Jan Paseka; Zdena Riečanová; Junde Wu

Almost orthogonality and Hausdorff interval topologies of atomic lattice effect algebras

Kybernetika, Vol. 46 (2010), No. 6, 953--970

Persistent URL: `http://dml.cz/dmlcz/141459`

KYBERNETIKA — VOLUME *46* (2010), NUMBER 6, PAGES 953–970

ALMOST ORTHOGONALITY AND HAUSDORFF INTERVAL TOPOLOGIES OF ATOMIC LATTICE EFFECT ALGEBRAS

Jan Paseka, Zdenka Riečanová and Wu Junde

We prove that the interval topology of an Archimedean atomic lattice effect algebra E is Hausdorff whenever the set of all atoms of E is almost orthogonal. In such a case E is order continuous. If moreover E is complete then order convergence of nets of elements of E is topological and hence it coincides with convergence in the order topology and this topology is compact Hausdorff compatible with a uniformity induced by a separating function family on E corresponding to compact and cocompact elements. For block-finite Archimedean atomic lattice effect algebras the equivalence of almost orthogonality and s-compact generation is shown. As the main application we obtain a state smearing theorem for these effect algebras, as well as the continuity of $\oplus$-operation in the order and interval topologies on them.

Keywords: non-classical logics, D-posets, effect algebras, MV-algebras, interval and order topology, states

Classification: 03G12, 06F05, 03G25, 54H12, 08A55

1. INTRODUCTION, BASIC DEFINITIONS AND FACTS

In the study of effect algebras (or more general, quantum structures) as carriers of states and probability measures, an important tool is the study of topologies on them. We can say that topology is practically equivalent with the concept of convergence. From the probability point of view the convergence of nets is the main tool in spite of that convergence of filters is easier to handle and preferred in the modern topology. It is because states or probabilities are mappings (functions) defined on elements but not on subsets of quantum structures. Note also, that connections between order convergence of filters and nets are not trivial. For instance, if a filter order converges to some point of a poset then the associated net need not order converge (see e. g., [12]).

On the other hand certain topological properties of studied structures characterize also their certain algebraic properties and conversely. For instance a known fact is that a Boolean algebra B is atomic iff the interval topology τ_i on B is Hausdorff (see [20, Corollary 3.4]). This is not more valid for lattice effect algebras (even MV-algebras). By Frink's Theorem the interval topology τ_i on B (more generally on any lattice L) is compact iff it is a complete lattice [5]. In [16] it was proved that if a lattice effect algebra E (more generally any basic algebra) is compactly generated then E is atomic.

We are going to prove that on an Archimedean atomic lattice effect algebra E the interval topology τ_i is Hausdorff and E is (o)-continuous if and only if E is almost orthogonal. Moreover, if E is complete then τ_i is compact and coincides with the order topology τ_o on E and this compact topology $\tau_i = \tau_o$ is compatible with a uniformity on E induced by a separating function family on E corresponding to compact and cocompact elements of E.

As the main corollary of that we obtain that every Archimedean atomic block-finite lattice effect algebra E has Hausdorff interval topology and hence both topologies τ_i and τ_o are Hausdorff and they coincide. In this case almost orthogonality of E and s-compact generation by finite elements of E are equivalent. As an application a state smearing theorem for these effect algebras is formulated. Moreover, continuity of $\oplus$-operation in τ_i and τ_o on them is shown.

Definition 1.1. A partial algebra $(E; \oplus, 0, 1)$ is called an effect algebra if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,

(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,

(Eiii) for every $a \in E$ there is a unique $b \in E$ such that $a \oplus b = 1$ (we put $a' = b$),

(Eiv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. In every effect algebra E we can define the partial order $\leq$ and the partial operation $\ominus$ by putting
$a \leq b$ and $b \ominus a = c$ iff $a \oplus c$ is defined and $a \oplus c = b$, we set $c = b \ominus a$.

If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (*a complete lattice effect algebra*).

Recall that a set $Q \subseteq E$ is called a *sub-effect algebra* of the effect algebra E if

(i) $1 \in Q$

(ii) if out of elements $a, b, c \in E$ with $a \oplus b = c$ two are in Q, then $a, b, c \in Q$.

If Q is simultaneously a sublattice of E then Q is called a *sub-lattice effect algebra* of E.

We say that a finite system $F = (a_k)_{k=1}^n$ of not necessarily different elements of an effect algebra $(E; \oplus, 0, 1)$ is $\oplus$-*orthogonal* if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ (written $\bigoplus_{k=1}^n a_k$ or $\bigoplus F$) exists in E. Here we define $a_1 \oplus a_2 \oplus \cdots \oplus a_n = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ supposing that $\bigoplus_{k=1}^{n-1} a_k$ exists and $\bigoplus_{k=1}^{n-1} a_k \leq a_n'$. An arbitrary system $G = (a_\kappa)_{\kappa \in H}$ of not necessarily different elements of E is $\oplus$-*orthogonal* if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a $\oplus$-orthogonal system $G = (a_\kappa)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K \mid K \subseteq G,\ K \text{ is finite}\}$ exists in E and then we put

$\bigoplus G = \bigvee\{\bigoplus K \mid K \subseteq G,\ K$ is finite$\}$ (we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (a_\kappa)_{\kappa \in H_1}$).

Recall that elements x and y of a lattice effect algebra are called *compatible* (written $x \leftrightarrow y$) if $x \vee y = x \oplus (y \ominus (x \wedge y))$ [13]. For $x \in E$ and $Y \subseteq E$ we write $x \leftrightarrow Y$ iff $x \leftrightarrow y$ for all $y \in Y$. If every two elements are compatible then E is called an *MV-effect algebra*. In fact, every MV-effect algebra can be organized into an MV-algebra (see [2]) if we extend the partial $\oplus$ to a total operation by setting $x \oplus y = x \oplus (x' \wedge y)$ for all $x, y \in E$ (also conversely, restricting a total $\oplus$ into partial $\oplus$ for only $x, y \in E$ with $x \le y'$ we obtain a MV-effect algebra).

Moreover, in [23] it was proved that every lattice effect algebra is a set-theoretical union of MV-effect algebras called blocks. *Blocks* are maximal subsets of pairwise compatible elements of E, under which every subset of pairwise compatible elements is by Zorn's Lemma contained in a maximal one. Further, blocks are sub-lattices and sub-effect algebras of E and hence maximal sub-MV-effect algebras of E. A lattice effect algebra is called *block-finite* if it has only finitely many blocks.

Finally note that lattice effect algebras generalize orthomodular lattices [10] (including Boolean algebras) if we assume existence of unsharp elements $x \in E$, meaning that $x \wedge x' \neq 0$. On the other hand the *set* $S(E) = \{x \in E \mid x \wedge x' = 0\}$ *of all sharp elements* of a lattice effect algebra E is an orthomodular lattice [8]. In this sense a lattice effect algebra is a "smeared" orthomodular lattice, while an MV-effect algebra is a "smeared" Boolean algebra. An orthomodular lattice L can be organized into a lattice effect algebra by setting $a \oplus b = a \vee b$ for every pair $a, b \in L$ such that $a \le b^\perp$.

For an element x of an effect algebra E we write $\mathrm{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n-times) exists for every positive integer n and we write $\mathrm{ord}(x) = n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E. An effect algebra E is *Archimedean* if $\mathrm{ord}(x) < \infty$ for all $x \in E$, $x \neq 0$. It is known that every complete effect algebra is Archimedean (see [22]).

An element a of an effect algebra E is an *atom* if $0 \le b < a$ implies $b = 0$ and E is called *atomic* if for every nonzero element $x \in E$ there is an atom a of E with $a \le x$. If $u \in E$ and either $u = 0$ or $u = p_1 \oplus p_2 \oplus \cdots \oplus p_n$ for some not necessarily different atoms $p_1, p_2, \ldots, p_n \in E$ then $u \in E$ is called *finite* and $u' \in E$ is called *cofinite*. If E is a lattice effect algebra then for $x \in E$ and an atom a of E we have $a \leftrightarrow x$ iff $a \le x$ or $a \le x'$. It follows that if a is an atom of a block M of E then a is also an atom of E. On the other hand if E is atomic then, in general, every block in E need not be atomic (even for orthomodular lattices [1]).

The following theorem is well known.

