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Mathematica Bohemica, Vol. 136 (2011), No. 1, 105–112

Persistent URL: http://dml.cz/dmlcz/141454

136 (2011) MATHEMATICA BOHEMICA No. 1, 105–112

SOME CHARACTERIZATIONS OF ORDER WEAKLY COMPACT OPERATOR

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(Received December 21, 2009)

Abstract. We introduce the notion of order weakly sequentially continuous lattice operations of a Banach lattice, use it to generalize a result regarding the characterization of order weakly compact operators, and establish its converse. Also, we derive some interesting consequences.

Keywords: order weakly compact operator, order continuous norm, discrete Banach lattice, weakly sequentially continuous lattice operations

MSC 2010: 46A40, 46B40, 46B42

1. Introduction and notation

Following Dodds [3], an operator T from a Banach lattice E into a Banach space X is called order weakly compact whenever $T[-x,x]$ is relatively weakly compact for every $x \in E^+$ (see Definition 3.4.1 of [4]). Thus, according to Eberlein-Šmulian Theorem [2, Theorem 3.40], the operator T is order weakly compact if and only if for every order bounded sequence (x_n) of E, $(T(x_n))$ has a weakly convergent subsequence in X.

In [4], some characterizations of order weakly compact operators are established. More precisely, it is proved that an operator T from an AM-space E into a Banach space X is order weakly compact if and only if $(T(x_n))$ is norm convergent for each order bounded $\sigma(E,E')$-Cauchy sequence (x_n) of E (see Corollary 3.4.10 in [4]). After that, an example proving that the last result is not true in arbitrary Banach lattices is given. Nonetheless, when we read the proof of Corollary 3.4.10 in [4], we observe that this result is still true when we assume only that the lattices operations of E are weakly sequentially continuous.

Our objective in this paper is to generalize this result of [4] and study its converse. To do this, we will need to introduce a new notion that we call "order weakly sequentially continuous lattice operations", which, we believe, is weaker than the well known notion "weakly sequentially continuous lattice operations", and we will show that if we replace in Corollary 3.4.10 of [4] the hypothesis that "E is an AM-space" by the weaker hypothesis "the lattice operations of E are order weakly sequentially continuous" we obtain the same characterization for order weakly compact operators. Next, we will establish the converse by proving that if for each order weakly compact operator T from a Banach lattice E into a Banach space X, the sequence $(T(x_n))$ is norm convergent whenever (x_n) is an order bounded $\sigma(E, E')$-Cauchy sequence, then E has order weakly sequentially continuous lattice operations or c_0 is not embbeding in X. As consequences, we will obtain a characterization of the discreteness of a Banach lattice with an order continuous norm and a characterization of a Banach lattice with order weakly sequentially continuous lattice operations.

To state our results, we need to fix some notation and recall some definitions. A vector lattice E is an ordered vector space in which $\sup(x, y)$ exists for every $x, y \in E$. A subspace F of a vector lattice E is called a sublattice if for every pair of elements a, b of F the supremum of a and b taken in E belongs to F. A subset B of a vector lattice E is said to be solid if it follows from $|y| \leqslant |x|$ with $x \in B$ and $y \in E$ that $y \in B$. An order ideal of E is a solid subspace. Let E be a vector lattice, then for each $x, y \in E$ with $x \leqslant y$, the set $[x, y] = \{z \in E\colon\ x \leqslant z \leqslant y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice E is discrete if it admits a complete disjoint system of discrete elements. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leqslant |y|$, we have $\|x\| \leqslant \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Banach lattice E is called an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$ we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. The Banach lattice E is an AL-space if its topological dual E' is an AM-space. We refer to [2] for unexplained terminology on Banach lattice theory.

2. Main results

We will use the term operator $T\colon E \longrightarrow F$ between two Banach lattices meaning a bounded linear mapping. It is positive if $T(x) \geqslant 0$ in F whenever $x \geqslant 0$ in E. The operator T is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. It is well known that each positive linear mapping on a Banach lattice is continuous. For more information on positive operators, we refer the reader to [2].

