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REALIZATION OF MULTIVARIABLE NONLINEAR SYSTEMS VIA THE APPROACHES OF DIFFERENTIAL FORMS AND DIFFERENTIAL ALGEBRA

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In this paper differential forms and differential algebra are applied to give a new definition of realization for multivariable nonlinear systems consistent with the linear realization theory. Criteria for the existence of realization and the definition of minimal realization are presented. The relations of minimal realization and accessibility and finally the computation of realizations are also discussed in this paper.

Keywords: realization, nonlinear system, differential ideal, differential form

Classification: 93B15, 93C10

1. INTRODUCTION

Various approaches have been developed to find a suitable definition of realization for nonlinear systems since the late 1970's ([9, 12, 16, 22, 46, 50, 51]). These definitions are not equivalent and some are not consistent with the linear theory as this review will show. The purpose of this paper is to study this problem and present a new definition of realization for nonlinear systems which is consistent with the linear realization theory.

In general a single input single output (SISO) linear time-invariant system can be written in its *state space* form as follows

$$\Sigma : \begin{cases} \dot{x} &= Ax + bu, \\ y &= cx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$ and $c \in \mathbb{R}^{1\times n}$. By using Laplace transform, this SISO system can also be rewritten in the *transfer function* form

$$H(s) = \frac{b_{k-1}s^{k-1} + \cdots + b_1 s + b_0}{s^k + a_{k-1}s^{k-1} + \cdots + a_1 s + a_0}. \tag{2}$$

The relation between the state space form and transfer function form is clear: every system in state space form Σ *admits* a transfer function $c(sI-A)^{-1}b$, and every transfer function $H(s)$ is *realized* by a system in a state space form Σ [3]. In the later case, Σ is called a *realization* of $H(s)$ if

$$H(s) = c(sI - A)^{-1}b.$$

The above realization theory between state space form and transfer function form for SISO systems is easily generalized to multi-input multi-output (MIMO) linear time-invariant systems [3]; however, the corresponding generalization to nonlinear systems has been a longstanding problem. For nonlinear systems, the realization problem is finding a suitable state equation for a given system of input-output equations. Recall that the success of the concept of transfer function in solving the linear realization problem is because those transfer functions could be computed either from the input-output equations or from the state space equations. However, for nonlinear systems it is impossible to define transfer functions by using the Laplace transform which results in great difficulty in the study of nonlinear realization theory. Furthermore, the problem of minimal realization for nonlinear systems is not adequately understood.

Since transfer function is powerful for linear realization theory, it is natural to consider its nonlinear generalizations. By using the differential 1-form method introduced in [13] and the noncommutative ring theory ([42, 43]), references [56] and [23] define transfer functions/matrices for nonlinear systems, with a focus on the SISO case. Reference [36] applies the same techniques to discuss the irreducibility of nonlinear systems; however, its further application in nonlinear realization theory is not discussed. [24] presents a general framework for the MIMO case, while a comprehensive study on the existence and computation of realization is lacking. These approaches based on transfer functions generally depend on state variables and are not helpful for our purpose.

An early attempt at nonlinear realization theory investigated whether a nonlinear state space system admits an input-output equation. Under some *regularity* conditions, the state variable could be expressed as a function of the input, the output and their derivatives. Though such state elimination procedures can be found here and there in the early literature on nonlinear observability and observers ([26, 38, 39, 46]), and later in books by Isidori [27] and by Nijmeijer and van der Schaft [41], some detailed descriptions, with a view of realization, were first given in [14, 22] and [49]. Different state elimination processes for the same set of state space equations sometimes end up with different sets of input-output equations, and the trajectories of the obtained input-output equations may also be different.

The *trajectories* (or *behavior* as referred to in [47]) Σ_d of a state space nonlinear system and the *trajectories* (or *behavior*) Σ_e of a system of input-output equations are defined and [54] calls the state space system *admits an input-output equation* if $\Sigma_d \subset \Sigma_e$. This is the case when the state space variables in the state space equations can be eliminated (via observability) to yield (or to generate as in [40]) the input-output equations. Besides this inclusion definition of realization, there are two other definitions: [47] defines realization through the equality condition $\Sigma_e = \Sigma_d$, while [54] defines realization by the inclusion $\Sigma_e \subset \Sigma_d$. From a similar point of view as in [54], bilinear realizability was investigated in [21] and [50], polynomial realizability in [2], rational realizability in [52], and the more general form of realizability in [9, 10] and [11]. Some necessary or sufficient conditions and constructive procedures were given for realizing an input-output equation in these approaches. These trajectory based definitions are, however, conceptually inconsistent with the linear realization as the subsequent example will show. In fact, it follows from the linear realization theory that any of the following three state space systems

$$\Sigma_1 : \begin{cases} \dot{x} &= u, \\ y &= x, \end{cases} \qquad \Sigma_2 : \begin{cases} \dot{x}_1 &= u, \\ \dot{x}_2 &= 0, \\ y &= x_1, \end{cases} \qquad \Sigma_3 : \begin{cases} \dot{x}_1 &= x_2 + u, \\ \dot{x}_2 &= 0, \\ y &= x_1, \end{cases} \tag{3}$$

is a realization of both the transfer functions $H_1(s) = \frac{1}{s}$ and $H_2(s) = \frac{s}{s^2}$. These transfer functions are equivalent to the following input-output equations (4) and (5) respectively:

$$\dot{y} - u = 0, \tag{4}$$

$$\ddot{y} - \dot{u} = 0. \tag{5}$$

It is easy to check that $\Sigma_d \subset \Sigma_e$ is not satisfied for systems Σ_3 and (4), $\Sigma_d = \Sigma_e$ is violated for Σ_2 and (4), and $\Sigma_e \subset \Sigma_d$ does not hold for Σ_1 and (5).

There is also the point of view to understand the realization theory as the relation between the input-output map and its state-space representation. By this understanding, a lot of work is done for nonlinear theory (see, for example, [4, 27, 28, 30, 31, 32, 53]). This paper focuses on the relations between the input-output equation and the corresponding state-space realization, therefore the input-output map approach is not adopted here. In the literature reviewed above, the minimal realization for nonlinear MIMO systems is still far from being adequately understood. It is therefore necessary to study the realization of nonlinear MIMO systems including the notion of minimality.

In this paper a new definition of realization is given for nonlinear MIMO systems in accordance with the linear theory. Differential algebra (see [7, 15, 17, 18, 19, 20, 33, 34, 44]) and the method of differential 1-form (see [6, 13]) are the main tools for this new approach. In Section 2 differential ideals, notations and conventions are introduced. In Section 3 the definition of realization is given. Criteria to check whether a system is realizable is presented in Section 4. In the subsequent section the definition of minimal realization is provided and the relation with accessibility is found. In Section 6 the general scheme for the computation of realization and minimal realization is presented. Conclusions are presented in Section 7. The definition of a differential ring and the proof that the new definition of nonlinear realization is consistent with the linear theory are given in the Appendix.

2. PRELIMINARIES

Terminologies used throughout this paper emanates from several sources. Definitions of observability, accessibility and integrability are described by [6]. *Differential 1-form* can be found in [35], the definitions of *ring* and *ideal* are referred to in [29]. The notions of *derivation operator*, *differential ring* and *differential ideal* are taken from [34]. For any given set $\mathcal{S}$ in a differential ring, the symbol $\langle \mathcal{S} \rangle$ denotes the differential ideal generated by $\mathcal{S}$ in this differential ring.

2.1. Notations and Hypotheses on General Input-Output Equations

For any given nonnegative integers k_0 and s_0, and a contractible open subset U_0 in $\mathbb{R}^{p(k_0+1)+m(s_0+1)}$, consider the system of input-output equations

$$\Phi_i(y, \dot{y}, \ldots, y^{(k_0)}, u, \dot{u}, \ldots, u^{(s_0)}) = 0, i = 1, 2, \ldots, n_0, \tag{6}$$

where $y = (y_1, \ldots, y_p)^T \in \mathbb{R}^p, u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$, and each Φ_i is real analytic in its arguments on U_0. Here a real analytic function on U_0 means that it can be expanded as a convergent power series with real coefficients at any point inside U_0. For any $k_0' \geq k_0, s_0' \geq s_0$ define the projection

$$(y, \dot{y}, \ldots, y^{(k_0')}, u, \dot{u}, \ldots, u^{(s_0')}) \mapsto (y, \dot{y}, \ldots, y^{(k_0)}, u, \dot{u}, \ldots, u^{(s_0)}),$$

which induces a natural projection from some contractible open set $U_0' \subseteq \mathbb{R}^{p(k_0'+1)+m(s_0'+1)}$ to $U_0 \subseteq \mathbb{R}^{p(k_0+1)+m(s_0+1)}$.

Let $R_1(U_0)$(or R_1 for simplicity) be the set of all the real analytic functions in some finite variables in $\{y_i^{(j)}, u_r^{(l)} : j, l \geq 0; i = 1, \ldots, p; r = 1, \ldots, m\}$ such that each element is analytic over a contractible open set U_0' for which U_0 is the projection of U_0'. That is, each element of R_1 is real analytic on a contractible open set U_0', and for different elements in R_1 the corresponding open sets U_0' may be different as they are possibly lying in different spaces. In Subsection A of the Appendix, operations are introduced in $R_1(U_0)$ so that it becomes an integral domain and a differential ring. Denote the fraction field of $R_1(U_0)$ by $\mathcal{K}(U_0)$ (or $\mathcal{K}$ when the corresponding open set is clear from the context).

For any linear space M over the field $\mathcal{K}(U_0)$ with generators $\{\beta_\mu : \mu \in \Lambda'\}$ for an index set Λ', the following expression is used to denote M:

$$M = \operatorname{span}_{\mathcal{K}(U_0)}\{\beta_\mu : \mu \in \Lambda'\}.$$

Let I_Φ be the differential ideal (see [33] or [34]) of R_1 which is generated by Φ_i, $i = 1, \ldots, n_0$, and denote it by $I_\Phi = \langle \Phi_1, \ldots, \Phi_{n_0} \rangle$. For any element $\phi(y, \dot{y}, \ldots, y^{(k)}, u, \dot{u}, \ldots, u^{(r)})$ in $\mathcal{K}$ define formally the derivative and the differential

$$\begin{aligned} \dot{\phi} &= \sum_{i=0}^{k} \frac{\partial \phi}{\partial y^{(i)}} y^{(i+1)} + \sum_{j=0}^{r} \frac{\partial \phi}{\partial u^{(j)}} u^{(j+1)}, \\ \mathrm{d}\phi &= \sum_{i=0}^{k} \frac{\partial \phi}{\partial y^{(i)}} \mathrm{d}y^{(i)} + \sum_{j=0}^{r} \frac{\partial \phi}{\partial u^{(j)}} \mathrm{d}u^{(j)}. \end{aligned}$$

In this paper the analytic version of Poincaré Lemma and Frobenius Theorem, which means that all the functions and differential forms in these results are analytic, will be used. The proofs are almost the same as the classical proofs of the smooth version, and therefore omitted and referred to results in [5].