Theorem 1.2. (Riečanová [25, Theorem 3.3]) Let $(E; \oplus, 0, 1)$ be an Archimedean atomic lattice effect algebra. Then to every nonzero element $x \in E$ there are mutually distinct atoms $a_\alpha \in E$, $\alpha \in \mathcal{E}$ and positive integers k_α such that

$$x = \bigoplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\} = \bigvee\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\}$$

under which $x \in S(E)$ iff $k_\alpha = n_{a_\alpha} = \mathrm{ord}(a_\alpha)$ for all $\alpha \in \mathcal{E}$.

Definition 1.3. (1) An element a of a lattice L is called *compact* iff, for any $D \subseteq L$ with $\bigvee D \in L$, if $a \leq \bigvee D$ then $a \leq \bigvee F$ for some finite $F \subseteq D$.

(2) A lattice L is called *compactly generated* iff every element of L is a join of compact elements.

The notions of *cocompact element* and *cocompactly generated lattice* can be defined dually. Note that compact elements are important in computer science in the semantic approach called domain theory, where they are considered as a kind of primitive elements.

2. CHARACTERIZATIONS OF INTERVAL TOPOLOGIES ON BOUNDED LATTICES

The order convergence of nets ((o)-convergence), interval topology τ_i and order-topology τ_o ((o)-topology) can be defined on any poset. In our observations we will consider only bounded lattices and we will give a characterization of interval topologies on them.

Definition 2.1. Let L be a bounded lattice. Let $\mathcal{H} = \{[a,b] \subseteq L | a, b \in L$ with $a \leq b\}$ and let $\mathcal{G} = \{\bigcup_{k=1}^{n}[a_k, b_k] | [a_k, b_k] \in \mathcal{H}, k = 1, 2, \ldots, n,\ n \in \mathbb{N}\}$. The *interval topology* τ_i *of* L is the topology of L with $\mathcal{G}$ as a closed basis, hence with $\mathcal{H}$ as a closed subbasis.

From definition of τ_i we obtain that $U \in \tau_i$ iff for each $x \in U$ there is $F \in \mathcal{G}$ such that $x \in L \backslash F \subseteq U$.

Definition 2.2. Let L, K be posets and $(\mathcal{E}, \leq)$ a directed poset.

(i) A net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of L *order converges* (*(o)-converges*, for short) to a point $x \in L$ if there are nets $(u_\alpha)_{\alpha \in \mathcal{E}}$ and $(v_\alpha)_{\alpha \in \mathcal{E}}$ of elements of L such that

$$x \uparrow u_\alpha \leq x_\alpha \leq v_\alpha \downarrow x,\ \alpha \in \mathcal{E}$$

where $x \uparrow u_\alpha$ means that $u_{\alpha_1} \leq u_{\alpha_2}$ for every $\alpha_1 \leq \alpha_2$ and $x = \bigvee\{u_\alpha \mid \alpha \in \mathcal{E}\}$. The meaning of $v_\alpha \downarrow x$ is dual.

We write $x_\alpha \stackrel{(o)}{\to} x, \alpha \in \mathcal{E}$ in L.

(ii) A topology τ_o on L is called the *order topology* on L iff

(a) for any net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of L and $x \in L$: $x_\alpha \stackrel{(o)}{\to} x$ in L $\Rightarrow x_\alpha \stackrel{\tau_o}{\to} x,\ \alpha \in \mathcal{E}$, where $x_\alpha \stackrel{\tau_o}{\to} x$ denotes that $(x_\alpha)_{\alpha \in \mathcal{E}}$ *converges* to x in the topological space (L, τ_o),

(b) if τ is a topology on L with property (a) then $\tau \subseteq \tau_o$.

Hence τ_o is the strongest (finest, biggest) topology on L with property (a).

(c) The symbol $\tau_o \equiv (o)$ means that $x_\alpha \stackrel{\tau_o}{\to} x$ iff $x_\alpha \stackrel{(o)}{\to} x,\ \alpha \in \mathcal{E}$, for every net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of L and $x \in L$.

(iii) An order preserving map $f : L \to K$ is called *order continuous* ((o)-continuous for brevity) if for any net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of L and $x \in L$, $x_\alpha \uparrow x \Rightarrow f(x_\alpha) \uparrow f(x)$.

(iv) A lattice L is called *order continuous* ((o)-continuous for brevity) if for any net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of L and $x, y \in L$, $x_\alpha \uparrow x \Rightarrow x_\alpha \wedge y \uparrow x \wedge y$ i.e., the maps $(-) \wedge y : L \to L$ are (o)-continuous for all $y \in L$.

Recall that, for a directed set $(\mathcal{E}, \leq)$, a subset $\mathcal{E}' \subseteq \mathcal{E}$ is called *cofinal* in $\mathcal{E}$ iff for every $\alpha \in \mathcal{E}$ there is $\beta \in \mathcal{E}'$ such that $\alpha \leq \beta$. A special kind of a *subnet* of a net $(x_\alpha)_{\alpha\in\mathcal{E}}$ is net $(x_\beta)_{\beta\in\mathcal{E}'}$ where $\mathcal{E}'$ is a cofinal subset of $\mathcal{E}$. This kind of subnets works in many cases of our considerations.

In what follows we often use the following useful characterization of topological convergence of nets:

Lemma 2.3. For a net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of a topological space (X, τ) and $x \in X$:

$$x_\alpha \xrightarrow{\tau} x, \alpha \in \mathcal{E} \quad \text{iff} \quad \begin{array}{l}\text{for all } \mathcal{E}' \subseteq \mathcal{E}, \text{ where } \mathcal{E}' \text{ is cofinal in } \mathcal{E} \text{ there is} \\ \mathcal{E}'' \subseteq \mathcal{E}',\ \mathcal{E}'' \text{ cofinal in } \mathcal{E}' \text{ such that } x_\gamma \xrightarrow{\tau} x, \gamma \in \mathcal{E}''.\end{array}$$

P r o o f. $\Rightarrow$: It is trivial.
$\Leftarrow$: Let for every $\mathcal{E}' \subseteq \mathcal{E}$, where $\mathcal{E}'$ is cofinal in $\mathcal{E}$ there is $\mathcal{E}'' \subseteq \mathcal{E}'$, $\mathcal{E}''$ cofinal in $\mathcal{E}'$ and $x_\gamma \xrightarrow{\tau} x, \gamma \in \mathcal{E}''$, and let $x_\alpha \stackrel{\tau}{\not\to} x, \alpha \in \mathcal{E}$. Then there is $U(x) \in \tau$ such that for all $\alpha \in \mathcal{E}$ there is $\beta_\alpha \in \mathcal{E}$ with $\beta_\alpha \geq \alpha$ and $x_{\beta_\alpha} \notin U(x)$. Let $\mathcal{E}' = \{\beta_\alpha \in \mathcal{E} | \alpha \in \mathcal{E}\}$ then $x_{\beta_\alpha} \stackrel{\tau}{\not\to} x, \beta_\alpha \in \mathcal{E}'$ and for all cofinal $\mathcal{E}'' \subseteq \mathcal{E}'$: $x_\gamma \stackrel{\tau}{\not\to} x, \gamma \in \mathcal{E}''$. Hence there is $\mathcal{E}' \subseteq \mathcal{E}$ cofinal in $\mathcal{E}$ and for all $\mathcal{E}'' \subseteq \mathcal{E}'$, $\mathcal{E}''$ cofinal in $\mathcal{E}'$: $x_\gamma \stackrel{\tau}{\not\to} x, \gamma \in \mathcal{E}''$ a contradiction. □

Further, let us recall the following well known facts:

Lemma 2.4. Let L be a bounded lattice. Then

(i) $F \subseteq L$ is τ_o-closed iff for every net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of L and $x \in L$: $(x_\alpha \in F, x_\alpha \xrightarrow{(o)} x, \alpha \in \mathcal{E}) \Rightarrow x \in F$.

(ii) For every $a, b \in L$ with $a \leq b$ the interval $[a, b]$ is τ_o-closed.

(iii) $\tau_i \subseteq \tau_o$.

(iv) For any net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of L and $x \in L$:

$$x_\alpha \xrightarrow{(o)} x, \alpha \in \mathcal{E} \implies x_\alpha \xrightarrow{\tau_i} x, \alpha \in \mathcal{E}.$$

(v) If τ_i is Hausdorff then $\tau_o = \tau_i$ (see [4]).

(vi) The interval topology τ_i of a lattice L is compact iff L is a complete lattice (see [5]).

(vii) Let $f : L \to \mathbb{R}$ be a real function. Then f is (o)-continuous iff f is τ_o-continuous.