To establish our first major result, we will need the following lemma:

Lemma 2.1. *A sequence (x_n) of a topological vector space (E,τ) is τ-Cauchy if and only if $(x_{k_n} - x_n)$ is τ-convergent to zero for every subsequence (x_{k_n}) of (x_n).*

Proof. Assume first that (x_n) is τ-Cauchy, let (x_{k_n}) be a subsequence of (x_n) and let V be a τ-neighborhood of zero. By definition, there exists n_0 such that for all $p, q > n_0$ we have $x_p - x_q \in V$. In particular, for $n > n_0$ we obtain $x_n - x_{k_n} \in V$. Hence, the sequence $(x_{k_n} - x_n)$ τ-converges to zero.

Conversely, assume by way of contradiction that (x_n) is not τ-Cauchy. To complete the proof, we have to construct a subsequence (x_{k_n}) of (x_n) such that $(x_{k_n} - x_n)$ does not τ-converge to zero. Since (x_n) is not τ-Cauchy, there exists a τ-neighborhood V of zero such that for each $n \in \mathbb{N}$ there exist $p, q > n$ satisfying $x_p - x_q \notin V$. $(*)$

Fix a circled τ-neighborhood W of zero with $W + W \subseteq V$. We have to construct a subsequence (x_{k_n}) of (x_n) such that $x_{k_n} - x_n \notin W$ for each n.

▷ For $n = 1$ there exist $p, q > 1$ satisfying $x_p - x_q \notin V$. This implies that $x_p - x_1 \notin W$ or $x_q - x_1 \notin W$ (if not, then $x_p - x_q = (x_p - x_1) - (x_q - x_1) \in W + W \subseteq V$ and this is impossible). Hence there exists $k_1 > 1$ such $x_{k_1} - x_1 \notin W$.

▷ Assume that $x_{k_1}, \ldots, x_{k_n}$ are constructed such that $x_{k_i} - x_i \notin W$ for each $1 \leqslant i \leqslant n$. It follows from $(*)$ that there exist $p, q > k_n$ such that $x_p - x_q \notin V$. This implies that $x_p - x_{n+1} \notin W$ or $x_q - x_{n+1} \notin W$. Hence, there exists $k_{n+1} > k_n$ such $x_{k_{n+1}} - x_{n+1} \notin W$. This completes the construction of the subsequence (x_{k_n}) of (x_n) such that $x_{k_n} - x_n \notin W$ for each n. Since W is a τ-neighborhood of zero, $(x_{k_n} - x_n)$ does not τ-converge to zero. This completes the proof. □

Let us recall that a Banach space E is said to have the Schur property if every sequence weakly convergent to 0 in E is norm convergent to zero. As an example, the Banach space l^1 has the Schur property.

Recall that a Banach lattice E is said to have weakly sequentially continuous lattice operations whenever $x_n \to 0$ for $\sigma(E, E')$ implies $|x_n| \to 0$ for $\sigma(E, E')$. Note that every AM-space has this property (see Theorem 4.31 of [2]). Also, any Banach lattice with the Schur property has weakly sequentially continuous lattice operations. Thus, Banach lattices $C[0,1]$, l^1 and $l^1 \oplus C[0,1]$ have weakly sequentially continuous lattice operations.

Now, we introduce a new notion that we call "order weakly sequentially continuous lattice operations".

Definition. A Banach lattice E is said to have order weakly sequentially continuous lattice operations if for every order bounded sequence (x_n) of E satisfying $x_n \to 0$ for $\sigma(E, E')$ we have $|x_n| \to 0$ for $\sigma(E, E')$.

It is clear that each Banach lattice with weakly sequentially continuous lattice operations has order weakly sequentially continuous lattice operations. We believe that the converse is false in general. However, right now we do not have an example.

Nonetheless, there exists a Banach lattice which does not have order weakly sequentially continuous lattice operations. In fact, $L^2[0,1]$, which does not have weakly sequentially continuous lattice operations, does not have order weakly sequentially continuous lattice operations, either. For instance, let (r_n) be the sequence of Rademacher functions in $L^2[0,1]$. This sequence is order bounded and weakly convergent to 0. However, $|r_n| = 1$ for all n, where 1 is the constant function on $[0,1]$ with value 1.

Now, we are in position to generalize Corollary 3.4.10 of [4].