Let $R_2 = R_1(D]$ be the set of polynomials in the variable D with coefficients in R_1, where D is the operator that computes the derivative of a function. Define the multiplication in R_2 as the composition of operators, where each element of R_2 is viewed as an operator, then R_2 is noncommutative [55]. Let

$$E = \mathrm{d}\mathcal{K} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}g : \ g \in \mathcal{K}\}.$$

Note that the above R_2 and E are actually defined over the same contractible open set U_0 as R_1. For any differential ideal L, define

$$\mathrm{d}L = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}g : \ g \in L\}.$$

It is clear that $\mathrm{d}L$ is a linear subspace of E, and one can define the quotient space $E/\mathrm{d}L$. Note that each element in the quotient space $E/\mathrm{d}L$ can be written as $[w]$ or $w + \mathrm{d}L$, where w is an element of E. This element $[w]$ or $w + \mathrm{d}L$ is also called the equivalent class of w. Although both notations $[w]$ and $w + \mathrm{d}L$ are adopted in literature, we often use the latter since it obviously states that the equivalent relation is defined by $\mathrm{d}L$. The following lemma is required to define the degree of a nonzero element in R_2, and its proof can easily be calculated and is thus omitted.

Lemma 2.1. For any $\lambda \in R_1$, the equality $D^k\lambda = \sum_{i=0}^{k}\binom{k}{i}\lambda^{(i)}D^{k-i}$ holds in R_2, where $k = 1, 2, 3, \ldots$, and $\lambda^{(i)}$ denotes the derivative of λ up to order i.

A general element of R_2 can be a finite sum of the terms like $a_1D^{i_1}a_2D^{i_2} \ldots a_kD^{i_k}a_{k+1}$, where $a_1, a_2, \ldots, a_{k+1} \in \mathcal{K}$, and $i_1, i_2, \ldots, i_k$ are nonnegative integers. By the above Lemma 2.1, any nonzero element $\varrho \in R_2$ can be written uniquely in the form

$$\varrho = \sum_{i=0}^{k} \lambda_i D^i, \lambda_i \in R_1, i = 0, \ldots, k; \lambda_k \neq 0.$$

Define the *degree* of the above ϱ to be $\deg \varrho = k$.

Without loss of generality, suppose that system (6) is right-invertible, i. e. that the functions $\{y_i^{(j)} : j \geq 0, i = 1, \ldots, p\}$ are *differentially analytically independent* on U_0 in the sense that $\{\mathrm{d}y_i^{(j)} + \mathrm{d}I_\Phi : j \geq 0, i = 1, \ldots, p\}$ is an independent set in the quotient space $E/\mathrm{d}I_\Phi$ on U_0. This assumption is also called the *differential analytical independence* of $\{y_1, \ldots, y_p\}$. It implies that we do not consider the realization problem for any equation which has only variables in $\{y_l^{(i)} : i \geq 0; l = 1, \ldots, p\}$ but has no variable in $\{u_r^{(j)} : j \geq 0; r = 1, \ldots, m\}$. Assume also that $\{u_1, \ldots, u_m\}$ are differentially analytically independent which in turn implies that $\{\mathrm{d}u_r^{(j)} + \mathrm{d}I_\Phi : j \geq 0; r = 1, \ldots, m\}$ are independent in $E/\mathrm{d}I_\Phi$. The two assumptions are fixed throughout the paper. The assumption about the differential analytic independence of $\{y_1, \ldots, y_p\}$ or $\{u_1, \ldots, u_m\}$ does not make the approach of the paper less general. For example, if $\{y_1, \ldots, y_p\}$ is differentially analytically dependent, then with some suitable initial values, some of $\{y_1, \ldots, y_p\}$ are functions of the other independent variables. Now these variables can be eliminated from the input-output equation, and one can define the original input-output equation to have the same realization as the reduced input-output equation.

Remark 2.2. In the above notion of differential analytical independence, the symbols $y_i^{(j)}$ and $u_r^{(l)}$ are considered as variables and not functions of t. Therefore, $y_i^{(j)}$ and $u_r^{(l)}$ are treated as variables instead of functions of t when we refer to input-output equations, differential analytical independence, and differential ideals. They are treated as functions of time t only when state equations are considered.

2.2. From General Input-Output Equations to Standard Form

A standard form hypothesis on (6) is now introduced. An elimination process which triangularizes a system of analytic functions is needed to obtain the standard form of input-output equations. This is a generalization of Gaussian elimination for linear system of equations and Ritt's elimination theory for differential algebraic equations [44]. Define the following local operations for the functions in R_1.

(i) Use the implicit function theorem for the equation $\phi\big(y_1^{(i_1)}, y_1^{(i_2)}, \dots\big) = 0$, $i_2 < i_1$, to obtain locally that $y_1^{(i_1)} - g_0(y_1^{(i_2)}, \dots) = 0$ when $\frac{\partial \phi}{\partial y_1^{(i_1)}}$ is nonzero on certain open set.

(ii) For $\phi'\big(y_1^{(i_1+k)}, y_1^{(i_1+k-1)}, \dots, y_1^{(i_1)}, y_1^{(i_2)}, \dots\big) = 0$, $i_2 < i_1$, and the $\phi = 0$ in the above (i), substitute $y_1^{(i_1)} = g_0$, $y_1^{(i_1+1)} = g_1$, $\dots, y_1^{(i_1+k)} = g_k$ into ϕ' and obtain $\phi''\big(g_k, g_{k-1}, \dots, g_0, y_1^{(i_2)}, \dots\big) = 0$, where $g_i = g_0^{(i)}$ for $i = 1, \dots, k$.

(iii) Permute two equations.

The above elementary operations differ from the transformation in [48]. In fact, [48] aims to eliminate latent variables so as to obtain a system which consists of input and output variables only. Such an algorithm will not work for (6) since there are no latent variables in (6). Furthermore, the aim of the elementary operations in this paper is to transform (6) into (7) where the function has an explicit leading term $y_i^{(m_i)}$, while [48] will not transform a function into such explicit form as $y_i^{(m_i)} + \phi_i$ with $LT(\phi_i) < y_i^{(m_i)}$ (see the text below for the notations about ordering).

For the set of functions $\mathcal{T} = \{y_i^{(j)}, u_r^{(l)} : i = 1, \dots, p; r = 1, \dots, m; j, l \geq 0\}$, define the following ordering:

$$
\begin{array}{ll}
y_i^{(j)} > u_r^{(l)} & \text{for all } i, j, l, r; \\
y_i^{(j)} > y_{i'}^{(j')} \text{ if and only if} & i < i' \text{ or } i = i', j > j'; \\
u_r^{(l)} > u_{r'}^{(l')} \text{ if and only if} & r < r' \text{ or } r = r', l > l'.
\end{array}
$$

Then the above ordering is well-defined (see [8] for more information about ordering). For any meromorphic function $\phi(y_i^{(j)}, u_r^{(l)} : i = 1, \dots, p; r = 1, \dots, m; j, l \geq 0)$, define its leading term $LT(\phi)$ to be the greatest variable in $\mathcal{T}$ such that the partial derivative of ϕ with respect to this variable is nonzero.

Now consider operations on the contractible open set U_0. If, in system (6), $LT(\Phi_i) = y_1^{(j_i)}$ for all $i = 1, \dots, n_0$, and $j_1 < j_2 < \dots < j_{n_0}$, then the first elementary operation is used to solve $y_1^{(j_1)}$ from $\Phi_1 = 0$ and substitute it into the other equations. Then the derivatives of y_1 with order higher than or equal to j_1 are eliminated. Since the first operation is only a local operation and may not hold on the whole U_0, we shrink the open set U_0 into some smaller, contractible open set $\overline{U_0}$ so that the resulting equation after the first operation is analytic on some $\overline{U_0}'$ whose projection equals $\overline{U_0}$. Repeating the process for other variables and shrinking the underlying open set when necessary, system (6) is transformed into the so-called *echelon form.* This is that, for any y_i, there is at most one nonzero function, whose leading term is some derivative of y_i, in the resulting set of functions obtained by the above operations. After using the first elementary operation, the nonzero element of the echelon form with leading term $y_i^{(j)}$ can be written as

$$y_i^{(m_i)} + \phi_i\Big(y_i^{(j_0)}, y_{j_1}^{(j_2)}, u_{j_3}^{(j_4)} : j_0 = 0, \ldots, m_i - 1; \\ j_1 = i+1, \ldots, p;\ j_3 = 1, \ldots, m; j_2 \le m_i, 0 \le j_4\Big). \tag{7}$$

Since $\{u_1, \ldots, u_m\}$ are differentially analytically independent, any nonzero element in the echelon form can not have a leading term $u_r^{(l)}$, that is, its leading form must be some $y_i^{(j)}$. Therefore, all the nonzero elements in the echelon form can be written as (7) and the number of the nonzero elements, denoted by p', may also be smaller than p. After back substitution, the following hypothesis is made on (6) such that $p' = p$. Note that the functions in the echelon form are analytic only on a contractible open subset V_0 which is derived from U_0. For simplicity, the following hypothesis is made.

Standard Form Hypothesis 1. Assume the functions in system (6) are real analytic on U_0 and (6) can be transformed on U_0 by the above three kinds of elementary operations and the rearrangement of $\{y_1, \ldots, y_p\}$ into the form

$$y_i^{(m_i)} + \zeta_i\Big(y_i^{(j_0)}, y_{j_1}^{(j_2)}, u_{j_3}^{(j_4)} : 0 \le j_0 \le m_i - 1; 0 \le j_2 < m_{j_1}; 0 \le j_4 < s_1; \\ i+1 \le j_1 \le p; 1 \le j_3 \le m\Big) = 0, i = 1, \ldots, p, \tag{8}$$

where s_1 is a positive integer. Furthermore, assume that there exists a point P_0, which is called an *operating point* of the above system of equations (8) or of the differential ideal $\langle y_i^{(m_i)} + \zeta_i : i = 1, \ldots, p\rangle$, such that P_0 belongs to U_0 and P_0 is a zero of all the equations in (8).

Note that the above hypothesis assumes that one can obtain the standard form from (6) on the set U_0. This hypothesis is quite general and does not loose any generality. In fact, an open set V_0 is obtained after performing the three kinds of elementary operations. This V_0 may lie in an Euclidean space whose dimension is higher than $p(k_0 + 1) + m(s_0 + 1)$; however, its projection to $\mathbb{R}^{p(k_0+1)+m(s_0+1)}$ is a subset of U_0. We can work on this V_0, perform the three kind of elementary operations, and obtain the standard form on V_0.