Proof. It is enough to check only (vii). Clearly, if f is τ_o-continuous then f is (o)-continuous since any (o)-convergent net is τ_o-convergent. Assume now that f is (o)-continuous. Let $D \subseteq \mathbb{R}$ be a closed subset and let $F = f^{-1}(D)$. It is enough to check that F is τ_o-closed. Using (i), assume that $(x_\alpha)_{\alpha\in\mathcal{E}}$ is a net of elements of L, $x \in L$ such that $x_\alpha \in F, x_\alpha \overset{(o)}{\to} x, \alpha \in \mathcal{E}$. Hence $f(x_\alpha) \in D, f(x_\alpha) \to f(x), \alpha \in \mathcal{E}$. Since D is closed we get $f(x) \in D$. Therefore $x \in F$. □

Finally, let us note that compact Hausdorff topological space is always normal. Thus separation axiom T_2, T_3 and T_4 are trivially equivalent for the interval topology of a complete lattice L.

Theorem 2.5. Let L be a complete lattice with interval topology τ_i. If $F \subseteq L$ is a complete sub-lattice of L then

(a) $\tau_i^F = \tau_i \cap F$ is the interval topology of F,

(b) for any net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of F and $x \in F$:

$$x_\alpha \overset{\tau_i^F}{\to} x, \alpha \in \mathcal{E} \iff x_\alpha \overset{\tau_i}{\to} x, \alpha \in \mathcal{E}.$$

Proof. (a): Let $\mathcal{H}$ and $\mathcal{H}_F$ be a closed subbasis of τ_i and τ_i^F respectively. Then evidently $\mathcal{H} \cap F = \{[a,b] \cap F | [a,b] \in \mathcal{H}\}$ is a closed subbasis of $\tau_i \cap F$. Further for $[c,d]_F \in \mathcal{H}_F$ we have $[c,d]_F = \{x \in F | c \le x \le d\} = [c,d] \cap F \in \mathcal{H} \cap F$. Conversely, since F is a complete sub-lattice of L, if $[a,b] \in \mathcal{H}$ then $[a,b] \cap F = \{x \in F | a \le x \le b\}$ and either $[a,b] \cap F = \emptyset$ or there is $c = \wedge\{x \in F | a \le x \le b\}$ and $d = \vee\{x \in F | a \le x \le d\}$ and $[a,b] \cap F = [c,d]_F \in \mathcal{H}_F$. This proves that $\tau_i^F = \tau_i \cap F$.
(b): This is an easy consequence of (a). □

3. HAUSDORFF INTERVAL TOPOLOGY OF ALMOST ORTHOGONAL ARCHIMEDEAN ATOMIC LATTICE EFFECT ALGEBRAS AND THEIR ORDER CONTINUITY

The atomicity of Boolean algebra B is equivalent with Hausdorffness of interval topology on B (see [11, 29] and [20, Corollary 3.4]). This is not more valid for lattice effect algebras, even also for MV-algebras.

Example 3.1. Let $M = [0,1] \subseteq \mathbb{R}$ be a standard MV-effect algebra, i. e., we define $a \oplus b = a + b$ iff $a + b \le 1$, $a, b \in M$. Then M is a complete (o)-continuous lattice with $\tau_i = \tau_o$ being Hausdorff and with (o)-convergence of nets coinciding with τ_o-convergence. Nevertheless, M is not atomic.

We have proved in [16] that a complete lattice effect algebra is atomic and (o)-continuous lattice iff E is compactly generated. Nevertheless, in such a case, the interval topology on E need not be Hausdorff.

Example 3.2. Let E be a horizontal sum of infinitely many finite chains $(P_i, \bigoplus_i, 0_i, 1_i)$ with at least 3 elements, $i = 1, 2, \ldots, n, \ldots$, (i. e., for $i = 1, 2, \ldots, n, \ldots$, we identify all 0_i and all 1_i as well, $\bigoplus_i$ on P_i are preserved and any $a \in P_i \backslash \{0_i, 1_i\}$, $b \in P_j \backslash \{0_j, 1_j\}$ for $i \neq j$ are noncomparable). Then E is an atomic complete lattice effect algebra, E is not block-finite and the interval topology τ_i on E is compact. Nevertheless, τ_i is not Hausdorff because e. g., for $a \in P_i, b \in P_j, i \neq j$, a, b noncomparable, we have $[a, 1] \cap [0, b] = \emptyset$ and there is no finite family $\mathcal{I}$ of closed intervals in E separating $[a, 1], [0, b]$ (i. e., the lattice E can not be covered by a finite number of closed intervals from $\mathcal{I}$ each of which is disjoint with at least one of the intervals $[a, 1]$ and $[0, b]$). This implies that τ_i is not Hausdorff by [20, Lemma 2.2]. Further E is *compactly generated by finite elements* (hence (o)-continuous). It follows by [16] that the order topology τ_o on E is a uniform topology and (o)-convergence of nets on E coincides with τ_o-convergence.

In what follows we shall need an extension of [26, Lemma 2.1 (iii)].

Lemma 3.3. Let E be a lattice effect algebra, $x, y \in E$, $k, l \in \mathbb{N}$. Then $x \wedge y = 0$ and $x \leq y'$ iff $kx \wedge ly = 0$ and $kx \leq (ly)'$, whenever kx and ly exist in E.

Proof. Let $x \leq y'$, $x \wedge y = 0$ and $2y$ exists in E. Then $x \oplus y = (x \vee y) \oplus (x \wedge y) = x \vee y \leq y'$ and hence there is $x \oplus 2y = (x \vee y) \oplus y = (x \oplus y) \vee 2y = x \vee y \vee 2y = x \vee 2y$, which gives that $x \leq (2y)'$ and $x \wedge 2y = 0$. By induction, if ly exists then $x \oplus ly = x \vee ly$ and hence $x \leq (ly)'$ and $x \wedge ly = 0$.

Now, $x \leq (ly)'$ iff $ly \leq x'$ and because $x \wedge ly = 0$, we obtain by the same argument as above that $ly \oplus kx = ly \vee kx$, hence $kx \leq (ly)'$ and $ly \wedge kx = 0$ whenever kx exists in E.

Conversely, $kx \wedge ly = 0$ implies that $x \wedge y = 0$ and $kx \leq (ly)'$ implies $x \leq kx \leq (ly)' \leq y'$. □

In next we will use the statement of Lemma 3.3 in the following form: For any $x, y \in E$ and $k, l \in \mathbb{N}$ with $x \wedge y = 0$, $x \not\leq y'$ iff $kx \not\leq (ly)'$, whenever kx and ly exist in E.

Definition 3.4. Let E be an atomic lattice effect algebra. E is said to be *almost orthogonal* if the set $\{b \in E \mid b \not\leq a', b \text{ is an atom}\}$ is finite for every atom $a \in E$.

Note that our definition of almost orthogonality coincides with the usual definition for orthomodular lattices (see e. g. [17, 18]).

Theorem 3.5. Let E be an Archimedean atomic lattice effect algebra. Then E is almost orthogonal if and only if for any atom $a \in E$ and any integer l, $1 \leq l \leq n_a$, there are finitely many atoms $c_1, \ldots, c_m$ and integers $j_1, \ldots, j_m$, $1 \leq j_1 \leq n_{c_1}, \ldots, 1 \leq j_m \leq n_{c_m}$ such that $j_k c_k \not\leq (la)'$ for all $k \in \{1, \ldots, m\}$ and, for all $x \in E$, $x \not\leq (la)'$ implies $j_{k_0} c_{k_0} \leq x$ for some $k_0 \in \{1, \ldots, m\}$.

Proof. $\Longrightarrow$: Assume that E is almost orthogonal. Let $a \in E$ be an atom, $1 \leq l \leq n_a$. We shall denote $A_a = \{b \in E \mid b \text{ is an atom}, b \not\leq a'\}$. Clearly, A_a is finite i.e. $A_a = \{b_1, \ldots, b_n\}$ for suitable atoms $b_1, \ldots, b_n$ from E.

Let $b \in E$ be an atom, $1 \leq k \leq n_b$ and $kb \not\leq (la)'$. Either $b = a$ or $b \neq a$ and in this case we have by Lemma 3.3 (iv) that $b \not\leq a'$. Hence either $b = a$ or $b \in A_a$. Let us put

$$\{c_1, \ldots, c_m\} = \begin{cases} A_a & \text{if } a \in S(E) \\ A_a \cup \{a\} & \text{otherwise.} \end{cases}$$

In both cases we have that $a \in \{c_1, \ldots, c_m\}$.

