P_1, \ldots, P_n), \tilde{A}(Q_1, \ldots, Q_m))$$

Definition 5.12. [Bisymmetry] A gagop $\tilde{A} : \cup_{n\in\mathbb{N}} F(X)^n \to F(X)$ is bisymmetric if

$$\forall n, m \in \mathbb{N}, \forall P_{11}, \ldots, P_{mn} \in F(X) :$$
$$\tilde{A}_{(mn)}(P_{11}, \ldots, P_{mn}) = \tilde{A}_{(m)}(\tilde{A}_{(n)}(P_{11}, \ldots, P_{1n}), \ldots, \tilde{A}_{(n)}(P_{m1}, \ldots, P_{mn}))$$
$$= \tilde{A}_{(n)}(\tilde{A}_{(m)}(P_{11}, \ldots, P_{m1}), \ldots, \tilde{A}_{(m)}(P_{1n}, \ldots, P_{mn}))$$

Proposition 5.13. Let A be a symmetric agop. An arbitrary T-extension $\tilde{A} : \cup_{n\in\mathbb{N}} F^*([0,1])^n \to F^*([0,1])$ of A given by (1) is a symmetric gagop.

Proposition 5.14. Let A be an associative agop. An arbitrary T-extension $\tilde{A} : \cup_{n\in\mathbb{N}} F^*([0,1])^n \to F^*([0,1])$ of A given by (1) is an associative gagop.

P r o o f. Consider $\tilde{A}(P_1, \ldots, P_n, Q_1, \ldots, Q_m)(z)$.
According to theorem 3.1 $\exists\ s_1^*, \ldots, s_n^*, t_1^*, \ldots, t_m^*$:

$$A(s_1^*, \ldots, s_n^*, t_1^*, \ldots, t_m^*) = z$$

and

$$\tilde{A}(P_1, \ldots, P_n, Q_1, \ldots, Q_m)(z) = T(P_1(s_1^*), \ldots, P_n(s_n^*), Q_1(t_1^*), \ldots, Q_m(t_m^*)).$$

Consider $\tilde{A}(\tilde{A}(P_1, \ldots, P_n), \tilde{A}(Q_1, \ldots, Q_m))(z)$. Let's assume that

$$\tilde{A}(P_1, \ldots, P_n) = P^*$$

and

$$\tilde{A}(Q_1, \ldots, Q_m) = Q^*,$$

where $P^*, Q^* \in F^*([0,1])$
Thus we consider $\tilde{A}(P^*, Q^*)(z)$: according to theorem 3.1 $\exists\ x^*, y^*$:

$$A(x^*, y^*) = z$$

and

$$\tilde{A}(P^*, Q^*)(z) = T(P^*(x^*), Q^*(y^*)). \tag{13}$$

Following the definition of P^*, Q^* and employing theorem 3.1 for an arbitrary $x^*, y^* \in [0,1]$ we can write:

$$P^*(x^*) = T(P_1(x_1^*), \ldots, P_n(x_n^*)), \tag{14}$$

where $x_1^*, \ldots, x_n^* : A(x_1^*, \ldots, x_n^*) = x^*$ and

$$Q^*(y^*) = T(Q_1(y_1^*), \ldots, Q_m(y_n^*)), \tag{15}$$

where $y_1^*, \ldots, y_m^* : A(y_1^*, \ldots, y_m^*) = y^*$.
We put (14) and (15) into (13) and obtain:

$$\tilde{A}(P^*, Q^*)(z) = T(T(P_1(x_1^*), \ldots, P_n(x_n^*)), T(Q_1(y_1^*), \ldots, Q_m(y_n^*))), \tag{16}$$

where

$$A(A(x_1^*, \ldots, x_n^*), A(y_1^*, \ldots, y_m^*)) = z. \tag{17}$$

Using associativity of T and A we continue (16) and (17) in the following way:

$$\tilde{A}(P^*, Q^*)(z) = T(P_1(x_1^*), \ldots, P_n(x_n^*), Q_1(y_1^*), \ldots, Q_m(y_n^*))$$

$$A(x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_m^*) = z.$$

By definition of $\tilde{A}$, we immediately deduce that

$$\tilde{A}(P_1, \ldots, P_n, Q_1, \ldots, Q_m)(z) \geq \tilde{A}(\tilde{A}(P_1, \ldots, P_n), \tilde{A}(Q_1, \ldots, Q_m))(z).$$