This hypothesis means that any nonzero function ϕ in the echelon form must contain a variable in the set $\{y_i^{(j)} : i = 1, \ldots, p; j \ge 0\}$ and there are exactly p nonzero elements in the echelon form. Let $\mathcal{I}$ be the differential ideal generated by the functions $\{y_1^{(m_1)}+\zeta_1, \ldots, y_p^{(m_p)}+\zeta_p\}$. The equations $y_i^{(m_i)}+\zeta_i=0$, $i=1,\ldots,p$, defined on U_0 with the operation point P_0, will be the starting point of the realization problem. That is, consider the equations $y_i^{(m_i)}+\zeta_i=0$, $i=1,\ldots,p$, over the contractible open set U_0 and an operation point $P_0\in U_0$, instead of the equations in (6). The reason why the integers m_i and s_1 are introduced in equation (8) is for the convenience of the computation of $V_{\max}$ which is defined at the beginning of Section 4.

The above hypothesis can also be expressed in another form which is easier to check. The following three types of invertible operations are needed for any k-tuple $(w_1,w_2,\ldots,w_k)$, where $w_i\in E, i=1,\ldots,k$.

(i′) Substitute $f(D)w_1+w_2$ to w_2, where $f(D)\in R_2$;

(ii′) Substitute λw_1 to w_1, where $\lambda\in\mathcal{K}$ and $\lambda\neq 0$;

(iii′) Permute two differential 1-forms.

The above three operations are obviously invertible. Assume that the functions $\Phi_i, i=1,\ldots,n_0$, in (6) satisfy the following condition.

Standard Form Hypothesis 1′. Suppose $(\mathrm{d}\Phi_1,\ldots,\mathrm{d}\Phi_{n_0})$ can be transformed on U_0 by the above three kind of elementary operations and the rearrangement of $(y_1,\ldots,y_p)$ into $(\widetilde{w}_1,\widetilde{w}_2, \ldots, \widetilde{w}_p, 0, \ldots, 0)$, where

$$\begin{aligned}\widetilde{w}_i &= \mathrm{d}(y_i^{(m_i)}+\zeta_i)\\ &= \mathrm{d}y_i^{(m_i)}+\sum_{j_1=0}^{m_i-1}a_{j_1}^i\mathrm{d}y_i^{(j_1)}+\sum_{j_1=i+1}^{p}\sum_{0\leq j_2<m_{j_1}}b_{j_1,j_2}\mathrm{d}y_{j_1}^{(j_2)}\\ &\quad+\sum_{j_1=1}^{m}\sum_{0\leq j_3<s_1}c_{j_1,j_3}\mathrm{d}u_{j_1}^{(j_3)}, i=1,\ldots,p,\end{aligned}\tag{9}$$

ζ_i is a real analytic function on U_0 and $LT(\zeta_i)<y_i^{(m_i)}$, $i=1,\ldots,p$, and s_1 is a positive integer. Furthermore, assume that there exists a point P_0 such that P_0 belongs to U_0 and is a zero of $y_i^{(m_i)}+\zeta_i=0$ for all $i=1,\ldots,p$.

Obviously the above two forms of hypotheses are equivalent. The first form will be used in the definition of realization, and the second form is used to check if the standard form hypothesis holds.

Definition 2.3. A differential ideal L of $R_1(U_0)$ is called standard with basis $\{\phi_1, \ldots, \phi_{n_1}\}$ and indices $(m_1, \ldots, m_p, s_1)$ on a contractible open set $U_3\subseteq U_0$ if it has a finite set of generators $\{\phi_1,\ldots,\phi_{n_1}\}$ which satisfies the Standard Form Hypothesis 1 on the set U_3, where U_3 contains also an operating point P_0. The set of nonzero elements in the corresponding echelon form (8) or (9) is called the standard form of L with basis $\{\phi_1,\ldots,\phi_{n_1}\}$ and indices $(m_1,\ldots,m_p,s_1)$ on U_3 or simply standard form. A real analytic function $g\in R_1(U_0)$ is called in standard form if there exists a function ζ and integers (i,j) or (r,l) such that $g=y_i^{(j)}+\zeta, LT(g)=y_i^{(j)}>LT(\zeta)$; or $g=u_r^{(l)}+\tilde{\zeta}, LT(g)=u_r^{(l)}>LT(\tilde{\zeta})$, where $1\leq i\leq p, j\geq 0, 1\leq r\leq m, l\geq 0$. For simplicity, denote $g=s(\xi)$.

Given any standard differential ideal L and the corresponding finite basis, it follows directly from the elimination process of the Standard Form Hypothesis 1 that the resulting standard form of L must be unique. That is, the above indices $(m_1, \ldots, m_p, s_1)$ are well defined for any given basis $\mathcal{B} := \{\phi_1, \ldots, \phi_{n_1}\}$ of L. Define $s(L, \mathcal{B})$ as the differential ideal which is generated by the standard form. Note that $s(L, \mathcal{B})$ depends on the basis $\mathcal{B}$. For simplicity the following convention is made:

When a system of input-output equations $\xi_1 = \ldots = \xi_{n_1} = 0$ is considered or when a differential ideal is defined by $\langle \xi_1, \ldots, \xi_{n_1} \rangle$, then the basis of the corresponding ideal will be taken as $\{\xi_1, \ldots, \xi_{n_1}\}$. This basis will be fixed to compute the standard form. The resulting $s(L, \mathcal{B})$ will be denoted by $s(L)$ for simplicity.

Therefore, $s(I_\Phi) = \mathcal{I}$. Note that $s(L) \neq L$ in general. For example, when L is generated by the function $\sin(\dot{y}) - u$ on the open set $\{(y, u) : -\frac{\pi}{2} < y < \frac{\pi}{2}, -1 < u < 1\}$, then $s(L) = \langle \dot{y} - \arcsin(u) \rangle \neq L$.

For the operating point P_0 of (6) and the contractible and open set U_0, consider any differential ideal $L(U_0)$ with a basis $\mathcal{B}$, and define its *differential closure* $\overline{L}(U_0, P_0, \mathcal{B})$ (sometimes write $\overline{L}(U_0, P_0)$, $\overline{L}(U_0)$, or $\overline{L}$, for simplicity), with respect to (6), as the differential ideal generated by the set

$$\Big\{ g \in R_1(U_0) : g \text{ is in standard form, } g(P_0) = 0, \text{ and there exist } k_1 \geq 0, \\ \alpha_i \in R_1(U_0), \text{ and } \beta_i \in R_1(U_0), \text{ such that } \alpha_i \notin s(L(U_0)), \\ \beta_i \notin s(L(U_0)), \text{ and } \textstyle\sum_{j=0}^{k_1} \beta_j D^j(\alpha_j g) \in s(L(U_0)), \text{ where } i = 0, 1, \ldots, k_1 \Big\}. \tag{10}$$

Now introduce another stronger hypothesis.

Standard Form Hypothesis 2. Assume that the ideal $\overline{s(I_\Phi)} = \overline{\mathcal{I}}$ is standard with respect to the basis $\{y_i^{(m_i)} + \zeta_i : i = 1, \ldots, p\}$ on some contractible open set U_0'' whose projection equals U_0.

The following convention is made throughout this paper on the differential analytical independence of the y_i's and u_r's since the standard form hypotheses are made:

Suppose that both $\{\mathrm{d}y_i^{(j)} + \mathrm{d}\overline{\mathcal{I}} : i = 1, \ldots, p; j \geq 0\}$ and $\{\mathrm{d}u_r^{(l)} + \mathrm{d}\overline{\mathcal{I}} : r = 1, \ldots, m; l \geq 0\}$ are linearly independent sets in $E/\mathrm{d}\overline{\mathcal{I}}$.

Define $\mathcal{U} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_r^{(l)} : r = 1, \ldots, m; l \geq 0\}$ which is a subspace of E. For any standard ideal L, it is clear that the standard form of L in the Standard Form Hypothesis 1′ contains no nonzero element in $\mathcal{U}$. For this standard ideal L, define the quotient space $\overline{\mathcal{U}} := \mathcal{U}/(\mathrm{d}L \cap \mathcal{U}) \cong (\mathcal{U} + \mathrm{d}L)/\mathrm{d}L$.

Proposition 2.4. Fix the notation $\mathcal{U}$ as above, and suppose L is a standard ideal on a contractible open set U_2, then the equivalent classes of 1-forms in the set $\{\mathrm{d}u_r^{(l)} : r = 1, \ldots, m; l \geq 0\}$ are linearly independent in the quotient space $\overline{\mathcal{U}} = \mathcal{U}/(\mathrm{d}L \cap \mathcal{U})$.

Proof. Suppose $\{[\mathrm{d}u_r^{(l)}] : r = 1, \ldots, m; l \geq 0\}$ is dependent, then there exist $\{\alpha_{rl} \in R_1(U_2) : r = 1, \ldots, m; l = 0, \ldots, n_1\}$, which are not all zeros, such that

$\sum_{r=1}^{m}\sum_{l=0}^{n_1}\alpha_{rl}\mathrm{d}u_r^{(l)} \in \mathrm{d}L$. Since L is standard, one can assume that there exists $g_1, g_2, \ldots, g_{n_2} \in L$ and nonzero $\beta_1, \beta_2, \ldots, \beta_{n_2} \in R_1(U_2)$ such that $LT(g_1) = y_{i_1}^{(j_1)} > LT(g_k)$ for all $k = 2, 3, \ldots, n_2$, and

$$\sum_{r=1}^{m}\sum_{l=0}^{n_1}\alpha_{rl}\mathrm{d}u_r^{(l)} = \sum_{i=1}^{n_2}\beta_i \mathrm{d}g_i \tag{11}$$

holds in E. Then the above equality can be rewritten locally as $\mathrm{d}y_{i_1}^{(j_1)} + \sum_{i,j}\gamma_{ij}\mathrm{d}y_i^{(j)} + \sum_{r,l}\delta_{rl}\mathrm{d}u_r^{(l)} = 0$ holds in E, where $LT(y_{i_1}^{(j_1)}) > LT(y_i^{(j)}) > LT(u_r^{(l)})$ for all possible indices i, j, r, l in (11). However, this is impossible since all the 1-forms $\{\mathrm{d}y_{i_2}^{(j_2)},\ \mathrm{d}u_{r_2}^{(l_2)}\colon\ 1 \le i_2 \le p,\ j_2 \ge 0,\ 1 \le r_2 \le m,\ l_2 \ge 0\}$ are independent in E over any open set. □

It follows from the above proposition that the equivalent classes of the 1-forms in $\{\mathrm{d}u_r^{(l)} :\ r = 1, \ldots, m; l \ge 0\}$ are independent on U_0 when they are considered in the quotient space $\mathcal{U}/(\mathrm{d}I_\Phi \cap \mathcal{U})$.