Now, let $x \in E$ and $x \not\leq (la)'$. By Theorem 1.2 there is an atom $c \in E$ and an integer $1 \leq j \leq n_c$ such that $jc \leq x$ and $jc \not\leq (la)'$. Either $c = a$ or $c \not\leq a$. In the first case we have that $j \geq (n_a - l + 1)$ i.e. $x \geq (n_a - l + 1)a$. In the second case we get that $c \not\leq a'$ i.e. $c \in A_a$ and $x \geq b_i$ for suitable $i \in \{1, \ldots, n\}$. Hence it is enough to put $j_k = 1$ if $c_k \in A_a$ and $j_k = (n_a - l + 1)$ if $c_k = a$.

$\Longleftarrow$: Conversely, let $a \in E$ be an atom. Then there are finitely many atoms $c_1, \ldots, c_m$ and integers $j_1, \ldots, j_m$, $1 \leq j_1 \leq n_{c_1}, \ldots, 1 \leq j_m \leq n_{c_m}$ such that $j_k c_k \not\leq a'$ for all $k \in \{1, \ldots, m\}$ and, for all $x \in E$, $x \not\leq a'$ implies $j_{k_0} c_{k_0} \leq x$ for some $k_0 \in \{1, \ldots, m\}$. Let us check that $A_a \subseteq \{c_1, \ldots, c_m\}$. Let $b \in A_a$. Then $b \geq j_{k_0} c_{k_0} \geq c_{k_0}$ for some $k_0 \in \{1, \ldots, m\}$. Hence $b = c_{k_0}$. This yields A_a is finite. $\square$

Lemma 3.6. Let E be an almost orthogonal Archimedean atomic lattice effect algebra. Then, for any atom $a \in E$ and any integer l, $1 \leq l \leq n_a$ there are finitely many atoms $b_1, \ldots, b_n$ and integers $j_1, \ldots, j_n$, $1 \leq j_1 \leq n_{b_1}, \ldots, 1 \leq j_n \leq n_{b_n}$ such that

$$E = [0, (la)'] \cup \left(\bigcup_{k=1}^{n} [j_k b_k, 1] \cup [(n_a + 1 - l)a, 1]\right)$$

and

$$[0, (la)'] \cap \left(\bigcup_{k=1}^{n} [j_k b_k, 1] \cup [(n_a + 1 - l)a, 1]\right) = \emptyset.$$

Hence $[0, (la)']$ is a clopen subset in the interval topology.

Proof. Let $a \in E$ be an atom, $1 \leq l \leq n_a$. By Definition 3.5, let $\{j_1 b_1, \ldots, j_n b_n\}$ be the finite set of non-orthogonal finite elements to la of the form $j_k b_k$, $1 \leq j_k \leq n_{b_k}$ minimal such that $b_1, \ldots, b_n$ are atoms different from a. We put $D = [0, (la)'] \cup (\bigcup_{k=1}^{n} [j_k b_k, 1] \cup [(n_a + 1 - l)a, 1])$. Let us check that $D = E$. Clearly, $D \subseteq E$. Now, let $z \in E$. Then by Theorem 1.2 there are mutually distinct atoms $c_\gamma \in E$, $\gamma \in \mathcal{E}$ and integers t_γ such that

$$z = \bigoplus \{t_\gamma c_\gamma \mid \gamma \in \mathcal{E}\} = \bigvee \{t_\gamma c_\gamma \mid \gamma \in \mathcal{E}\}.$$

Either $t_\gamma c_\gamma \leq (la)'$ for all $\gamma \in \mathcal{E}$ and hence $z \in [0, (la)']$ or there is $\gamma_0 \in \mathcal{E}$ such that $t_{\gamma_0} c_{\gamma_0} \not\leq (la)'$. Hence, by almost orthogonality, either $j_{k_0} b_{k_0} \leq t_{\gamma_0} c_{\gamma_0} \leq z$ for some $k_0 \in \{1, \ldots, n\}$ or $(n_a + 1 - l)a \leq t_{\gamma_0} c_{\gamma_0} \leq z$. In both cases we get that $z \in D$.

Now, assume that $y \in [0, (la)'] \cap (\bigcup_{k=1}^{n} [j_k b_k, 1] \cup [(n_a + 1 - l)a, 1])$. Then $(n_a + 1 - l)a \leq y \leq (la)'$ or $j_k b_k \leq y \leq (la)'$ for some $k \in \{1, \ldots, n\}$. In any case we have a contradiction. $\square$

Proposition 3.7. Let E be an almost orthogonal Archimedean atomic lattice effect algebra. Then, for any not necessarily different atoms $a, b \in E$ and any integers l, k; $1 \leq l \leq n_a$, $1 \leq k \leq n_b$, the interval $[kb, (la)']$ is clopen in the interval topology.

Proof. From Lemma 3.6 we have that $[0, (la)']$ is a clopen subset. Since a dual of an almost orthogonal Archimedean atomic lattice effect algebra is an almost orthogonal Archimedean atomic lattice effect algebra as well, we have that $[kb, 1]$ is again clopen in the interval topology. Hence also $[kb, (la)']$ is clopen in the interval topology. $\square$

Theorem 3.8. Let E be an almost orthogonal Archimedean atomic lattice effect algebra. Then the interval topology τ_i on E is Hausdorff.

Proof. Let $x, y \in E$ and $x \neq y$. Then (without loss of generality) we may assume that $x \not\leq y$. Then by [25, Theorem 3.3] there is an atom b from E and an integer k, $1 \leq k \leq n_b$ such that $kb \leq x$ and $kb \not\leq y$. Applying the dual of [25, Theorem 3.3] there is an atom a from E and an integer l, $1 \leq l \leq n_a$ such that $y \leq (la)'$ and $kb \not\leq (la)'$. Clearly, $x \in [kb, 1]$, $y \in [0, (la)']$.

Assume that there is an element $z \in E$ such that $z \in [kb, 1] \cap [0, (la)']$. Then $kb \leq z \leq (la)'$, a contradiction. Hence by Proposition 3.7, $[kb, 1]$ and $[0, (la)']$ are disjoint open subsets separating x and y. $\square$

Theorem 3.9. Let E be an almost orthogonal Archimedean atomic lattice effect algebra. Then E is compactly generated and therefore (o)-continuous.

Proof. It is enough to check that, for any atom $a \in E$ and any integer l, $1 \leq l \leq n_a$ the element la is compact in E since any element of E is a join of such elements (see Theorem 1.2 resp. [25, Theorem 3.3]).

Let $x = \bigvee_{\alpha \in \mathcal{E}} x_\alpha$ for some net $(x_\alpha)_{\alpha \in \mathcal{E}}$ in E, $la \leq x$, i. e., $(la)' \geq x' \downarrow x_\alpha'$.

By Lemma 3.6 we have $E = [0, (la)'] \cup (\bigcup_{k=1}^{n} [j_k b_k, 1] \cup [(n_a + 1 - l)a, 1])$, $[0, (la)'] \cap (\bigcup_{k=1}^{n} [b_k, 1] \cup [(n_a + 1 - l)a, 1]) = \emptyset$, $b_1, \ldots, b_n$ are atoms of E, $1 \leq j_k \leq n_{b_k}$, $1 \leq k \leq n$.