Further, by the associativity of T, we have that

$$\begin{aligned} & \tilde{A}(P_1, \ldots, P_n, Q_1, \ldots, Q_m)(z) = T(P_1(s_1^*), \ldots, P_n(s_n^*), Q_1(t_1^*), \ldots, Q_m(t_m^*)) \\ = \quad & T(T(P_1(s_1^*), \ldots, P_n(s_n^*)), T(Q_1(t_1^*), \ldots, Q_m(t_m^*))). \end{aligned}$$

If we set $s := A(s_1^*, \ldots, s_n^*)$ and $t := A(t_1^*, \ldots, t_m^*)$, we know that $A(s,t) = z$, hence, by definition of $\tilde{A}$ and the monotonicity of T, we derive that

$$\begin{aligned} & T(T(P_1(s_1^*), \ldots, P_n(s_n^*)), T(Q_1(t_1^*), \ldots, Q_m(t_m^*))) \\ \leq \quad & T(\tilde{A}(P_1, \ldots, P_n)(s), \tilde{A}(Q_1, \ldots Q_m)(t)) \leq \tilde{A}(P^*, Q^*)(z). \end{aligned}$$

$\square$

Proposition 5.15. Let A be a bisymmetric agop. An arbitrary T-extension $\tilde{A} : \cup_{n \in \mathbb{N}} F^*([0,1])^n \to F^*([0,1])$ of A given by (1) is an bisymmetric gagop.

5.5. Idempotence of a T-extension

We consider the idempotence property of $\tilde{A}$ in this subsection.

Definition 5.16. [IDEMPOTENCE] An element $P \in F(X)$ is called $\tilde{A}$-idempotent element whenever $\tilde{A}_{(n)}(P, \ldots, P) = P, \forall n \in \mathbb{N}$. $\tilde{A}$ is called an idempotent gagop if each $P \in F(X)$ is an idempotent element of $\tilde{A}$.

In general, i. e., if we take an idempotent T and an idempotent A, the corresponding extension is not idempotent. The convexity of input values is crucial here. Let's consider $A(x_1, x_2) = \frac{x_1+x_2}{2}$ and not convex set P such that $P(x^*) = 0$ and $P(x) > 0$ $\forall x \in [a, x^*) \cup (x^*, b] \subseteq [0, 1]$. It is intuitively clear that result of aggregation in the point x^* may have value greater than 0.

Proposition 5.17. Let A be an idempotent agop. T_M-extension $\tilde{A} : \cup_{n\in\mathbb{N}} FQ([0, 1])^n \to FQ([0, 1])$ of A given by (1) is an idempotent gagop.

Proof. We consider an arbitrary $P(x) \in FQ([0, 1])$ and $x^* \in [0, 1]$, then according to the definition of a T-extension and theorem 3.1 $\exists\ x_1^*, \dots, x_n^*$ s.t.:

$$A(x_1^*, \dots, x_n^*) = x^*$$

and

$$\tilde{A}_{(n)}(P, \dots, P)(x^*) = T_M(P(x_1^*), \dots, P(x_n^*)).$$

We denote $S = \{(x_1, \dots, x_n) : A(x_1, \dots, x_n) = x^*\}$ and $(x_1^*, \dots, x_n^*) \in S$.
A is an idempotent agop, therefore $(x^*, \dots, x^*) \in S$ and $T_M(P(x^*), \dots, P(x^*)) = P(x^*)$.
Idempotence and monotonicity of A imply compensation property, i.e. for an arbitrary $(x_1, \dots, x_n) \in S$ the following hold:

$$\min_{i=1,\dots,n} x_i \leq A(x_1, \dots, x_n) \leq \max_{i=1,\dots,n} x_i \tag{18}$$

$$\min_{i=1,\dots,n} x_i \leq x^* \leq \max_{i=1,\dots,n} x_i. \tag{19}$$

For an arbitrary $(x_1, \dots, x_n) \in S$ s.t. $(x_1, \dots, x_n) \neq (x^*, \dots, x^*)$ one of the following properties holds:

(i) $\exists\ x_{i_1}, \dots, x_{i_k}, 1 \leq k \leq n-1$: $x_{i_j} < x^*$ $\forall j = 1, \dots, k$ and for the rest x_{i_s} $s \notin \{1, \dots, k\}$: $x_{i_s} \geq x^*$

(ii) $\exists\ x_{i_1}, \dots, x_{i_k}, 1 \leq k \leq n-1$: $x_{i_j} > x^*$ $\forall j = 1, \dots, k$ and for the rest x_{i_s} $s \notin \{1, \dots, k\}$: $x_{i_s} \leq x^*$.

If neither (i) no (ii) holds then $(x_1, \dots, x_n)$ s.t.

$$x_i < x^*\ \forall i$$

or

$$x_i > x^*\ \forall i.$$

But in the first case according to the compensation property ((18), (19)) we obtain

$$A(x_1, \dots, x_n) \leq \max(x_1, \dots, x_n) < x^*$$

in the second case we obtain

$$x^* < \min(x_1, \dots, x_n) \leq A(x_1, \dots, x_n)$$

but then $(x_1, \dots, x_n) \notin S$.

Now we take an arbitrary $(x_1, \ldots, x_n) \in S$ and assume that (i) holds:
for an arbitrary $x_l < x^*$ and arbitrary $x_k \geq x^*$ convexity of P implies that

$$P(x^*) \geq T_M(P(x_l), P(x_k)). \tag{20}$$

If we add the rest coordinates of the vector we can only reduce the minimum thus we can continue formula (20) in the following way:

$$P(x^*) \geq T_M(P(x_1), \ldots, P(x_n)).$$

So, we have obtained that for an arbitrary vector $(x_1, \ldots, x_n) \in S$

$$T_M(P(x^*), \ldots, P(x^*)) = P(x^*) \geq T_M(P(x_1), \ldots, P(x_n)).$$

The assumption that (ii) holds will lead us to the same result.
and this means that

$$\tilde{A}_{(n)}(P, \ldots, P)(x^*) = T_M(P(x^*), \ldots, P(x^*)) = P(x^*).$$

□

Remark 5.18. Recall that T_M is the only idempotent t-norm. Now we show that only T_M-extension ensures idempotence of $\tilde{A}$ (given conditions of Proposition 5.17). If we take an arbitrary t-norm $T < T_M$ then according to the proof of Proposition 5.17

$$\begin{aligned} & \forall (x_1, \ldots, x_n) \in S,\ P(x^*) = T_M(P(x^*), \ldots, P(x^*)) \\ = \quad & \max\{T_M(P(x_1), \ldots, P(x_n)) | (x_1, \ldots, x_n) \in S\}. \end{aligned} \tag{21}$$

Now applying the upper bound of the class of t-norms we obtain

$$\begin{aligned} & \max\{T_M(P(x_1), \ldots, P(x_n)) | (x_1, \ldots, x_n) \in S\} \\ > \quad & \max\{T(P(x_1), \ldots, P(x_n)) | (x_1, \ldots, x_n) \in S\} = \tilde{A}_{(n)}(P, \ldots, P)(x^*). \end{aligned} \tag{22}$$

Combining the result of formulas (21) and (22) we obtain:

$$P(x^*) > \tilde{A}_{(n)}(P, \ldots, P)(x^*).$$

5.6. Neutral and absorbing elements

Now we study neutral and absorbing elements of a T-extension. First we provide definitions and results on uniqueness of neutral and absorbing elements. After that we provide constructions of the corresponding elements.