For a standard ideal L with indices $(m_1, \ldots, m_p, s_1)$ on any contractible open set $U_3 \subseteq U_0$ which contains also the operating point P_0, define the linear space $H_0(L)$ (or H_0) on U_3 by

$$\begin{aligned} H_0(L) \quad = \quad & \mathrm{span}_{\mathcal{K}}\Big\{\mathrm{d}y_i^{(j)} + \mathrm{d}L, \mathrm{d}u_r^{(l)} + \mathrm{d}L :\ j = 0, 1, \ldots, m_i - 1; l = 0, 1, \ldots, \\ & \qquad s_1 - 1; i = 1, \ldots, p; r = 1, \ldots, m\Big\}. \end{aligned} \tag{12}$$

Obviously $H_0(L)$ is a subspace of $E/\mathrm{d}L$ on U_3.

2.3. Notations and Conventions on State Equations

Consider the following system of state equations

$$\dot{x} = \quad f(x, u), \tag{13}$$

$$y = \quad h(x), \tag{14}$$

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $y = (y_1, \ldots, y_p)^T \in \mathbb{R}^p$, $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$, f and h are real analytic functions of their arguments on a contractible open set U_0' whose projection equals U_0.

If the system (13−14) is observable [6] with observability indices $(k_1, \ldots, k_p)$ and $k_1 \ge \cdots \ge k_p$ (see [49]), then x can be solved as

$$x = \xi(y, u) = \xi\big(y_i^{(j)}, u_r^{(l)} :\ 0 \le j \le k_i - 1; 0 \le l \le k_1 - 1; 1 \le i \le p; 1 \le r \le m\big).$$

Denote by J the differential ideal of R_1 on U_0 which is generated by the standard forms of the components of the vector functions $\dot{x} - f(x, u)$ and $y - h(x)$ and their derivatives, where $x = \xi(y, u)$. For simplicity write $J = \langle s(\dot{\xi} - f(\xi, u)), s(y - h(\xi))\rangle$ or $J = \langle \dot{\xi} - f(\xi, u), y - h(\xi)\rangle$, where $s(\dot{\xi} - f(\xi, u))$ denotes the vector whose components are the standard forms of the components of $\dot{\xi} - f(\xi, u)$. Similar convention is made on $s(y - h(\xi))$ as well.

If the system (13)−(14) is not observable, then let $x = ((x')^T, (x'')^T)^T$, $f = ((f')^T, (f'')^T)^T$, such that x' is the maximal observable part, $\dot{x}' = f'(x', u), y = h(x')$, and assume that the rank of $\frac{\partial(y_i^{(j)}:\ j=0,1,\ldots,k_i-1;i=1,\ldots,p)}{\partial x}$ is constant on some contractible open subset $U_0' \subseteq U_0$. Then there exists ξ' such that $x' = \xi'(y_i^{(j)}, u_r^{(l)} : j = 0, 1, \ldots, k_i - 1; l = 0, 1, \ldots, k_1 - 1; i = 1, \ldots, p; r = 1, \ldots, m)$ on U_0' for some integers $k_i, i = 1, \ldots, p$, with $k_1 \geq k_2 \geq \cdots \geq k_p$. Now consider the realization problem on U_0' instead of U_0, and define similarly the ideal J which is generated by the components of the vector functions $\dot{x}' - f'(x', u)$ and $y - h(x')$ and their derivatives, where $x' = \xi'$.

Assume that the functions $\{y_1, \ldots, y_p\}$ in (13)−(14) are differentially analytically independent on U_0 with respect to the differential ideal J. Then it is clear that $\text{rank}\frac{\partial h}{\partial x} = p$ on U_0.

Remark 2.5. In the above definition of J, the direct substitution of ξ into $\dot{x} - f(x, u)$ and $y - h(x)$ will sometimes give zeros; however, the substitution of ξ into the derivatives of $\dot{x} - f(x, u)$ and $y - h(x)$ may not result in zeros (see the computation of J in Example 3.3 in Section 3). Therefore, the definition of J assumes that we compute the derivatives before the substitution of ξ; and the above $J = \langle \dot{\xi} - f(\xi, u), y - h(\xi) \rangle$ is simply a notation. However, in the linear case one can substitute ξ in $\dot{\xi} - f(\xi, u)$, $y - h(\xi)$ first and then compute the derivatives, the resulting J's are the same. This is due to the fact that $(\dot{\xi} - A\xi - Bu)^{(i)} = \xi^{(i+1)} - A\xi^{(i)} - Bu^{(i)}$, $(y - C\xi)^{(i)} = y^{(i)} - C\xi^{(i)}$, $i = 0, 1, 2, \ldots$. Thus, it makes no difference whether or not ξ is substituted before or after the computation of the derivatives. In the proof of Lemma B.2 this observation is used without any comment.

The following lemma is clear since the ideal J is generated by $\{s(\dot{\xi} - f(\xi, u)), s(y - h(\xi))\}$.

Lemma 2.6. For any system of equations (13−14) and the ideal J defined above, suppose the functions y, u also satisfy system (6) on U_0, then there exists a finite set of generators of J such that J is standard on U_0 with respect to this basis.

Therefore, $s(J)$ and $H_0(J)$ can be defined for the basis $\{y_i^{(l)} - g_i^l : x = \xi, l \geq k_i, i = 1, \ldots, p\}$ of J, and this basis will be fixed for the definitions of $s(J)$ and $H_0(J)$.

All the assumptions and notations in this section are fixed throughout the paper. For the convenience of the reader the main symbols are listed.

n, p, m	the dimensions of x, y and u respectively ;
n_0	the number of equations in (6);
k_0, s_0	the highest orders of y and u, respectively, in equation (6);
$k_1, \ldots, k_p$	the observability indices of (13−14);
$m_1, \ldots, m_p, s_1$	m_i is the highest orders of y in the ith equation of the standard form, while s_1 is the highest order of u in the p equations of the standard form, see Standard Form Hypothesis 1 and Definition 2.3;

I_Φ	the differential ideal generated by $\{\Phi_1, \dots, \Phi_{n_0}\}$, see the fourth paragraph of Section 2.1;
$\mathcal{I}$	the differential ideal generated by the functions in the standard form of system (6), see the second paragraph after Standard Form Hypothesis 1 in Section 2.2;
$\overline{\mathcal{I}}$	the differential closure of $\mathcal{I}$, see (10);
J	the differential ideal generated by the components of $\dot{x} - f(x, u)$ and $y - h(x)$ and their derivatives, where $x = \xi(y, u)$, see the second paragraph of Section 2.3;
$s(J), \overline{J}$	the standard form and closure of J, respectively;
$H_0(L), H_0$	the linear space defined in (12);
R_1	the ring of real analytic functions, see the second paragraph of Section 2.1;
R_2	the noncommutative ring of operators $R_1(D]$;
R_0	the linear space defined in (32);
$V_{\max}(H_0, L)$	the linear space defined in the beginning of Section 4;
$\mathcal{U}^{(i)}$	the linear space spanned by $\{\mathrm{d}u_r^{(l)} : r = 1, \dots, m, 0 \le l \le i\}$, see the beginning of Section 4;
$\mathcal{U}$	the linear space spanned by $\{\mathrm{d}u_r^{(l)} : r = 1, \dots, m, l \ge 0\}$, see the paragraph before Proposition 2.4 in Section 2.2.

3. DEFINITION OF REALIZATION

Definition 3.1. Given system (6), the system (13–14) is called a realization of (6) on (U_0, P_0) (or U_0 for short) if $\overline{s(J)} = \overline{\mathcal{I}}$ holds on U_0.

In Subsection B of the Appendix, it is proved that in the special case of linear systems the above definition reduces to equality of transfer function matrices computed either from the input-output equations or from the state equations.

Now consider the three systems in (3). Suppose the corresponding ideals J for the three systems are J_1, J_2, J_3 respectively. Take U_0 to be the whole Euclidean space, and its origin as the operating point P_0. Let $I_{\Phi_1} = \langle \dot{y} - u \rangle$, $I_{\Phi_2} = \langle \ddot{y} - \dot{u} \rangle$, then the two ideals are standard and $\mathcal{I}_1 := s(I_{\Phi_1}) = I_{\Phi_1}$, $\mathcal{I}_2 := s(I_{\Phi_2}) = I_{\Phi_2}$. Now the two input-output equations have the same $\overline{\mathcal{I}}$ which equals $\mathcal{I}_1 = \overline{\mathcal{I}_1} = \overline{\mathcal{I}_2} = \langle \dot{y} - u \rangle$. It is easy to compute that $J_1 = s(J_1) = \langle \dot{y} - u \rangle = \overline{\mathcal{I}_1}$, $J_2 = s(J_2) = \langle \dot{y} - u \rangle = \overline{\mathcal{I}_1}$, $J_3 = s(J_3) = \langle \ddot{y} - \dot{u} \rangle \subset \overline{\mathcal{I}_1}$, $\overline{s(J_1)} = \overline{s(J_2)} = \overline{s(J_3)} = \overline{\mathcal{I}_1}$. Thus, all three systems are realizations of both input-output equations.

The following example shows how the realization varies with respect to different open sets.

Example 3.2. Consider the input-output equation $\Phi = (\dot{y} - u)(\dot{y} - 2u) = 0$, then $I_\Phi = \langle (\dot{y}-u)(\dot{y}-2u) \rangle$. Define four contractible open sets $U_1 = \{(\dot{y}, y, u) : \dot{y}-2u > 0\}$, $U_2 = \{(\dot{y}, y, u) : \dot{y}-u > 0\}$, $U_3 = \{(\dot{y}, y, u) : \dot{y}-2u < 0\}$, $U_4 = \{(\dot{y}, y, u) : \dot{y}-u < 0\}$. It is clear that the standard form of I_Φ can not be defined on the whole space, therefore consider on the open sets U_i separately, $i = 1, 2, 3, 4$. Take the operating points $P_1 = (-1, 0, -1), P_2 = (2, 0, 1), P_3 = (1, 0, 1), P_4 = (-2, 0, -1)$, then $P_i \in U_i$, where $i = 1, 2, 3, 4$. Define the following two systems

$$\Sigma : \begin{cases} \dot{x} = u, \\ y = x, \end{cases} \qquad \Sigma' : \begin{cases} \dot{x} = 2u, \\ y = x. \end{cases}$$

Then the corresponding ideals J for the two systems are, respectively, $J = \langle \dot{y} - u \rangle = \overline{s(J)}$ and $J' = \langle \dot{y} - 2u \rangle = \overline{s(J')}$. On the set U_1, $I_\Phi = \langle \dot{y} - u \rangle$, $\overline{\mathcal{I}} = \overline{s(J)}$. Thus, Σ is a realization of $\Phi = 0$ on (U_1, P_1). Similarly, Σ is also a realization on (U_3, P_3); and Σ' is a realization of $\Phi = 0$ on (U_2, P_2), and on (U_4, P_4) as well. Therefore, the system has no realization on the whole space but it has realizations on smaller open sets.