Since $\mathcal{E}$ is directed upwards, there is a cofinal subset $\mathcal{E}' \subseteq \mathcal{E}$ such that $x_\beta' \in [0, (la)']$ for all $\beta \in \mathcal{E}'$ or there is $k_0 \in \{1, 2, \ldots, n\}$ such that $x_\beta' \in [j_{k_0} b_{k_0}, 1]$ for all $\beta \in \mathcal{E}'$ or $x_\beta' \in [(n_a + 1 - l)a, 1]$ for all $\beta \in \mathcal{E}'$. If $x_\beta' \in [0, (la)']$ for all $\beta \in \mathcal{E}'$ then clearly $la \leq x_\beta$ for all $\beta \in \mathcal{E}'$. If there is $k_0 \in \{1, 2, \ldots, n\}$ such that $x_\beta' \in [j_{k_0} b_{k_0}, 1]$ for all $\beta \in \mathcal{E}'$ or $x_\beta' \in [(n_a + 1 - l)a, 1]$ for all $\beta \in \mathcal{E}'$ we obtain that $x' \in [j_{k_0} b_{k_0}, 1]$ or $x' \in [(n_a + 1 - l)a, 1]$ which is a contradiction with $x' \in [0, (la)']$. $\square$

Let E be an Archimedean atomic lattice effect algebra. We put $\mathcal{U} = \{x \in E \mid x = \bigvee_{i=1}^{n} l_i a_i, a_1, \ldots, a_n \text{ are atoms of } E, 1 \leq l_i \leq n_{a_i}, 1 \leq i \leq n, n \text{ natural number}\}$ and $\mathcal{V} = \{x \in E \mid x' \in \mathcal{U}\}$. Then by [25, Theorem 3.3], for every $x \in L$, we have that

$$x = \bigvee \{u \in \mathcal{U} \mid u \leq x\} = \bigwedge \{v \in \mathcal{V} \mid x \leq v\}.$$

Consider the function family $\Phi = \{f_u \mid u \in \mathcal{U}\} \cup \{g_v \mid v \in \mathcal{V}\}$, where $f_u, g_v : L \to \{0,1\}$, $u \in \mathcal{U}, v \in \mathcal{V}$ are defined by putting

$$f_u(x) = \begin{cases} 1 & \text{iff} \quad u \leq x \\ 0 & \text{iff} \quad u \not\leq x \end{cases} \quad \text{and} \quad g_v(y) = \begin{cases} 1 & \text{iff} \quad x \leq v \\ 0 & \text{iff} \quad x \not\leq v \end{cases}$$

for all $x, y \in L$.

Further, consider the family of pseudometrics on L: $\Sigma_\Phi = \{\rho_u \mid u \in \mathcal{U}\} \cup \{\pi_v \mid v \in \mathcal{V}\}$, where $\rho_u(a,b) = |f_u(a) - f_u(b)|$ and $\pi_v(a,b) = |g_v(a) - g_v(b)|$ for all $a, b \in L$.

Let us denote by $\mathcal{U}_\Phi$ the uniformity on L induced by the family of pseudometrics Σ_Φ (see e. g. [3]). Further denote by τ_Φ the topology compatible with the uniformity $\mathcal{U}_\Phi$.

Then for every net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of L

$$x_\alpha \xrightarrow{\tau_\Phi} x \text{ implies } \varphi(x_\alpha) \to \varphi(x) \text{ for any } \varphi \in \Phi.$$

This implies, since f_u, $u \in \mathcal{U}$, and g_v, $v \in \mathcal{V}$, is a separating function family on L, that the topology τ_Φ is Hausdorff. Moreover, the intervals $[u,v] = [u,1] \cap [0,v] = f_u^{-1}(\{1\}) \cap g_v^{-1}(\{1\})$ are clopen sets in τ_Φ.

Theorem 3.10. Let E be an almost orthogonal Archimedean atomic lattice effect algebra. Then $\tau_i = \tau_o = \tau_\Phi$.

P r o o f. Since by Theorem 3.8, τ_i is Hausdorff we obtain by [4] that $\tau_i = \tau_o$. Further if $O \in \tau_o$ and $x \in O$ then by Theorem 1.2 we have $x = \bigvee\{u \in \mathcal{U} \mid u \leq x\} = \bigwedge\{v \in \mathcal{V} \mid x \leq v\}$, which by [12] implies that there are finite sets $F \subseteq \mathcal{U}$, $G \subseteq \mathcal{V}$ such that $x \in [\bigvee F, \bigwedge G] \subseteq O$. Hence $\tau_o \subseteq \tau_\Phi$. To show the reverse inclusion it is enough to check that $x_\alpha \overset{(o)}{\to} x$ implies $\varphi(x_\alpha) \to \varphi(x)$ for any $\varphi \in \Phi$. This is equivalent by Lemma 2.4 (vii) that $x_\alpha \overset{\tau_o}{\to} x$ implies $\varphi(x_\alpha) \to \varphi(x)$ for any $\varphi \in \Phi$. Then, since τ_Φ is the coarsest topology with this property, we get $\tau_\Phi \subseteq \tau_o$.

Now, let us show that $x_\alpha \xrightarrow{(o)} x$ implies $\varphi(x_\alpha) \to \varphi(x)$ for any $\varphi \in \Phi$. Assume that $u_\alpha \leq x_\alpha \leq v_\alpha$ for all α such that $u_\alpha \uparrow x$ and $v_\alpha \downarrow x$. Let $u \in \mathcal{U}$. If $f_u(x) = 0$ we have that $u \not\leq x$. Therefore $u \not\leq u_\alpha$ for all α i. e. $f_u(u_\alpha) = 0$. Moreover there is an index α_0 such that $u \not\leq v_{\alpha_0}$ i. e. $f_u(v_\alpha) = 0$ for all $\alpha \geq \alpha_0$. If $f_u(x) = 1$ we have that $u \leq x$. By Theorem 3.9 we have that u is compact and hence there is an index α_0 such that $u \leq u_{\alpha_0}$. This immediately implies that for all $\alpha \geq \alpha_0$ we have $u \leq x_\alpha$ i. e. $f_u(u_\alpha) = 1$. Clearly, $u \leq v_\alpha$ for all α i. e. $f_u(v_\alpha) = 1$.

Hence in both cases we have that $f_u(x_\alpha)$ is eventually constant. Therefore $f_u(x_\alpha) \to f_u(x)$. The case $v \in \mathcal{V}$ can be proved dually. Hence we have, for all $u \in \mathcal{U}$ and for all $v \in \mathcal{V}$, $f_u(x_\alpha) \to f_u(x)$ and $g_v(x_\alpha) \to g_v(x)$. □

Theorem 3.11. Let E be an Archimedean atomic block-finite lattice effect algebra. Then $\tau_i = \tau_o$ is a Hausdorff topology.

P r o o f. As in [18], it suffices to show that for every $x, y \in E$, $x \not\leq y$ there are finitely many intervals, none of which contains both x and y and the union of which covers E.

By [15], E is a union of finitely many atomic blocks M_i, $i = 1, 2, \dots, n$. Choose $i \in \{1, 2, \dots, n\}$. If $x, y \in M_i$ then there is an atom $a_i \in M_i$ and an integer l_i, $1 \le l_i \le n_{a_i}$ such that $l_i a_i \le x$, $l_i a_i \not\le y$. Let us put $k_i = n - l_i + 1$. Since M_i is almost orthogonal (the only possible non-orthogonal kb to la for an atom a, $1 \le l \le n_a$ is that $a = b$) we have by Lemma 3.6 that $M_i = ([0, (k_i a_i)'] \cap M_i) \cup ([(n_{a_i} + 1 - k_i)a_i, 1] \cap M_i)$. Hence $M_i \subseteq [0, (k_i a_i)'] \cup [(n_{a_i} + 1 - k_i)a_i, 1]$. Let us check that $[0, (k_i a_i)'] \cap [(n_{a_i} + 1 - k_i)a_i, 1] = \emptyset$. Assume that $(n_{a_i} + 1 - k_i)a_i \le z \le (k_i a_i)'$. Then $(n_{a_i} + 1 - k_i)a_i \le (k_i a_i)'$, a contradiction. Put $J_i = [0, (k_i a_i)']$, $K_i = [(n_{a_i} + 1 - k_i)a_i, 1]$. This yields $x \in K_i$, $y \in J_i$, $M_i \subseteq J_i \cup K_i$ and $J_i \cap K_i = \emptyset$. Let $x \notin M_i$. Then there is an atom $a_i \in M_i$ that is not compatible with x. Let us check that $x \notin [0, (a_i)'] \cup [n_{a_i} a_i, 1]$. Assume that $x \in [0, (a_i)']$ or $x \in [n_{a_i} a_i, 1]$. Then $x \le (a_i)'$ or $a_i \le n_{a_i} a_i \le x$, i. e., in both cases we get that $x \leftrightarrow a_i$, a contradiction. Let us put $J_i = [0, (a_i)']$, $K_i = [n_{a_i} a_i, 1]$. As above, $M_i \subseteq J_i \cup K_i$, $J_i \cap K_i = \emptyset$ and moreover $x \notin J_i \cup K_i$. The remaining case $y \notin M_i$ can be checked by similar considerations. We obtain $E = \bigcup_{i=1}^n M_i \subseteq \bigcup_{i=1}^n (J_i \cup K_i) \subseteq E$ and none of the intervals J_i, K_i, $i = 1, 2, \dots, n$ contains both x and y. $\square$

4. ORDER AND INTERVAL TOPOLOGIES OF COMPLETE ATOMIC BLOCK-FINITE LATTICE EFFECT ALGEBRAS

We are going to show that on every complete atomic block-finite lattice effect algebra E the interval topology is Hausdorff. Hence both topologies τ_i and τ_o are in this case compact Hausdorff and they coincide. Moreover, a necessary and sufficient condition for a complete atomic lattice algebra E to be almost orthogonal is given.