Definition 5.19. [Neutral element] An element $E \in F(X)$ is called a neutral element of $\tilde{A}$ if $\forall n \in \mathbb{N}, \forall P_1, \ldots, P_n, \in F(X)$ if $P_i = E$ for some $i \in \{1, \ldots, n\}$ then

$$\tilde{A}(P_1, \ldots, P_n) = \tilde{A}_{(n-1)}(P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n).$$

The following result hold for the neutral element of a gagop:

Proposition 5.20. If $\tilde{A}$ is a gagop w.r.t. $\leq$ and $E \in F(X)$ is a neutral element of $\tilde{A}$ then it is unique.

P r o o f. Let's assume that E and E^* are neutral elements of $\tilde{A}$ and $E \neq E^*$. We consider an arbitrary $n \in \mathbb{N}$ and vector $(P_1, \ldots, P_n)$ s.t.:

$$P_i = \begin{cases} E, & \text{if } i \in I_1 \\ E^*, & \text{if } i \in I_2, \end{cases}$$

where $I_1 = \{1, \ldots, k\}$, $I_2 = \{k+1, \ldots, n\}$.
Using neutrality of $\tilde{A}$ versus E we obtain:

$$\begin{aligned} & \tilde{A}_{(n)}(P_1, \ldots, P_n) = \tilde{A}_{(n-1)}(P_2, \ldots, P_n) = \ldots = \tilde{A}_{(n-k)}(P_{k+1}, \ldots, P_n) \\ = \; & \tilde{A}_{(n-k)}(E^*, \ldots, E^*) \end{aligned} \tag{23}$$

now we apply neutrality of E^*, convention $\tilde{A}_{(1)}(P) = P$ and continue (23):

$$\tilde{A}_{(n-k)}(E^*, \ldots, E^*) = \tilde{A}_{(n-k-1)}(E^*, \ldots, E^*) = \ldots = \tilde{A}_{(1)}(E^*) = E^*.$$

In the same way first employing neutrality of E^* and then neutrality of E we obtain:

$$\begin{aligned} & \tilde{A}(P_1, \ldots, P_n) = \tilde{A}_{(n-1)}(P_1, \ldots, P_{n-1}) = \ldots = \tilde{A}_{(k)}(P_1, \ldots, P_k) = \tilde{A}_{(k)}(E, \ldots, E) \\ = \; & \tilde{A}_{(k)}(E, \ldots, E) = \tilde{A}_{(k-1)}(E, \ldots, E) = \ldots = \tilde{A}_{(1)}(E) = E. \end{aligned}$$

We have obtained contradiction. □

Definition 5.21. [Absorbing element] An element $R \in F(X)$ is called an absorbing element of $\tilde{A}$ if

$$\forall n \in \mathbb{N}, \forall P_1, \ldots, P_n, \in F(X) : R \in \{P_1, \ldots, P_n\} \Rightarrow \tilde{A}(P_1, \ldots, P_n) = R.$$

Absorbing element like neutral element is unique if it exists:

Proposition 5.22. If $\tilde{A}$ is a gagop w.r.t. $\leq$ and $R \in F(X)$ is an absorbing element of $\tilde{A}$ then it is unique.

P r o o f. Let's assume that R and R^* are absorbing elements of $\tilde{A}$ and $R \neq R^*$.

We consider an arbitrary $n \in \mathbb{N}$ and vector $(P_1, \ldots, P_n)$ s.t. $R, R^* \in \{P_1, \ldots, P_n\}$. R is an absorbing element therefore according to definition 5.21

$$\tilde{A}(P_1, \ldots, P_n) = R.$$

R^* as well is an absorbing element of $\tilde{A}$, therefore:

$$\tilde{A}(P_1, \ldots, P_n) = R^*$$

We have obtained contradiction, thus our assumption on existence of R^* is incorrect. □

Further formulated results outline the nature of neutral and absorbing elements of $\tilde{A}$.