Theorem 2.2. *Let E be a Banach lattice with order weakly sequentially continuous lattice operations (in particular, with weakly sequentially continuous lattice operations), and let X be a Banach space. Then for every operator $T\colon E \to X$, the following assertions are equivalent:*

i) *T is order weakly compact.*

ii) *For each order bounded $\sigma(E, E')$-Cauchy sequence (x_n) of E, the sequence $(T(x_n))$ is norm convergent in X.*

P r o o f. i) $\Rightarrow$ ii) Let $(x_n) \subset E$ be an order bounded weakly Cauchy sequence. We have to show that $(T(x_n))$ is norm convergent. This is equivalent to proving that $(T(x_n))$ is a norm Cauchy sequence. By Lemma 2.1, it suffices to establish that $(T(x_{k_n} - x_n))$ is norm convergent to zero for every subsequence (x_{k_n}) of (x_n). Let (x_{k_n}) be a subsequence of (x_n). Since (x_n) is $\sigma(E, E')$-Cauchy, it follows from Lemma 2.1 that $x_{k_n} - x_n \to 0$ for $\sigma(E, E')$. On the other hand, the sequence $(x_{k_n} - x_n)$ is order bounded and E has order weakly sequentially continuous lattice operations, hence $|x_{k_n} - x_n| \to 0$ for $\sigma(E, E')$. Consequently, both $((x_{k_n} - x_n)^+)$ and $((x_{k_n} - x_n)^-)$ are weakly convergent to zero. Now, from Corollary 3.4.9 of [4] it follows that both $(T(x_{k_n} - x_n)^+)$ and $(T(x_{k_n} - x_n)^-)$ are norm convergent to zero. So $(T(x_{k_n} - x_n))$ is convergent in the norm to zero. This completes the proof of i) $\Rightarrow$ ii).

ii) $\Rightarrow$ i) We have to show that T is order weakly compact. By Theorem 3.4.4 of [4], it suffices to show that $(T(x_n))$ is norm convergent to zero for every order bounded disjoint sequence $(x_n) \subset E^+$. Let (x_n) be such a sequence. It is easy to see that $x_n \to 0$ for $\sigma(E, E')$. So, (x_n) is $\sigma(E, E')$-Cauchy. Then, by ii), $(T(x_n))$ is norm convergent to zero (necessarily to 0 because $T(x_n) \to 0$ weakly). □

R e m a r k. Observe that the implication ii) ⇒ i) of Theorem 2.2 is true without any hypothesis on E, but the implication i) ⇒ ii) of Theorem 2.2 is false if the Banach lattice E does not have order weakly sequentially continuous lattice operations. In fact, let us take the example after Corollary 3.4.10 of [4], i.e. let (r_n) be the sequence of Rademacher functions in $L^2[0,1]$. This sequence is order bounded and weakly convergent to 0. Note that the identity operator $T = \mathrm{Id}_{L^2[0,1]}\colon L^2[0,1] \to L^2[0,1]$ is order weakly compact and such that $\|T(r_n)\| = \|r_n\| = 1$ for each n. This proves that $(T(r_n))$ is not norm convergent.

As a consequence, we obtain the following characterization of discrete Banach lattices:

Corollary 2.3. *Let E be a Banach lattice with an order continuous norm. Then the following assertions are equivalent:*

i) *E is discrete.*

ii) *E has weakly sequentially continuous lattice operations.*

iii) *E has order weakly sequentially continuous lattice operations.*

iv) *Each order bounded $\sigma(E, E')$-convergent sequence of E is norm convergent.*

P r o o f. i) ⇒ ii) Follows from Proposition 2.5.23 of [4].

ii) ⇒ iii) Obvious.

iii) ⇒ iv) Let (x_n) be an order bounded sequence of E such that $x_n \to x$ weakly. It is clear that $(x_n - x)$ is a $\sigma(E, E')$-Cauchy sequence. On the other hand, since the norm of E is order continuous, the identity operator $T = \mathrm{Id}_E\colon E \to E$ is order weakly compact. By hypothesis, the lattice operations of E are order weakly sequentially continuous, hence it follows from Theorem 2.2 that $(T(x_n - x)) = ((x_n - x))$ is norm convergent to zero. So, (x_n) is norm convergent to x and we are done.

iv) ⇒ i) It suffices to show that the condition (iv) of Corollary 21.13 of [1] holds. We note that the first part of the condition (iv) of Corollary 21.13 of [1] is just our condition iv) and the second part of the condition (iv) of Corollary 21.13 [1] is obviously satisfied. Hence, the condition (i) of Corollary 21.13 of [1] implies that E is discrete. □

Now, we look at the converse.