For the proof of Theorems 4.2 and 4.3 we will use the following statement, firstly proved in the equivalent setting of D-posets in [19].

Theorem 4.1. (Riečanová [19, Theorem 1.7]) Suppose that $(E; \oplus, 0, 1)$ is a complete lattice effect algebra. Let $\emptyset \neq D \subseteq E$ be a sub-lattice effect algebra. The following conditions are equivalent:

(i) For all nets $(x_\alpha)_{\alpha \in \mathcal{E}}$ such that $x_\alpha \in D$ for all $\alpha \in \mathcal{E}$

$$x_\alpha \xrightarrow{(o)} x \text{ in } E \text{ if and only if } x \in D \text{ and } x_\alpha \xrightarrow{(o)} x \text{ in } D.$$

(ii) For every $M \subseteq D$ with $\bigvee M = x$ in E it holds $x \in D$.

(iii) For every $Q \subseteq D$ with $\bigwedge Q = y$ in E it holds $y \in D$.

(iv) D is a complete sub-lattice of E.

(v) D is a closed set in order topology τ_o on E.

Each of these conditions implies that $\tau_o^D = \tau_o^E \cap D$, where τ_o^D is an order topology on D.

Important sub-lattice effect algebras are blocks, $S(E)$, $B(E) = \bigcap\{M \subseteq E \mid M \text{ block of } E\}$ and $C(E) = B(E) \cap S(E)$ (see [6, 7, 13, 21, 23]).

Theorem 4.2. Let E be a complete lattice effect algebra. Then for every $D \in \{S(E), C(E), B(E)\}$ or $D = M$, where M is a block of E, we have:

(1) $x_\alpha \xrightarrow{\tau_i^E} x \iff x_\alpha \xrightarrow{\tau_i^D} x$, for all nets $(x_\alpha)_{\alpha\in\mathcal{E}}$ in D and all $x \in D$.

(2) If τ_i^E is Hausdorff then

$$x_\alpha \xrightarrow{\tau_i^E} x \iff x_\alpha \xrightarrow{\tau_i^D} x, \quad \text{for all nets } (x_\alpha)_{\alpha\in\mathcal{E}} \text{ in } D \text{ and all } x \in E.$$

Proof. The first part of the statement follows by Theorem 2.5 and the fact that if E is a complete lattice effect algebra then M, $S(E)$, $C(E)$ and $B(E)$ are complete sub-lattices of E (see [9, 24]). The second part follows by [4] since τ_i is Hausdorff implies $\tau_i = \tau_o$ and by Theorem 4.1. □

Theorem 4.3. (i) The interval topology τ_i on every Archimedean atomic MV-effect algebra M is Hausdorff and $\tau_i = \tau_o = \tau_\Phi$.
(ii) For every complete atomic MV-effect algebra M and for any net (x_α) of M and any $x \in M$,

$$x_\alpha \xrightarrow{\tau_o} x \text{ if and only if } x_\alpha \xrightarrow{(o)} x \quad (\text{briefly } \tau_o \equiv (o)).$$

Moreover, τ_o is a uniform compact Hausdorff topology on M.
(iii) For every atomic block-finite lattice effect algebra E, E is a complete lattice iff $\tau_i = \tau_o$ is a compact Hausdorff topology.

Proof. (i), (ii): This follows from the fact that every pair of elements of M is compatible, hence every pair of atoms is orthogonal. Thus for (i) we can apply Theorem 3.10 and for (ii) we can use (i) and [16, Theorem 2] since M is compactly generated by finite elements and τ_i is compact.
(iii) From Theorem 3.11 we know that $\tau_i = \tau_o$ is a Hausdorff topology. By Lemma 2.4 (vi) the interval topology τ_i on E is compact iff E is a complete lattice. □

In what follows we will need Corollary 4.5 of Lemma 4.4.

Lemma 4.4. Let E be an Archimedean atomic lattice effect algebra. Then

(i) If $c, d \in E$ are compact elements with $c \leq d'$ then $c \oplus d$ is compact.

(ii) If $u = \bigoplus G$, where G is a $\oplus$-orthogonal system of atoms of E, and u is compact then G is finite.

Proof. (i) Let $c \oplus d \leq \bigvee D$. Let $\mathcal{E} = \{F \subseteq D : F \text{ is finite}\}$ be directed by set inclusion and let for every $F \in \mathcal{E}$ be $x_F = \bigvee F$. Then $x_F \uparrow x = \bigvee D$. Since $c \leq \bigvee D$ and $d \leq \bigvee D$ there is a finite subset $F_1 \subseteq D$ such that $c \vee d \leq \bigvee F_1$. Therefore, for $F \supseteq F_1$, $x_F \ominus c \uparrow x \ominus c, d \leq x \ominus c$. Then there is a finite subset $F_2 \subseteq D$, $F_1 \subseteq F_2$ such that $d \leq x_{F_2} \ominus c$. Hence $c \oplus d \leq x_{F_2}$.

(ii) Let $u \in E$, $u = \bigoplus G = \bigvee\{\bigoplus K \mid K \subseteq G \text{ is finite}\}$ where $G = (a_\kappa)_{\kappa \in H}$ is a $\oplus$-orthogonal system of atoms. Clearly if $K_1, K_2 \subseteq G$ are finite such that $K_1 \subseteq K_2$ then $\bigoplus K_1 \leq \bigoplus K_2$.

Assume that u is compact. Hence there are finite $K_1, K_2, \ldots, K_n \subseteq G$ such that $u \leq \bigvee\{\bigoplus K_i \mid i = 1, 2, \ldots, n\}$. Let $K_0 = \bigcup\{K_i \mid i = 1, 2, \ldots, n\}$. Then $K_0 \subseteq G$, K_0 is finite and $\bigoplus K_i \leq \bigoplus K_0$, $i = 1, 2 \ldots, n$, which gives that $\bigvee\{\bigoplus K_i \mid i = 1, 2, \ldots, n\} \leq \bigoplus K_0$. It follows that $u \leq \bigoplus K_0 \leq u = \bigvee\{\bigoplus K \mid K \subseteq G \text{ is finite}\}$. Hence $u = \bigoplus K_0$, $K_0 \subseteq G$ is finite. Further, for every finite $K \subseteq G \setminus K_0$ we have $\bigoplus K_0 \subseteq \bigoplus(K_0 \cup K) = \bigoplus K_0 \oplus \bigoplus K \leq u = \bigoplus K_0$, which gives that $\bigoplus K = 0$. Hence $K = \emptyset$ and thus $G \setminus K_0 = \emptyset$ which gives that $K_0 = G$. □

Corollary 4.5. Let E be an (o)-continuous Archimedean atomic lattice effect algebra. Then every finite element of E is compact.

Proof. Clearly, by [16, Theorem 7] we know that E is compactly generated. Therefore, any atom of E is compact. The compactness of every finite element follows by an easy induction. □

Theorem 4.6. Let E be an Archimedean atomic lattice effect algebra. Then the following conditions are equivalent:

(i) $\tau_i = \tau_o = \tau_\Phi$.

(ii) E is (o)-continuous and τ_i is Hausdorff.

(iii) E is almost orthogonal.

Proof. (i) $\Longrightarrow$ (ii): Since $\tau_o = \tau_\Phi$ we have by [16, Theorem 1] that E is compactly generated and hence (o)-continuous. The condition $\tau_i = \tau_\Phi$ implies that τ_i is Hausdorff because τ_Φ is Hausdorff.
(ii) $\Longrightarrow$ (i), (iii): Since τ_i is Hausdorff we obtain $\tau_i = \tau_o$ by [4]. Moreover, from [16, Theorem 7] and Corollary 4.5 the (o)-continuity of E implies that E is compactly generated by the elements from $\mathcal{U}$. This gives $\tau_o = \tau_\Phi$ from [16, Theorem 1].