Proposition 5.23. Let A be an agop with neutral element e. An arbitrary T-extension $\tilde{A} : \cup_{n\in\mathbb{N}} F^*([0,1])^n \to F^*([0,1])$ of A given by (1) has a neutral element given by

$$E(x) = \begin{cases} 1, & \text{if } x = e \\ 0, & \text{if } x \neq e. \end{cases}$$

P r o o f. Consider vector $(P_1, \dots, P_n)$, such that $P_i = E$, and $\tilde{A}(P_1, \dots, P_n)(x)$.

According to theorem 3.1 $\exists\ x_1^*, \dots, x_n^*$:

$$A(x_1^*, \dots, x_n^*) = x \tag{24}$$

and

$$\tilde{A}(P_1, \dots, P_n)(x) = T(P_1(x_1^*), \dots, P_n(x_n^*)). \tag{25}$$

Let us assume that $x_i^* = e$ in the formula (24), therefore applying neutrality of A versus e we can continue (24):

$$A(x_1^*, \dots, x_n^*) = A(x_1^*, \dots, x_{i-1}^*, e, x_{i+1}^*, \dots x_n^*)$$

$$= A_{(n-1)}(x_1^*, \dots, x_{i-1}^*, x_{i+1}^*, \dots x_n^*) = x.$$

Since $P_i = E$ and $x_i^* = e$ we have $P_i(x_i^*) = P_i(e) = E(e) = 1$ thus applying neutrality of t-norm T versus 1 we continue formula (25) in the following way:

$$\begin{aligned} & \tilde{A}(P_1, \dots, P_n)(x) = T(P_1(x_1^*), \dots, P_n(x_n^*)) \\ = \quad & T(P_1(x_1^*), \dots, P_{i-1}(x_{i-1}^*), 1, P_{i+1}(x_{i+1}^*), \dots, P_n(x_n^*)) \\ = \quad & T_{(n-1)}(P_1(x_1^*), \dots, P_{i-1}(x_{i-1}^*), P_{i+1}(x_{i+1}^*), \dots, P_n(x_n^*)). \end{aligned} \tag{26}$$

Now we consider $\tilde{A}_{n-1}(P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n)(x)$.
according to theorem 3.1 $\exists\ y_1^*, \dots, y_{i-1}^*, y_{i+1}^*, \dots, y_{n-1}^*$:

$$A(y_1^*, \dots, y_{i-1}^*, y_{i+1}^*, \dots, y_{n-1}^*) = x$$

and

$$\begin{aligned} & \tilde{A}_{n-1}(P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n)(x) \\ = \quad & T(P_1(y_1^*), \dots, P_{i-1}(y_{i-1}^*), P_{i+1}(y_{i+1}^*), \dots, P_n(y_{n-1}^*)). \end{aligned} \tag{27}$$

If we assume that $\tilde{A}(P_1, \dots, P_n)(x) > \tilde{A}_{n-1}(P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n)(x)$ then according to formulas (26) and (27) we can take vector $(x_1^*, \dots, x_n^*), x_i^* = e$ instead of $(y_1^*, \dots, y_{i-1}^*, y_{i+1}^*, \dots, y_{n-1}^*)$ and we obtain higher value of $\tilde{A}_{n-1}(P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n)(x)$ than obtained previously, but this contradicts definition of vector $(y_1^*, \dots \dots, y_{i-1}^*, y_{i+1}^*, \dots, y_{n-1}^*)$.

To the similar contradiction will lead us the assumption

$$\tilde{A}(P_1, \dots, P_n)(x) < \tilde{A}_{n-1}(P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n)(x).$$

□

According to Proposition 5.20 $E(x)$ is the unique neutral element.