Theorem 2.4. *Let E be a Banach lattice and X a Banach space. If for each order weakly compact operator $T\colon E \to X$, the sequence $(T(x_n))$ is norm convergent whenever (x_n) is an order bounded $\sigma(E, E')$-Cauchy sequence, then one of the following assertions is valid:*

i) *E has order weakly sequentially continuous lattice operations.*

ii) *c_0 is not embbeded in X.*

P r o o f. Assume by way of contradiction that E does not have order weakly sequentially continuous lattice operations and c_0 is embbeded in X. Let (x_n) be an order bounded sequence which is weakly convergent to 0 but its module $(|x_n|)$ is not weakly convergent to 0. We will use the same proof as for Theorem 2 of Wickstead [5] to find $f \in (E')^+$, a sequence (g_n) of E' and $g \in E'$ with $g_n, g \in [-f, f]$ and $g_n \to g$ for $\sigma(E', E)$ satisfying $g_n(x_n) \geqslant \varepsilon$ for all n and some $\varepsilon > 0$. (In fact, there is $f \in E'$ with $f(|x_n|)$ not convergent to 0. By choosing a subsequence and, if necessary, replacing f by $-f$ we may suppose that there is ε with $f(|x_n|) \geqslant \varepsilon > 0$ for all $n \in \mathbb{N}$. As the same is true for $|f|$, we may assume that $f \geqslant 0$.

Equip E with the seminorm $x \longmapsto f(|x|)$ and denote the completion of its Hausdorff quotient (which is an AL-space) by Z. Let $J\colon E \longrightarrow Z$ be the canonical embedding and let Z_0 denote the separable closed sublattice of Z generated by $\{Jx_n\colon n \in \mathbb{N}\}$. There is a positive contractive projection P of Z onto Z_0. We know that $\|Jx_n\| \geqslant \varepsilon$ for each n, so there exists $\varphi_n \in Z'$ with $\|\varphi_n\| = 1$ and $\varphi_n(Jx_n) \geqslant \varepsilon$. Since Z_0 is separable we may pass to subsequences (which we continue to denote by (x_n) and (φ_n), respectively) with $(P'\varphi_n)$ converging weak* to φ (say). If we let $g_n = J'P'\varphi_n$ and $g = J'\varphi$ then $g_n, g \in [-f, f]$ and $g_n \to g$ weakly* and $g_n(x_n) \geqslant \varepsilon$ for all $n \in \mathbb{N}$.)

Now we consider the linear map S_1 defined by

$$S_1\colon E \to c_0, \quad x \longmapsto S_1(x) = (g_n(x) - g(x))_1^\infty.$$

Since $g_n \to g$ for $\sigma(E', E)$, we have $g_n(x) - g(x) \to 0$ for all $x \in E$. So $(g_n(x) - g(x))_1^\infty \in c_0$ and

$$\|S_1(x)\|_\infty = \sup_n |g_n(x) - g(x)| \leqslant 2\|x\| \cdot \|f\| \quad \text{for all } x \in E.$$

This shows that S_1 is well defined and continuous. Moreover, S_1 is order weakly compact. In fact, let (z_n) be an order bounded disjoint sequence of E^+ and note that

$$\|S_1(z_n)\|_\infty = \sup_k |g_k(z_n) - g(z_n)| \leqslant 2f(z_n) \quad \text{for all } n.$$

It is easy to see that $f(z_n) \to 0$ (see [2, p. 192]). So $\|S_1(z_n)\|_\infty \to 0$. Thus by Theorem 5.57 of [2] S_1 is order weakly compact.