Let $a \in E$ be an atom, $1 \leq l \leq n_a$. Then the interval $[0, (la)']$ is a clopen set in the order topology $\tau_o = \tau_\Phi = \tau_i$. Hence there is a finite set of intervals in E such that $0 \in E \setminus \bigcup_{i=1}^{n}[u_i, v_i] \subseteq [0, (la)']$. Thus $E \subseteq [0, (la)'] \cup \bigcup_{i=1}^{n}[u_i, v_i] \subseteq [0, (la)'] \cup \bigcup_{i=1}^{n}[k_i b_i, 1]$, where $b_i \in E$ are atoms such that $k_i b_i \leq u_i$, $1 \leq k_i \leq n_{b_i}$, $i = 1, \ldots, n$. This yields that E is almost orthogonal.
(iii) $\Longrightarrow$ (ii): From Theorems 3.8 and 3.9 we have that τ_i is Hausdorff and E is compactly generated, hence (o)-continuous. □

Corollary 4.7. Let E be a complete atomic lattice effect algebra. Then the following conditions are equivalent:

(i) E is almost orthogonal.

(ii) $\tau_i = \tau_o = \tau_\Phi \equiv (o)$.

(iii) E is (o)-continuous and τ_i is Hausdorff.

Proof. It follows from Theorems 4.6 and the fact that by (o)-continuity of E [27, Theorem 8] we have $\tau_o \equiv (o)$. □

The next example shows that a complete block-finite atomic lattice effect algebra need not be (o)-continuous and almost orthogonal in spite of that $\tau_i = \tau_o$ is a compact Hausdorff topology.

Example 4.8. Let E be a horizontal sum of finitely many infinite complete atomic Boolean algebras $(B_i, \bigoplus_i, 0_i, 1_i)$, $i = 1, 2, \ldots, n$. Then E is an atomic complete lattice effect algebra, E is not almost orthogonal, E is not compactly generated by finite elements (hence $\tau_o \neq \tau_\Phi$), E is block-finite, $\tau_i = \tau_o$ is Hausdorff by Theorem 3.11, and the interval topology τ_i on E is compact.

5. APPLICATIONS

Theorem 5.1. Let E be a block-finite complete atomic lattice effect algebra. Then the following conditions are equivalent:

(i) E is almost orthogonal.

(ii) E is compactly generated.

(iii) E is (o)-continuous.

(iv) $\tau_i = \tau_o = \tau_\Phi \equiv (o)$.

Proof. By Theorem 3.11, $\tau_i = \tau_o$ is a Hausdorff topology. This by [16, Theorem 7] gives that (ii) $\iff$ (iii) and by Corollary 4.7 we obtain that (i) $\iff$ (iii) $\iff$ (iv). □

In Theorem 5.1, the assumption that E is atomic can not be omitted. For instance, every non-atomic complete Boolean algebra is (o)-continuous but it is not compactly generated, because in such a case E must be atomic by [16, Theorem 6].

Remark 5.2. If a $\oplus$-operation on a lattice effect algebra E is continuous with respect to its interval topology τ_i meaning that $x_\alpha \leq y_\alpha^{'}$, $x_\alpha \xrightarrow{\tau_i} x$, $y_\alpha \xrightarrow{\tau_i} y$, $\alpha \in \mathcal{E}$ implies $x_\alpha \oplus y_\alpha \xrightarrow{\tau_i} x \oplus y$, then τ_i is Hausdorff (see [14]). Hence $\oplus$-operation on complete (o)-continuous atomic lattice effect algebras which are not almost orthogonal cannot be τ_i-continuous, by [14] and Corollary 4.7.

Theorem 5.3. Let E be a block-finite complete atomic lattice effect algebra. Let $(x_\alpha)_{\alpha\in\mathcal{E}}$ and $(y_\alpha)_{\alpha\in\mathcal{E}}$ be nets of elements of E such that $x_\alpha \leq y_\alpha^{'}$ for all $\alpha \in \mathcal{E}$.

If $x_\alpha \xrightarrow{\tau_i} x$, $y_\alpha \xrightarrow{\tau_i} y$, $\alpha \in \mathcal{E}$ then $x \leq y^{'}$ and $x_\alpha \oplus y_\alpha \xrightarrow{\tau_i} x \oplus y$, $\alpha \in \mathcal{E}$. Moreover, $\tau_i = \tau_o$.

P r o o f. Since, by Theorem 3.11, τ_i is Hausdorff, we obtain that $\tau_i = \tau_o$ by [4]. Let $\{M_1, \dots, M_n\}$ be the set of all blocks of E. Further, for every $\alpha \in \mathcal{E}$, elements of the set $\{x_\alpha, y_\alpha, x_\alpha \oplus y_\alpha\}$ are pairwise compatible. It follows that for every $\alpha \in \mathcal{E}$ there is a block M_{k_α} of E, $k_\alpha \in \{1, \dots, n\}$ such that $\{x_\alpha, y_\alpha, x_\alpha \oplus y_\alpha\} \subseteq M_{k_\alpha}$. Let $\mathcal{E}'$ be any cofinal subset of $\mathcal{E}$. Since $\mathcal{E}'$ is directed upwards, there is a block M_{k_0} of E and a cofinal subset $\mathcal{E}''$ of $\mathcal{E}'$ such that $\{x_\beta, y_\alpha, x_\beta \oplus y_\beta\} \subseteq M_{k_0}$ for all $\beta \in \mathcal{E}''$. Otherwise we obtain a contradiction with the finiteness of the set $\{M_1, \dots, M_n\}$. Further, by Theorem 2.5, we obtain that $\tau_i^{M_{k_0}} = \tau_i \cap M_{k_0}$, as M_{k_0} is a complete sublattice of E (see Theorem 4.2). It follows that the interval topology $\tau_i^{M_{k_0}}$ on the complete MV-effect algebra M_{k_0} is Hausdorff. The last by [14, Theorem 3.6] gives that $x_\beta \oplus y_\beta \xrightarrow{\tau_i^{M_{k_0}}} x \oplus y$, $\beta \in \mathcal{E}''$ and hence $x_\beta \oplus y_\beta \xrightarrow{\tau_i} x \oplus y$, $\beta \in \mathcal{E}''$, as $\tau_i^{M_{k_0}} = \tau_i \cap M_{k_0}$. It follows that $x_\alpha \oplus y_\alpha \xrightarrow{\tau_i} x \oplus y$, $\alpha \in \mathcal{E}$ by Lemma 2.3. □

In [22, Theorem 4.5] it was proved that a block-finite lattice effect algebra $(E; \oplus, 0, 1)$ has a MacNeille completion which is a complete effect algebra $(MC(E); \oplus, 0, 1)$ containing E as a (join-dense and meet-dense) sub-lattice effect algebra iff E is Archimedean. In what follows we put $\widehat{E} = MC(E)$.

Corollary 5.4. Let E be a block-finite Archimedean atomic lattice effect algebra. Then for any nets $(x_\alpha)_{\alpha \in \mathcal{E}}$ and $(y_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E with $x_\alpha \le y_\alpha'$, $\alpha \in \mathcal{E}$: $x_\alpha \xrightarrow{\tau_i} x$, $y_\alpha \xrightarrow{\tau_i} y$, $\alpha \in \mathcal{E}$ implies $x_\alpha \oplus y_\alpha \xrightarrow{\tau_i} x \oplus y$, $\alpha \in \mathcal{E}$.

P r o o f. By [20, Lemma 1.1], for interval topologies $\widehat{\tau_i}$ on $\widehat{E}$ and τ_i on E, we have $\widehat{\tau_i} \cap E = \tau_i$. Thus for $x_\alpha, y_\alpha, x, y \in E$ we obtain $x_\alpha \oplus y_\alpha \xrightarrow{\widehat{\tau_i}} x \oplus y$, $\alpha \in \mathcal{E}$ which gives $x_\alpha \oplus y_\alpha \xrightarrow{\tau_i} x \oplus y$, $\alpha \in \mathcal{E}$ by the fact that $\widehat{\tau_i} \cap E = \tau_i$. □

Definition 5.5. Let E be a lattice.Then

(i) An element u of E is called *strongly compact* (briefly *s-compact*) iff, for any $D \subseteq E$: $u \le c \in E$ for all $c \ge \bigvee D$ implies $u \le \bigvee F$ for some finite $F \subseteq D$.

(ii) E is called *s-compactly generated* iff every element of E is a join of s-compact elements.

Theorem 5.6. Let E be a block-finite Archimedean atomic lattice effect algebra. Then the following conditions are equivalent:

(i) E is almost orthogonal.