Since $FN([0,1]) \subset FI([0,1]) \subset FQ([0,1])$ and $E(x) \in FN([0,1])$ then defining T-extension on the classes $FN([0,1]), FI([0,1])$ or $FQ([0,1])$ we obtain a gagop with the neutral element given in Proposition 5.23.

Now we consider an absorbing element of an arbitrary T-extension.

Proposition 5.24. Let A be an agop. An arbitrary T-extension $\tilde{A} : \cup_{n\in\mathbb{N}} F^*([0,1])^n \to F^*([0,1])$ of A given by (1) has an absorbing element given by

$$R(x) = 0 \ \forall x \in [0,1].$$

Proof. Vector $(P_1, \ldots, P_n)$ such that $P_i = R$ is given, consider $\tilde{A}(P_1, \ldots, P_n)(x)$.

According to theorem 3.1 $\exists\ x_1^*, \ldots, x_n^*$ s.t.:

$$A(x_1^*, \ldots, x_n^*) = x$$

and

$$\tilde{A}(P_1, \ldots, P_n)(x) = T(P_1(x_1^*), \ldots, R(x_i^*), \ldots, P_n(x_n^*)). \tag{28}$$

For an arbitrary $x_i^* \in [0,1]$ $R(x_i^*) = 0$ using this fact and applying absorbing property of 0 for an arbitrary t-norm T we continue (28):

$$\tilde{A}(P_1, \ldots, P_n)(x) = 0.$$

We have shown that for an arbitrary $x \in [0,1]$ $\tilde{A}(P_1, \ldots, P_n)(x) = 0$ and thus the assertion holds. $\square$

There is no other absorbing elements as the uniqueness of $R(x)$ is ensured by Proposition 5.22.

The question how to interpret $R(x)$ arises. On the one hand element R belongs to the class $F^*([0,1])$, but on the other hand it does not have any real value, i.e. any point is possible with value 0. Thus question on nature of R is rather philosophical. We skip the philosophical part of this question and consider that absorbing element of T-extension exists, it is from the class $F^*([0,1])$ and it is given in Proposition 5.24.

Since $R(x) \in FQ([0,1])$ T-extension defined on the class $FQ([0,1])$ has the same absorbing element.

$R(x) \notin FI([0,1])$ and there is no other absorbing element in $FI([0,1])$. If we assume that there exists $R^*(x) \in FI([0,1])$ and it differs from $R(x)$ then $R^*(x) \in F^*([0,1])$, but this contradicts result of Proposition 5.22. Thus defining a T-extension on the class $FI([0,1])$ or $FN([0,1])$ we deal with a gagop without the absorbing element.

The interesting fact should be noticed here: T-extension of an agop without an absorbing element can result in a gagop with absorbing element.

6. CONCLUDING REMARKS

Properties of a T-extension are tightly related to properties of the corresponding t-norm and sometimes we can manage to obtain a desired property by choosing an appropriate t-norm, but not always it is possible. For an example in the case of absorbing element, the form of absorbing element can not be changed, because 0 is the only absorbing element of any t-norm. Thus such cases can not be managed by changing t-norm. But substituting t-norm in the definition of a T-extension by e. g. a nullnorm ([4]) N we obtain an N-extension:

$$\tilde{A}(P_1, \ldots, P_n)(x) = \sup\{N(P_1(x_1), \ldots, P_n(x_n)|(A(x_1, \ldots, x_n) = x)\}.$$

Performing U-extension, i.e. extension via a uninorm U ([4]) we can obtain a neutral element with different properties and also we can use compensation property of uninorms.

The definition of a T-extension can be generalized via an arbitrary agop A^* (with desired properties):

$$\tilde{A}(P_1, \ldots, P_n)(x) = \sup\{A^*(P_1(x_1), \ldots, P_n(x_n)|A(x_1, \ldots, x_n) = x\}.$$

Extension (via a t-norm, an u-norm or other function) is a flexible construction method as far as it relates to properties of a gagop. The research of properties of a gagop constructed in this way is a direction of further study.

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