Example 3.3. Consider the equation $\Phi = \ddot{y}(2\dot{y} - 3u) - \dot{u}(3\dot{y} - 4u) = 0$, then $\Phi = \frac{d((\dot{y}-u)(\dot{y}-2u))}{dt}$. Let $U_0 = \{(\ddot{y}, \dot{y}, y, \dot{u}, u) : 2\dot{y} - 3u > 0\}$, and take an operating point $P_0 = (0, 1, 0, 0, 0)$ in U_0. Then $\mathcal{I}(U_0, P_0) = \langle \ddot{y} - \dot{u}(3\dot{y} - 4u)/(2\dot{y} - 3u) \rangle$, $\overline{\mathcal{I}}(U_0, P_0) = \overline{\langle \dot{y} - u, \dot{y} - 2u \rangle} = \langle y, u \rangle$. Define $U_1 = \{(y, u) : 2\dot{y} - 3u > 0, \dot{y} - u > 0\}$, $U_2 = \{(y, u) : 2\dot{y} - 3u > 0, \dot{y} - 2u > 0\}$. Then $\overline{\mathcal{I}}(U_1, P_0) = \langle \dot{y} - u \rangle$, $\overline{\mathcal{I}}(U_2, P_0) = \langle \dot{y} - 2u \rangle$, and system Σ (respectively, Σ') is a realization on (U_1, P_0) (respectively, (U_2, P_0)). The following system is a realization of Φ on (U_0, P_0).

$$\begin{cases} \dot{x}_1 = & f_1(x, u) = & (3u + \sqrt{u^2 + 4x_2})/2, \\ \dot{x}_2 = & f_2(x, u) = & 0, \\ y = & h(x) = & x_1. \end{cases} \tag{15}$$

This system is obtained following the detail in Section 6. It can be shown that (15) is a realization on U_0. In fact, $x_1 = \xi_1(y, u) = y, x_2 = \xi_2(y, u) = \dot{y}^2 - 3u\dot{y} + 2u^2$. Then $\dot{\xi} - f(\xi, u) = 0, y - h(\xi) = 0$, $\frac{\mathrm{d}(\dot{\xi}_1 - f_1(\xi, u))}{\mathrm{d}t} = \ddot{y} - \dot{u}(3\dot{y} - 4u)/(2\dot{y} - 3u)$. Now $J = \mathcal{I}(U_0)$, $\overline{s(J)} = \overline{\mathcal{I}}(U_0)$, and one concludes that (15) is a realization on U_0.

4. CRITERIA OF REALIZABILITY

In the following, criteria are sought for the existence of realization. For any standard differential ideal L of R_1 on a contractible open set $U_3 \subseteq U_0$ which contains also the operating point P_0, define the linear space $V_{\max}(H_0, L)$ on U_3 with respect to $H_0 = H_0(L)$ on U_3 (see (12) for definition) by

$$V_{\max}(H_0, L) = \max\{V : \text{ The set } V \text{ is a } \mathcal{K}\text{-subspace of } H_0 \text{ and } \dot{V} \subseteq V + \overline{\mathcal{U}^0}\}.$$

The maximum means the maximal subspace with respect to inclusion; the symbol $\dot{V}$ denotes the linear space which is generated by the equivalent classes of the time derivatives of all the elements of V; $\mathcal{U}^k := \operatorname{span}_{\mathcal{K}}\{\mathrm{d}u_r^{(l)} : 0 \le l \le k, r = 1, \ldots, m\}$; and $\overline{\mathcal{U}^{(k)}} = \mathcal{U}^{(k)}/(\mathcal{U}^{(k)} \cap \mathrm{d}L) \cong (\mathcal{U}^{(k)} + \mathrm{d}L)/\mathrm{d}L$, $k = 0, 1, \ldots$.

Now it is shown that the above $V_{\max}(H_0, L)$ remains unchanged if H_0 is replaced by any bigger $H_0' \supseteq H_0$.

Lemma 4.1. Suppose $L \subseteq \overline{\mathcal{I}}$ is a standard differential ideal of R_1 on some contractible open set $U_2 \subseteq U_0$ with indices $(m_1, \ldots, m_p, s_1)$ for some fixed finite set of generators, where U_2 contains also the operating point P_0. Define H_0^1 to be the right hand side of the equality (12). Also define

$$H_0^2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i^{(j)} + \mathrm{d}L, \mathrm{d}u_r^{(l)} + \mathrm{d}L : 0 \le j \le k_0'; 0 \le l \le s_0'; 1 \le i \le p; 1 \le r \le m\},$$

where s_0' and k_0' are arbitrary integers such that $s_0' \ge s_1 - 1, k_0' \ge \max\{m_i - 1 : i = 1, \ldots, p\}$, and $H_0^1 \ne H_0^2$. Let $V_1 = V_{\max}(H_0^1, L)$, $V_2 = V_{\max}(H_0^2, L)$, then $V_1 = V_2$ on U_2.

Proof. For any $\mathrm{d}y_{i'}^{(j')} + \mathrm{d}L \in H_0^2 \backslash H_0^1$, the standard form in Standard Form Hypothesis 1′ can be used to represent $\mathrm{d}y_{i'}^{(j')}$ as a linear combination of $\mathcal{T}_0 := \{\mathrm{d}y_i^{(j)}, \mathrm{d}u_r^{(l)} : 0 \le j \le m_i - 1;\ 0 \le l \le s_0'';\ 1 \le i \le p;\ 1 \le r \le m\}$ on U_2, where $s_0'' \ge s_0'$. Thus, without loss of generality suppose that H_0^2 is the span of the equivalent classes of the functions in $\mathcal{T}_0$ on U_2. Let $Y_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i^{(j)} : 0 \le j \le m_i - 1; 1 \le i \le p\}$. The inclusion $V_1 \subseteq V_2$ is clear. If $V_1 \ne V_2$ then $V_2 = V_1 \oplus V_0$ where V_1 has a basis $\{w_1 + \mathrm{d}L, \ldots, w_a + \mathrm{d}L\}$, V_0 has a basis $\{\tau_1 + \mathrm{d}L, \ldots, \tau_b + \mathrm{d}L\}$, $w = (w_1, \ldots, w_a)^T$, $\tau = (\tau_1, \ldots, \tau_b)^T$, $w_1, \ldots, w_a \in E$, and $\tau_1, \ldots, \tau_b \in (\mathcal{U}^{s_0''} + Y_1) \backslash (\mathcal{U}^{s_1 - 1} + Y_1)$. Then there exist matrices A, B, C, S, F, with elements in $\mathcal{K}$, such that $\dot{w} \equiv Aw + B\mathrm{d}u (\mathrm{mod}\ \mathrm{d}L)$, $\dot{\tau} \equiv Cw + S\tau + F\mathrm{d}u (\mathrm{mod}\ \mathrm{d}L)$, where $\mathrm{d}u = (\mathrm{d}u_1, \ldots, \mathrm{d}u_m)^T$, and the notation $X \equiv Y (\mathrm{mod}\ \mathrm{d}L)$ means each component of the vector $(X - Y)$ belongs to $\mathrm{d}L$. Thus,

$$\dot{\tau} - F\mathrm{d}u \equiv Cw + S\tau (\mathrm{mod}\ \mathrm{d}L). \tag{16}$$

Suppose $i_0 = \max\{i :$ there exists some r, $1 \le r \le m$, and some function $\alpha \in \mathcal{K}$ such that $\alpha \mathrm{d}u_r^{(i)}$ is a nonzero term of $\tau\}$, then $i_0 > s_1 - 1$. It follows from (16) that $v_{i_0+1} := \dot{\tau} - F\mathrm{d}u \in (\mathcal{U}^{i_0+1} + Y_1) \backslash (\mathcal{U}^{i_0} + Y_1)$, $v_{i_0+1} + \mathrm{d}L \in V_2$, where the notation '$v_{i_0+1} \in$' means that each component of v_{i_0+1} belongs to the set in the right hand side. By computing the derivative of (16) one has $v_{i_0+2} := \dot{v}_{i_0+1} - (CB + SF)\mathrm{d}u = (\dot{C} + CA + SC)w + (\dot{S} + S^2)\tau$. There may be some term $\mathrm{d}y_i^{(m_i)}$ in the above v_{i_0+2}. Use the standard form again and note that $i_0 > s_1 - 1$; one obtains $v_{i_0+2} \in ((\mathcal{U}^{i_0+2} + Y_1) \backslash (\mathcal{U}^{i_0+1} + Y_1))$, and $v_{i_0+2} + \mathrm{d}L \in V_2$. A similar process results in the existence of $v_k \in (\mathcal{U}^k + Y_1) \backslash (\mathcal{U}^{k-1} + Y_1)$ which satisfies also $v_k + \mathrm{d}L \cap V_2$, where k is any integer greater than i_0. By Proposition 2.4, the 1-forms $\{\mathrm{d}u_r^{(l)} + \mathrm{d}L : l \ge 0, r = 1, \ldots, m\}$ are independent. Then it follows that $\dim_{\mathcal{K}} \mathrm{span}_{\mathcal{K}}\{v_k + \mathrm{d}L : k \ge i_0 + 1\} = \infty$. This contradicts the fact that V_2 is finite dimensional. □

By Lemma 4.1 one can use $V_{\max}(L)$(or $V_{\max}$) to denote $V_{\max}(H_0^2, L)$ for any H_0^2 in which the order of $\mathrm{d}y$ (respectively, $\mathrm{d}u$) is not less than $\max\{m_1 - 1, \ldots, m_p - 1\}$ (respectively, $s_1 - 1$).

Lemma 4.2. For any observable system (13–14) which is a realization of (6) on some contractible open set $U_2 \subseteq U_0$ with $P_0 \in U_2$, define the observability indices $k_1, \ldots, k_p$ and the functions $\xi_1, \ldots, \xi_n$ as that in Section 2.3. Let

$$\begin{aligned} H_0' = \ & H_0(J) + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i^{(j)} + \mathrm{d}J, \mathrm{d}u_r^{(l)} + \mathrm{d}J : 0 \le j \le k_0'; \\ & \qquad 0 \le l \le s_0'; 1 \le i \le p; 1 \le r \le m\}, \end{aligned}$$

where $k_0' > k_1, s_0' > k_0'$. Then $V_{\max}(H_0', J) = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_1 + \mathrm{d}J, \ldots, \mathrm{d}\xi_n + \mathrm{d}J\}$ on U_2.

P r o o f. Let W_0 be the space $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_1+\mathrm{d}J,\dots,\mathrm{d}\xi_n+\mathrm{d}J\}$, then it is a subspace of H_0' and $\dot{W}_0 \subseteq W_0+\overline{\mathcal{U}^0}$. The maximality of $V_{\max}(H_0',J)$ implies $V_{\max}(H_0',J) \supseteq W_0$.