On the other hand, since c_0 is embbeded in X, there exists an embedding $S_2\colon\ c_0 \to X$ and hence there exist two constants $K,\ M > 0$ such that

$$K \left\|(\alpha_n)_{n=1}^{\infty}\right\|_{\infty} \leqslant \left\|S_2((\alpha_n)_{n=1}^{\infty})\right\| \leqslant M \left\|(\alpha_n)_{n=1}^{\infty}\right\|_{\infty} \text{ for all } (\alpha_n)_{n=1}^{\infty} \in c_0$$

Finally, we consider the composed operator $S = S_2 \circ S_1\colon\ E \to X$. Since S_1 is order weakly compact then S is also order weakly compact. Note that (x_n) is an order bounded sequence which is weakly convergent to 0. So, (x_n) is $\sigma(E, E')$-Cauchy. Now, for every n we have

$$\begin{aligned} \|S(x_n)\| &= \left\|S_2((g_k(x_n) - g(x_n))_{k=1}^{\infty})\right\| \\ &\geqslant K \left\|(g_k(x_n) - g(x_n))_{k=1}^{\infty}\right\|_{\infty} \\ &\geqslant K \left|g_n(x_n) - g(x_n)\right|. \end{aligned}$$

Then $(S(x_n))$ certainly does not converge to 0 as $|g_n(x_n) - g(x_n)|$ is eventually bounded away from zero (because $g(x_n) \to 0$ and $g_n(x_n) \geqslant \varepsilon$ for all n). Clearly, $S(x_n) \to 0$ weakly. Hence, $(S(x_n))$ is not norm convergent, which contradicts with our hypothesis. This completes the proof of the theorem. $\square$

As a consequence of Theorem 2.2 and Theorem 2.4, we obtain the following characterization of Banach lattices with order weakly sequentially continuous lattice operations.

Corollary 2.5. *Let E be a Banach lattice. Then the following assertions are equivalent:*

i) *If $T\colon\ E \to c_0$ is an order weakly compact operator, then $(T(x_n))$ is norm convergent for every order bounded $\sigma(E, E')$-Cauchy sequence (x_n) of E.*
ii) *E has order weakly sequentially continuous lattice operations.*

To give another result we need to recall that an operator T from a Banach lattice E into a Banach space F is said to be AM-compact if it carries each order bounded subset of E onto a relatively compact subset of F. It is clear that each AM-compact operator is order weakly compact. However, the converse is false in general. In fact, the identity operator of $L^2[0,1]$ is weakly compact, and hence order weakly compact. However, it is not AM-compact (if it were, then each order interval of $L^2[0,1]$ would be compact. But the interval $[-\mathbf{1}, \mathbf{1}]$ contains Rademacher functions, which is a weakly convergent sequence to 0 and all its elements have norm 1, so it has no norm convergent subsequence. This gives a contradiction).

We end our paper by the following characterization of AM-compact operators.

Proposition 2.6. *Let E be a Banach lattice with an order continuous norm. Then the following assertions are equivalent:*

i) *Each operator $T\colon E \to c_0$ is* AM-*compact.*

ii) *For each operator $T\colon E \to c_0$, $(T(x_n))$ is norm convergent for every order bounded $\sigma(E,E')$-Cauchy sequence (x_n) of E.*

iii) *E has weakly sequentially continuous lattice operations.*

iv) *E is discrete.*

Proof. i) $\Rightarrow$ ii) Let $T\colon E \to c_0$ be an operator. Assume by way of contradiction that there exists an order bounded $\sigma(E,E')$-Cauchy sequence (x_n) of E such that $(T(x_n))$ is not norm convergent. Then there exists a subsequence (x_{k_n}) of (x_n) such that $(T(x_{k_n} - x_n))$ is not norm convergent to zero. By choosing a subsequence of (x_{k_n}), we may suppose that there is $\varepsilon > 0$ such that $\|T(x_{k_n} - x_n)\| \geqslant \varepsilon$ for all $n \in \mathbb{N}$. As $(T(x_{k_n} - x_n))$ is weakly convergent to zero, we conclude that $(T(x_{k_n} - x_n))$ has no norm convergent subsequence. So, since $((x_{k_n} - x_n))$ is order bounded, T is not AM-compact, which contradicts i).

ii) $\Rightarrow$ iii) By Corollary 2.5, E has order weakly sequentially continuous lattice operations. As the norm of E is order continuous, it follows from Corollary 2.3 that E has weakly sequentially continuous lattice operations.

iii) $\Leftrightarrow$ iv) Since the norm of E is order continuous, the result follows from Corollary 2.3.

iv) $\Rightarrow$ i) Let $T\colon E \to c_0$ be an operator. Since E is discrete and its norm is order continuous, it follows from Corollary 21.13 of [1] that each order interval $[-x, x]$ is norm compact. So, $T([-x, x])$ is norm compact and hence T is AM-compact. □

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