(ii) $\widehat{E} = MC(E)$ is almost orthogonal.

(iii) $\widehat{E} = MC(E)$ is compactly generated.

(iv) E is s-compactly generated.

P r o o f . By J. Schmidt [30] a MacNeille completion $\widehat{E}$ of E is (up to isomorphism) a complete lattice such that for every element $x \in \widehat{E}$ there is $P, Q \subseteq E$ such that $x = \bigvee_{\widehat{E}} P = \bigwedge_{\widehat{E}} Q$ (taken in $_{\widehat{E}}$). Here we identify E with $\varphi(E)$, where $\varphi : E \to \widehat{E}$ is the embedding (meaning that E and $\varphi(E)$ are isomorphic lattice effect algebras). It follows that E and $\widehat{E}$ have the same set of all atoms and coatoms and hence also the same set of all finite and cofinite elements, which implies that (i) $\Longleftrightarrow$ (ii).

Moreover, for any $A \subseteq E$ and $u \in E$, we have ($d \in E$, $A \leq d$ implies $u \leq d$) iff $u \leq \bigvee_{\widehat{E}} A$. Then u is s-compact in E iff u is compact in $\widehat{E}$, which gives (iii) $\Longleftrightarrow$ (iv).

Finally (ii) $\Longleftrightarrow$ (iii) by Theorem 5.1. □

Definition 5.7. Let E be an effect algebra. A map $\omega : E \to [0,1]$ is called a *state* on E if $\omega(0) = 0$, $\omega(1) = 1$ and $\omega(x \oplus y) = \omega(x) + \omega(y)$ whenever $x \oplus y$ exists in E.

Theorem 5.8. (State smearing theorem for almost orthogonal block-finite Archimedean atomic lattice effect algebras) Let $(E; \oplus, 0, 1)$ be a block-finite Archimedean atomic lattice effect algebra. If E is almost orthogonal then:

(i) $E_1 = \{x \in E \mid x \text{ or } x' \text{ is finite}\}$ is a sub-lattice effect algebra of E.

(ii) If there is an (o)-continuous state ω on E_1 (or on $S(E_1) = S(E) \cap E_1$, or on $S(E)$) then there is an (o)-continuous state $\widetilde{\omega}$ on E extending ω and an (o)-continuous state $\widehat{\omega}$ on $\widehat{E} = MC(E) = MC(E_1)$ extending $\widetilde{\omega}$.

P r o o f . (i) By Theorem 5.6, E is s-compactly generated and thus by [28, Theorem 2.7] E_1 is a sub-lattice effect algebra of E.
(ii) Since E is s-compactly generated, we obtain the existence of (o)-continuous extensions $\widetilde{\omega}$ on E and $\widehat{\omega}$ on $\widehat{E}$ by [28, Theorem 4.2]. □

ACKNOWLEDGEMENT

Financial Support by the Ministry of Education of the Czech Republic under the project MSM0021622409 and by the Grant Agency of the Czech Republic under the grant No. 201/06/0664 is gratefully acknowledged by the first author. The second author was supported by the Slovak Resaerch and Development Agency under the contract No. APVV–0071–06 and the grant VEGA-1/3025/06 of MŠ SR. We also thank the anonymous referees for the very thorough reading and contributions to improve our presentation of the paper.

(Received May 25, 2010)

REFERENCES

[1] E. G. Beltrametti and G. Cassinelli: The Logic of Quantum Mechanics. Addison-Wesley, Reading 1981.

[2] C. C. Chang: Algebraic analysis of many-valued logics. Trans. Amer. Math. Soc. *88* (1958) 467–490.

[3] A. Császár: General Topology. Akadémiai Kiadó, Budapest 1978.

[4] M. Erné and S. Weck: Order convergence in lattices. Rocky Mt. J. Math. *10* (1980), 805–818.

[5] O. Frink: Topology in lattices. Trans. Amer. Math. Soc. *51* (1942), 569–582.

[6] R. J. Greechie, D. J. Foulis, and S. Pulmannová: The center of an effect algebra. Order *12* (1995), 91–106.

[7] S. P. Gudder: Sharply dominating effect algebras. Tatra Mt. Math. Publ. *15* (1998), 23–30.

[8] G. Jenča and Z. Riečanová: On sharp elements in lattice ordered effect algebras. BUSEFAL *80* (1999), 24–29.

[9] G. Jenča and Z. Riečanová: A survey on sharp elements in unsharp Qquantum logics. J. Electr. Engrg. *52* (2001), 7–8, 237-239.

[10] G. Kalmbach: Orthomodular Lattices. Kluwer Academic Publ. Dordrecht 1998.

[11] M. Katětov: Remarks on Boolean algebras. Colloq. Math. *11* (1951), 229–235.

[12] H. Kirchheimová and Riečanová: Note on order convergence and order topology. In: B. Riečan and T. Neubrunn: Measure, Integral and Order, Appendix B, Ister Science (Bratislava) and Kluwer Academic Publishers, Dordrecht – Boston – London 1997.

[13] F. Kôpka: Compatibility in D-posets. Interernat. J. Theor. Phys. *34* (1995), 1525–1531.

[14] Lei Qiang, Wu Junde, and Li Ronglu: Interval topology of lattice effect algebras. Appl. Math. Lett. *22* (2009), 1003–1006.

[15] K. Mosná: Atomic lattice effect algebras and their sub-lattice effect algebras. J. Electr. Engrg. *58* (2007), 7/S, 3–6.

[16] J. Paseka and Z. Riečanová: Compactly generated de Morgan lattices, basic algebras and effect algebras. Internat. J. Theor. Phys. *49* (2010), 3216–3223.

[17] S. Pulmannová and Z. Riečanová: Compact topological orthomodular lattices. In: Contributions to General Algebra 7, Verlag Hölder – Pichler – Tempsky, Wien, Verlag B.G. Teubner, Stuttgart 1991, pp. 277–282.

[18] S. Pulmannová and Z. Riečanová: Blok finite atomic orthomodular lattices. J. Pure and Applied Algebra *89* (1993), 295–304.

[19] Z. Riečanová: On Order Continuity of Quantum Structures and Their Homomorphisms. Demonstratio Mathematica *29* (1996), 433–443.

[20] Z. Riečanová: Lattices and Quantum Logics with Separated Intervals Atomicity, Internat. J. Theor. Phys. *37* (1998), 191–197.

[21] Z. Riečanová: Compatibility and central elements in effect algebras. Tatra Mt. Math. Publ. *16* (1999), 151–158.

[22] Z. Riečanová: Archimedean and block-finite lattice effect algebras. Demonstratio Mathematica *33* (2000), 443–452.

[23] Z. Riečanová: Generalization of blocks for D-lattices and lattice-ordered effect algebras. Internat. J. of Theor. Phys. *39* (2000), 231–237.

[24] Z. Riečanová: Orthogonal Sets in Effect Algebras. Demonstratio Mathematica *34* (2001), 525–532.

[25] Z. Riečanová: Smearings of states defined on sharp elements onto effect algebras. Internat. J. of Theor. Phys. *41* (2002), 1511–1524.

[26] Z. Riečanová: Continuous Lattice Effect Algebras Admitting Order-Continuous States. Fuzzy Sests and Systems *136* (2003), 41–54.

[27] Z. Riečanová: Order-topological lattice effect algebras. In: Contributions to General Algebra 15, Proc. Klagenfurt Workshop 2003 on General Algebra, Klagenfurt 2003, pp. 151–160.

[28] Z. Riečanová and J. Paseka: State smearing theorems and the existence of states on some atomic lattice effect algebras. J. Logic and Computation, Advance Access, published on March 13, 2009, doi:10.1093/logcom/exp018.

[29] T.A. Sarymsakov, S.A. Ajupov, Z. Chadzhijev and V.J. Chilin: Ordered algebras. FAN, Tashkent, (in Russian), 1983.

[30] J. Schmidt: Zur Kennzeichnung der Dedekind-Mac Neilleschen Hülle einer Geordneten Menge. Archiv d. Math. *7* (1956), 241–249.

Jan Paseka, Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno. Czech Republic.
e-mail: paseka@math.muni.cz

Zdenka Riečanová, Department of Mathematics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Ilkovičova 3, 812 19 Bratislava. Slovak Republic.
e-mail: zdenka.riecanova@stuba.sk

Junde Wu, Department of Mathematics, Zhejiang University, Hangzhou 310027. People's Republic of China.
e-mail: wjd@zju.edu.cn