If $V_{\max}(H_0',J) \neq W_0$ then $V_{\max}(H_0',J)$ has a basis $\{\mathrm{d}\xi_1+\mathrm{d}J,\ \dots,\ \mathrm{d}\xi_n+\mathrm{d}J,\ v_1+\mathrm{d}J,\ \dots, v_r+\mathrm{d}J\}$. It follows from $y=h(x)$ that $\mathrm{d}y_i+\mathrm{d}J \in W_0, i=1,\dots,p$, and $\mathrm{d}y_i^{(j)}+\mathrm{d}J \in W_0+\overline{\mathcal{U}^{j-1}}$, $i=1,\dots,p$; $j=1,2,\dots$. Thus, one can suppose that each v_i has the form $v_i=\sum_{i_1,i_2} c^i_{i_1,i_2}\mathrm{d}u_{i_1}^{(i_2)}, i=1,\dots,r$. Define $j_0=\max\{i_2$: there exists a nonzero term $c^i_{i_1,i_2}\mathrm{d}u_{i_1}^{(i_2)}$ in some $v_i\}$, then it corresponds to some nonzero term $c^i_{i_0,j_0}\mathrm{d}u_{i_0}^{(j_0)}$ in some v_i, say v_1. Hence $\dot{v}_1$ contains the term $c^i_{i_0,j_0}\mathrm{d}u_{i_0}^{(j_0+1)}$, and $\dot{v}_1+\mathrm{d}J \in V_{\max}(H_0',J)+\overline{\mathcal{U}^0}$. Thus, there exists a nonzero element $\tau_1 \in V_{\max}(H_0',J)\cap(\overline{\mathcal{U}^{j_0+1}}\backslash\overline{\mathcal{U}^{j_0}})$. It follows from Proposition 2.4 that $\{\mathrm{d}u_r^{(l)}+\mathrm{d}J : 1\le r\le m; l\ge 0\}$ are independent on U_2, therefore a similar process as that of the proof of Lemma 4.1 completes the proof. □

Lemma 4.3. Suppose L is a standard differential ideal of R_1 on some contractible open set $U_2 \subseteq U_0$ with indices $(m_1,\dots,\ m_p,\ s_1)$, P_0 belongs to U_2, and $s(L)=L$ is generated by the standard basis $y_1^{(m_1)}+\zeta_1,\dots,y_p^{(m_p)}+\zeta_p$. Assume that ϕ is an analytic function in the variables in $\{y_i^{(j)},u_r^{(l)} : 0\le j\le m_i-1, 1\le i\le p, 1\le r\le m, l\ge 0\}$ on some contractible open set U_2' whose projection equals U_2, $\phi(P_0)=0$, and $\mathrm{d}\phi \in \mathrm{d}L$. Then $\phi\equiv 0$ on U_2'.

P r o o f. Suppose $\phi \not\equiv 0$, then ϕ does not equal any constant. If $LT(\phi)=u_{r_0}^{(l_0)}$ for some (r_0,l_0), then it contradicts the result of Proposition 2.4, therefore $LT(\phi)=y_{i_0}^{(j_0)}$ for some (i_0,j_0). It follows that $j_0\le\min\{m_1-1,\dots,m_p-1\}$. Consider the system of equations determined by the standard basis of $s(L)$, then $\phi=0$ on the trajectories of this system of equations. Assume that u is given, then by the Implicit Function Theorem there exists a function ρ such that $y_{i_0}^{(j_0)}=\rho(y_i^{(j)} : y_i^{(j)}<y_{i_0}^{(j_0)})$ on some open subset $U_3'\subseteq U_2'$.

Note that $s(L)$ determines a system of ordinary differential equations $y_i^{(m_i)}+\zeta_i=0, i=1,\dots,p$. The system has a unique local solution for any initial condition $\{y_i^{(j_i)}(t_0) : 0\le j_i\le m_i-1; 1\le i\le p\}$ in U_3'. However, the relation $y_{i_0}^{(j_0)}=\rho(y_i^{(j)} : y_i^{(j)}<y_{i_0}^{(j_0)})$ yields that $\{y_i^{(j_i)}(t_0) : 0\le j_i\le m_i-1$ for $i=1,\dots,i_0-1,i_0+1,\dots,p$, and $j_{i_0}=0,1,\dots,j_0-1\}$ is enough to determine y uniquely, this is a contradiction. □

Theorem 4.4. Suppose Standard Form Hypothesis 1 and 2 hold, then (6) has a realization on U_0 if and only if there exists a differential ideal L in R_1, which is standard with respect to some finite basis $\mathcal{S}$ of L on U_0 and P_0 is a zero of all the functions of $\mathcal{S}$, such that the following three conditions

(i) $\overline{s(L)}=\overline{\mathcal{I}}$;

(ii) $V_{\max}(H_0'',L)$ is integrable;

(iii) $\mathrm{d}y_1+\mathrm{d}L,\dots,\mathrm{d}y_p+\mathrm{d}L \in V_{\max}(H_0'',L)$

hold on U_0, where H_0'' is any linear space such that $H_0'' \supseteq H_0(L)$ and $H_0(L)$ is defined by the indices corresponding to the basis $\mathcal{S}$ as (12), and the integrability of $V_{\max}(H_0'', L)$ means that there exist functions $\zeta_1, \ldots, \zeta_{n_1} \in \mathcal{K}$ such that $V_{\max}(H_0'', L) = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\zeta_1 + \mathrm{d}L, \ldots, \mathrm{d}\zeta_{n_1} + \mathrm{d}L\}$.

Remark 4.5. In the linear case, condition (ii) is trivially satisfied and condition (iii) implies the strict properness condition (see Example 6.3 in Section 6). Condition (i) reduces to equality of transfer matrices computed either from the input-output equations or from the state equations. An integrability condition similar to condition (ii) was established in [37] for discrete-time systems.

Proof of Theorem 4.4 Suppose (13–14) is an observable realization of (6) then J is standard and $\overline{s(J)} = \overline{\mathcal{I}}$ on U_0. By Lemma 4.2 one has that $V_{\max}(H_0'', J) = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\xi_1 + \mathrm{d}J, \ldots, \mathrm{d}\xi_n + \mathrm{d}J\}$ which is integrable on U_0. The equation $y = h(\xi)$ implies that $\mathrm{d}y_1 + \mathrm{d}J, \ldots, \mathrm{d}y_p + \mathrm{d}J \in V_{\max}(H_0'', J)$. Thus, the necessity follows.

Now prove the sufficiency. Assume that $s(L)$ is generated by the standard basis $\{y_1^{(m_1)} + \zeta_1, \ldots, y_p^{(m_p)} + \zeta_p\}$. Let $V_{\max}(H_0'', L) = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\xi_1 + \mathrm{d}L, \ldots, \mathrm{d}\xi_n + \mathrm{d}L\}$ for some functions ξ_i on U_0, $\xi = (\xi_1, \ldots, \xi_n)^T$, and $x_i = \xi_i$, $i = 1, \ldots, n$. By $\dot{V}_{\max}(H_0'', L) \subseteq V_{\max}(H_0'', L) + \overline{\mathcal{U}^0}$, $\mathrm{d}y_1 + \mathrm{d}L, \ldots, \mathrm{d}y_p + \mathrm{d}L \in V_{\max}(H_0'', L)$, and Poincaré Lemma [5], there exist functions $f(\cdot, u), h(\cdot)$ on U_0 such that

$$\mathrm{d}(\dot{\xi} - f(\xi, u)) \in \mathrm{d}L, \mathrm{d}(y - h(\xi)) \in \mathrm{d}L, (\dot{\xi} - f(\xi, u))|_{P_0} = 0, (y - h(\xi))|_{P_0} = 0. \quad (17)$$

Consider the following state equation

$$\dot{\xi} - f(\xi, u) = 0, y - h(\xi) = 0. \quad (18)$$

Let $J_1 = \langle \dot{\xi} - f(\xi, u), y - h(\xi) \rangle$ be the differential ideal in R_1, one only needs to prove that $\overline{s(J_1)} = \overline{\mathcal{I}}$. It suffices to prove the stronger result $s(J_1) = s(L)$. In the following, the fact in (17) is used when applying Lemma 4.3.

By Lemma 2.6 there exist functions $g_i^l(\xi, u, \dot{u}, \ldots, u^{(l-1)})$ such that $J_1 = \langle y_i^{(l)} - g_i^l : i = 1, \ldots, p; l \geq 0 \rangle$. For $l < m_i, y_i^{(l)} - g_i^l \equiv 0$ by Lemma 4.3. For $l = m_i$, $y_i^{(l)} - g_i^l = (y_i^{(m_i)} + \zeta_i) - (\zeta_i + g_i^l)$. By Lemma 4.3 one has $\zeta_i + g_i^l \equiv 0$, hence $y_i^{(l)} - g_i^l \in s(L)$ for $l = m_i$. For $l = m_i + 1$, $y_i^{(l)} - g_i^l = (y_i^{(m_i)} + \zeta_i)^{(l-m_i)} - \dot{\zeta}_i - g_i^l = (y_i^{(m_i)} + \zeta_i)^{(l-m_i)} - \sum_{j=1}^p \frac{\partial \zeta_i}{\partial y_j^{(m_j-1)}}(y_j^{(m_j)} + \zeta_j) + \sum_{j=1}^p \frac{\partial \zeta_i}{\partial y_j^{(m_j-1)}}\zeta_j - \sum_{j=1}^p \sum_{k=0}^{m_j-2} \frac{\partial \zeta_i}{\partial y_j^{(k)}} y_j^{(k+1)} - \sum_{l\geq 0} \frac{\partial \zeta_i}{\partial u^{(l)}} u^{(l+1)} - g_i^l$. It follows from Lemma 4.3 that $\sum_{j=1}^p \frac{\partial \zeta_i}{\partial y_j^{(m_j-1)}}\zeta_j - \sum_{j=1}^p \sum_{k=0}^{m_j-2} \frac{\partial \zeta_i}{\partial y_j^{(k)}} y_j^{(k+1)} - \sum_{l\geq 0} \frac{\partial \zeta_i}{\partial u^{(l)}} u^{(l+1)} - g_i^l \equiv 0$, hence $y_i^{(l)} - g_i^l \in s(L)$ for $l = m_i + 1$. Similarly one proves that $y_i^{(l)} - g_i^l \in s(L)$ for all $l \geq 0$ and $i = 1, \ldots, p$. Therefore, $J_1 \subseteq s(L)$.

From the above $\zeta_i + g_i^l \equiv 0$ for $l = m_i$. Therefore, $y_i^{(m_i)} + \zeta_i \equiv y_i^{(m_i)} - g_i^l \in J_1$. Thus, $s(L) \subseteq J_1$, and $s(L) = J_1$. Then $s(J_1) = J_1 = s(L)$. □

The system of state equations (18) constructed in the proof of Theorem 4.4 is called *the system determined by* $V_{\max}(L)$ *or* L.

Remark 4.6. It is worth noting that for any differential ideal L, which satisfies the conditions in Theorem 4.4, the dimension of $V_{\max}(L)$ on U_0 must be greater than or equal to p. In fact, suppose $\dim V_{\max}(L) < p$, then it follows from $\mathrm{d}y_i + \mathrm{d}L \in \dim V_{\max}(L)$, $i = 1, \ldots, p$, that there exist analytic functions $\lambda_1, \ldots, \lambda_p$, which are not all zero, such that $\sum_{i=1}^{p} \lambda_i \mathrm{d}y_i \in \mathrm{d}L$. This contradicts the hypothesis in Section 2.1 that $\{y_1, \ldots, y_p\}$ are differentially analytically independent. Therefore, one must have $\dim \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1, \ldots, \mathrm{d}y_p\} = p$ and $\dim V_{\max}(L) \geq p$. The dimension of the state space of the corresponding realization is greater than or equal to p.

Lemma 4.7. Suppose that, on a contractible open set $U_2 \subseteq U_0$ with $P_0 \in U_2$, L_1 and L_2 are two standard differential ideals of R_1, $L_1 = s(L_1) \subset L_2 = s(L_2)$, $\overline{s(L_1)} = \overline{s(L_2)} = \overline{\mathcal{I}}$, $H_0 = H_0(L_1) + H_0(L_2)$, $V_{\max}(H_0, L_1)$ is integrable, $\mathrm{d}y_i + \mathrm{d}L_1 \in V_{\max}(H_0, L_1)$, $i = 1, \ldots, p$. Assume also $V_{\max}(H_0, L_1) = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_j + \mathrm{d}L_1 : j = 1, \ldots, r\}, \xi_j \in R_1, j = 1, \ldots, r$. Denote $V_{\max}(L_1) = V_{\max}(H_0, L_1)$, $V_{\max}(L_2) = V_{\max}(H_0, L_2)$, $W_0 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_j + \mathrm{d}L_2 : j = 1, \ldots, r\}$, then $V_{\max}(L_2) = W_0$.

Proof. By the definition of $V_{\max}(H_0, L_1)$ and the condition $\mathrm{d}L_1 \subseteq \mathrm{d}L_2$, it is easy to obtain $\dot{W}_0 \subseteq W_0 + (\mathcal{U}^0 + \mathrm{d}L_2)/\mathrm{d}L_2$. Thus, $W_0 \subseteq V_{\max}(L_2)$. If the two are not equal, then take any nonzero $w + \mathrm{d}L_2 = \sum_{ij} a_{ij} \mathrm{d}y_i^{(j)} + \sum_{rl} b_{rl} \mathrm{d}u_r^{(l)} + \mathrm{d}L_2 \in V_{\max}(L_2) \backslash W_0$. Note that $V_{\max}(L_1) = \mathrm{span}\{\mathrm{d}\xi_1 + \mathrm{d}L_1, \ldots, \mathrm{d}\xi_r + \mathrm{d}L_1\}$ determines a system of state equations, say (13−14), then one has $\mathrm{d}y_i^{(j)} + \mathrm{d}L_1 \in V_{\max}(L_1) + (\mathcal{U} + \mathrm{d}L_1)/\mathrm{d}L_1$. Thus, $w + \mathrm{d}L_2 \in W_0 + (\mathcal{U} + \mathrm{d}L_2)/\mathrm{d}L_2$ and $W_0 \subset V_{\max}(L_2) \subseteq W_0 + (\mathcal{U} + \mathrm{d}L_2)/\mathrm{d}L_2$.

Suppose $V_{\max}(L_2) = \mathrm{span}\{w_1 + \mathrm{d}L_2, \ldots, w_s + \mathrm{d}L_2\}$, $w = (w_1, \ldots, w_s)^T$, $\xi = (\xi_1, \ldots, \xi_r)^T$. Then there exists a matrix A such that $w \equiv A\mathrm{d}\xi + \bar{u} (\mathrm{mod}\, \mathrm{d}L_2)$, where $\bar{u}$ is a nonzero vector whose components belong to $\mathcal{U}$. Assume that $\dot{w} \equiv Bw + C\mathrm{d}u (\mathrm{mod}\, \mathrm{d}L_2)$, $\mathrm{d}\dot{\xi} \equiv S\mathrm{d}\xi + F\mathrm{d}u (\mathrm{mod}\, \mathrm{d}L_2)$ for some matrices B, C, S, F. Then $\dot{w} \equiv \dot{A}\mathrm{d}\xi + AS\mathrm{d}\xi + AF\mathrm{d}u + \dot{\bar{u}} (\mathrm{mod}\, \mathrm{d}L_2)$, $\dot{w} \equiv BA\mathrm{d}\xi + B\bar{u} + C\mathrm{d}u (\mathrm{mod}\, \mathrm{d}L_2)$. Therefore,

$$(BA - \dot{A} - AS)\mathrm{d}\xi \equiv \dot{\bar{u}} - B\bar{u} + (AF - C)\mathrm{d}u (\mathrm{mod}\ \ \mathrm{d}L_2). \tag{19}$$

The highest order of $\mathrm{d}u$ in the above equality is in $\dot{\bar{u}}$, and the above equality means that the subspace $W_0 \cap ((\mathcal{U} + \mathrm{d}L_2)/\mathrm{d}L_2))$ is nonzero. Let the highest order of $\mathrm{d}u$ in $\dot{\bar{u}}$ be i_0, then $W_0 \cap ((\mathcal{U}^{i_0} + \mathrm{d}L_2)/\mathrm{d}L_2) \backslash ((\mathcal{U}^{i_0 - 1} + \mathrm{d}L_2)/\mathrm{d}L_2)) \neq 0$. Compute the derivatives of (19) and use again $\mathrm{d}\dot{\xi} \equiv S\mathrm{d}\xi + F\mathrm{d}u (\mathrm{mod}\, \mathrm{d}L_2)$ one obtains similarly $W_0 \cap ((\mathcal{U}^{i_0+1} + \mathrm{d}L_2)/\mathrm{d}L_2) \backslash ((\mathcal{U}^{i_0} + \mathrm{d}L_2)/\mathrm{d}L_2)) \neq 0$. A similar process as the one in the proof of Lemma 4.1 ends the proof. □

The following theorem follows directly from Theorem 4.4 and Lemma 4.7.

Theorem 4.8. Suppose Standard Form Hypothesis 1 and 2 hold, then (6) admits a realization on U_0 if and only if $V_{\max}(L_0)$ is integrable and $\mathrm{d}y_i + \mathrm{d}L_0 \in V_{\max}(L_0)$, $i = 1, \ldots, p$, on U_0, where L_0 is a maximal differential ideal in the set $\{L : L$ is a standard differential ideal on U_0, $L = s(L)$ and $\overline{s(L)} = \overline{\mathcal{I}}\}$, and the maximum is in the sense of inclusion.

When $s(\overline{\mathcal{I}}) = \overline{\mathcal{I}}$ it is clear that the above L_0 is unique and $L_0 = \overline{\mathcal{I}}$.

5. MINIMAL REALIZATION

Definition 5.1. The system (13−14) is called a minimal realization of (6) on a contractible open subset $U_1 \subseteq U_0$ with respect to the operating point P_0 if it is an observable realization of (6) on the set U_1, $P_0 \in U_1$, and the ideal J generated by (13−14) satisfies

$$\begin{aligned}&\dim V_{\max}(H_0(J), J)\\ &= \min\big\{\dim V_{\max}(H_0(L), L) : \text{On the open set } U_1,\ L \text{ is standard}, s(J) \subseteq L = s(L),\\ &\overline{s(J)} = \overline{s(L)} = \overline{\mathcal{I}}, V_{\max}(H_0(L), L) \text{ is integrable}, \mathrm{d}y_i \in V_{\max}(H_0(L), L), i = 1, \dots, p\big\}.\end{aligned}$$

By the above definition, a minimal realization on some contractible open set U_1 is just the one which has the minimal dimension of states in all the possible realizations on U_1. It is worth noting that a possible realization should satisfy the hypothesis in Section 2.3, that is, the functions $\{y_1, \dots, y_p\}$ must be differentially analytically independent. In Example 3.2, Σ is a minimal realization on U_1, Σ' is a minimal realization on U_2. In Example 3.3, Σ and Σ' in Example 3.2 are minimal realizations respectively on U_1 and U_2. It is easy to prove that the system (15) is a minimal realization on U_0. In fact, if it is not a minimal realization, then there exists an ideal $L = s(L) \supset s(J)$ (i.e. $L \supseteq s(J)$ and $L \neq s(J)$), such that $\overline{s(L)} = \overline{\mathcal{I}}$, $V_{\max}(L)$ is integrable, $\mathrm{d}y + \mathrm{d}L \in V_{\max}(L)$, $\dim V_{\max}(L) < 2$. Thus, $V_{\max}(L) = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y + \mathrm{d}L\}$, and it follows that the indices of L are $(m_1, s_1) = (1, 1)$, the standard form of L must be $\dot{y} - g(y, u) = 0$ for some analytic function g. Since $\overline{L} = \overline{\mathcal{I}} = \langle y, u\rangle$ on U_0 (see Example 3.3), one must have $g(y, u) = 0$, and the standard form of L is $\dot{y} = 0$, which contradicts the differential algebraic independence hypothesis on y. Therefore, (15) is a minimal realization on U_0. This example shows that a system of input-output equations may have different minimal realizations on different open sets, and the dimensions of the minimal realizations can be different as well.

The following proposition follows directly from Definition 5.1 and Theorem 4.8.

Proposition 5.2. Suppose (6) has a realization on the contractible open set U_0 and L_0 is the differential ideal defined in Theorem 4.8. Then the system of state equations determined by $V_{\max}(L_0)$ is a minimal realization of (6) on U_0.

For any standard differential ideal L which satisfies the conditions of Theorem 4.4, it determines a realization of (6). Suppose (13−14) is the corresponding system of state equations and it is observable. For the state equations, [1] defines $\mathcal{K}_1$ to be the field of meromorphic functions of $\{x_1, \dots, x_n, u_r^{(l)} : 1 \le r \le m; l \ge 0\}$, and

$$\begin{aligned}\widetilde{H}_0 &= \mathrm{span}_{\mathcal{K}_1}\big\{\mathrm{d}x_1 + \mathrm{d}J, \dots, \mathrm{d}x_n + \mathrm{d}J, \mathrm{d}u_1 + \mathrm{d}J, \dots, \mathrm{d}u_m + \mathrm{d}J\big\},\\ \widetilde{H}_k &= \big\{w + \mathrm{d}J \in \widetilde{H}_{k-1} : \dot{w} + \mathrm{d}J \in \widetilde{H}_{k-1}\big\}, \quad k \ge 1.\end{aligned}$$

Then there exists a k^* such that $\widetilde{H}_{k^*+1} = \widetilde{H}_{k^*+2} = \dots$. Define $\widetilde{H}_\infty = \widetilde{H}_{k^*+1}$. $\widetilde{H}_\infty$ is always integrable [1] and (13−14) is accessible if and only if $\widetilde{H}_\infty = 0$. This result is now used to find the relation between minimal realization and accessibility.

Theorem 5.3. Suppose $p \geq m$, and Standard Form Hypothesis 1 and 2 hold. Then the system (13 − 14) is a minimal realization of (6) on a contractible open set $U_1 \subseteq U_0$ with respect to the operating point $P_0 \in U_1$ if it is both observable and accessible on U_1.

Proof. Suppose the system (13 − 14) is both observable and accessible but not a minimal realization of (6) on U_1. Then one has $x = \xi(y, u)$, $V_{\max}(J) = \text{span}\{\mathrm{d}\xi_1 + \mathrm{d}J, \ldots, \mathrm{d}\xi_n + \mathrm{d}J\}$, $\dim \underline{V_{\max}(J) = n}$, and a standard differential ideal $L = s(L) \supseteq s(J)$ such that $\overline{\mathcal{I}} = \overline{s(L)} = \overline{s(J)}$ and L corresponds to a minimal realization. By Lemma 4.7, $V_{\max}(L)$ can be written as $\text{span}_{\mathcal{K}}\{\mathrm{d}\xi_1 + \mathrm{d}L, \ldots, \mathrm{d}\xi_n + \mathrm{d}L\}$, $\dim V_{\max}(L) < \dim V_{\max}(J)$. Thus, $\{\mathrm{d}\xi_1 + \mathrm{d}L, \ldots, \mathrm{d}\xi_n + \mathrm{d}L\}$ are linearly dependent. Without loss of generality suppose $\dim V_{\max}(L) = n - r$, $w = (w_1, \ldots, w_r)^T, \xi' = (\xi'_1, \ldots, \xi'_{n-r})^T$ such that $\text{span}\{w_1 + \mathrm{d}J, \ldots, w_r + \mathrm{d}J, \mathrm{d}\xi'_1 + \mathrm{d}J, \ldots, \mathrm{d}\xi'_{n-r} + \mathrm{d}J\} = V_{\max}(J)$, $\text{span}\{\mathrm{d}\xi'_1 + \mathrm{d}L, \ldots, \mathrm{d}\xi'_{n-r} + \mathrm{d}L\} = V_{\max}(L)$ and $w_i \in \mathrm{d}L, i = 1, \ldots, r$. It is clear that there exist matrices $A_{r\times n}, B_{(n-r)\times n}, A_1, B_1, C, S_{n\times m}$ such that $w \equiv A\mathrm{d}\xi (\text{mod } \mathrm{d}J), \mathrm{d}\xi' \equiv B\mathrm{d}\xi (\text{mod } \mathrm{d}J), \mathrm{d}\xi \equiv A_1 w + B_1 \mathrm{d}\xi' (\text{mod } \mathrm{d}J), \mathrm{d}\dot{\xi} \equiv C\mathrm{d}\xi + S\mathrm{d}u (\text{mod } \mathrm{d}J)$, and (A_1, B_1) is the inverse of the matrix $(A^T, B^T)^T$, where A and B are both of full row rank. Note that the matrices C and S are known while A and B are to be determined. It clearly follows that $\dot{w} \equiv \dot{A}\mathrm{d}\xi + A\mathrm{d}\dot{\xi} \equiv (\dot{A} + AC)A_1 w + (\dot{A} + AC)B_1 \mathrm{d}\xi' + AS\mathrm{d}u (\text{mod } \mathrm{d}J)$.

Since $\mathrm{d}y_1 + \mathrm{d}L, \ldots, \mathrm{d}y_p + \mathrm{d}L \in V_{\max}(L)$ one has $n - r \geq p$. Assume that $\text{rank} S = s$ and denote by $\mathcal{A}$ the subspace spanned by the rows of A. Let $\mathcal{D}$ be the subspace spanned by the columns of S, and $\mathcal{D}^\perp$ the subspace orthogonal to $\mathcal{D}$. Then $\dim \mathcal{D}^\perp = n - s$. Since $p \geq m$ one has $n - r \geq p \geq m \geq s$ which implies immediately that $n - s \geq r$ or equivalently $\dim \mathcal{D}^\perp \geq \dim \mathcal{A}$. Suppose the $n - s$ rows of the matrix $G_{(n-s)\times n}$ consists of a basis of $\mathcal{D}^\perp$, and let $A = VG$, where V is $r \times (n - s)$ and of full row rank. It follows that $AS = VGS = 0$ and $(\dot{A} + AC)B_1 = (\dot{V}G + V\dot{G} + VGC)B_1$.

The ordinary differential system $(\dot{V}G + V\dot{G} + VGC)B_1 = 0$ with respect to V has r equations and $r(n - s)$ unknowns. Note that G is a known matrix which can be determined by S, and B_1 is a function of $A = VG$ and B. Thus, for any given initial condition $V(t_0) = V_0$ with $\text{rank} V_0 = r$ the system has always a solution locally such that $V(t)$ is of full row rank. Hence for the corresponding $A(t)$, one has $\dot{w} \equiv (\dot{A} + AC)A_1 w (\text{mod } \mathrm{d}J)$, and $w_i^{(j)} + \mathrm{d}J \in \text{span}\{w_1 + \mathrm{d}J, \ldots, w_r + \mathrm{d}J\}$ for all $j \geq 0$ and $i = 1, \ldots, r$. Therefore, $\text{span}\{w_1 + \mathrm{d}J, \ldots, w_r + \mathrm{d}J\} \subseteq \widetilde{H}_\infty$, this contradicts the accessibility of the system. □

The system (15) in Example 3.3 shows that a minimal realization may not be accessible. The following is another example to illustrate this point again, and furthermore it shows that the ideal generated by the minimal realization may be strictly smaller than the differential closure $\overline{\mathcal{I}}$.

Example 5.4. Consider an input-output equation $u\ddot{y} - \dot{u}\dot{y} = 0$ on the open set $U_0 = \{(\ddot{y}, \dot{y}, y, \dot{u}, u) : u > 0\}$ with an operating point $P_0 = (0, 0, 0, 0, 1)$. Then

$$\begin{cases} \dot{x}_1 &= u, \quad \dot{x}_2 &= 0, \\ y &= x_1 x_2, \end{cases} \tag{20}$$

is a realization since $J = \langle u\ddot{y} - \dot{u}\dot{y}\rangle = \langle \ddot{y} - \dot{u}\dot{y}/u\rangle = s(J) = \mathcal{I}$ on U_0. Suppose it is not a minimal realization on U_0, then there exists a standard differential ideal $L = s(L) \supseteq s(J)$, such that $V_{\max}(L)$ is integrable, $\mathrm{d}y + \mathrm{d}L \in V_{\max}(L)$, $\dot{V}_{\max}(L) \subseteq V_{\max}(L) + \mathrm{span}\{\mathrm{d}u + \mathrm{d}L\}$, and $\dim V_{\max}(L) < 2$. By Remark 4.6, one has $\mathrm{d}y \notin \mathrm{d}L$ and $\dim V_{\max}(L) \neq 0$. Thus, $\dim V_{\max}(L) = 1$ and $V_{\max}(L) = \mathrm{span}\{\mathrm{d}y + \mathrm{d}L\}$. It is easy to compute $\overline{\mathcal{I}} = \overline{L} = \overline{< \ddot{y} - \dot{u}\dot{y}/u >} = \overline{< \frac{\mathrm{d}}{\mathrm{d}t}(\dot{y}/u) >} = \overline{\langle \dot{y}\rangle} = \langle y\rangle$ on U_0. Note that in the above computation, $\overline{\langle \frac{\mathrm{d}}{\mathrm{d}t}(\dot{y}/u)\rangle} \neq \overline{\langle \dot{y}/u + c\rangle}$ for any nonzero constant c, because P_0 is not a zero of $\dot{y}/u + c$ when $c \neq 0$. If $L = \langle y\rangle$ or $L = \langle \dot{y}\rangle$, then it contradicts the hypothesis that $\{y\}$ is differentially analytically independent. Similarly L can not be generated by elements which are analytic functions of $\{y^{(k)} : k \geq 0\}$ only. That is, u or its derivatives has to appear in the generators of L. It follows from the condition $\dot{V}_{\max}(L) \subseteq V_{\max}(L) + \mathrm{span}\{\mathrm{d}u + \mathrm{d}L\}$ that there exists a nonzero analytic function θ such that $\dot{y} - \theta(y, u)$ is the standard form of L. By the condition $\overline{L} = \langle y\rangle$, there exists a differential operator $g(D) \in R_2$ such that $g(D)(y) = \dot{y} - \theta(y, u)$. Then $\theta(y, u) = y\mu(y, u)$ for a nonzero analytic function μ. By $L \supseteq s(J) = \mathcal{I}$, there exist analytic functions ς_1 and ς_2 such that $\varsigma := \varsigma_1 D + \varsigma_2 \in R_2$, $\varsigma(\dot{y} - y\mu(y, u)) = \ddot{y} - \dot{u}\dot{y}/u$. Then $\varsigma_1(\ddot{y} - \dot{y}\mu - y\dot{\mu}) + \varsigma_2(\dot{y} - y\mu) = \ddot{y} - \dot{u}\dot{y}/u$, or equivalently $\ddot{y}\varsigma_1 + \dot{y}(-\varsigma_1\mu + \varsigma_2) - y(\varsigma_1\dot{\mu} + \varsigma_2\mu) = \ddot{y} - \dot{u}\dot{y}/u$. Since the right-hand side of the above equality contains no y, one must have $\varsigma_1\dot{\mu} + \varsigma_2\mu \equiv 0$, or equivalently $\varsigma_2 = -\varsigma_1\dot{\mu}/\mu$. Then $\ddot{y}\varsigma_1 + \dot{y}(-\varsigma_1\mu + \varsigma_2) = \varsigma_1(\ddot{y} - \dot{y}(\mu + \dot{\mu}/\mu)) = \ddot{y} - \dot{u}\dot{y}/u$. By expanding ς_1 into the power series of y and its derivatives, and identifying the coefficients of $\ddot{y}$ in the above equalities, one has $\varsigma_1 = 1$. Thus, $\mu + \dot{\mu}/\mu = \dot{u}/u$. It is clear that μ must be a function of u, that is, $\mu = \mu(u)$. Now $\mu(u) + \frac{\mu^{(1)}(u)\dot{u}}{\mu(u)} = \frac{\dot{u}}{u}$. Then $\mu(u) = 0$, $\frac{\mu^{(1)}(u)}{\mu(u)} = \frac{1}{u}$, which is impossible. The contradiction shows that $\dim V_{\max}(L) = 2$ and the system (20) is a minimal realization on U_0. It is clearly not accessible.

6. COMPUTATION OF $V_{\max}$

In this section the computation of $V_{\max}(L)$ is discussed for any given standard differential ideal L on U_0 with indices $(m_1, \ldots, m_p, s_1)$. Let $H_0 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i^{(j)} + \mathrm{d}L, \mathrm{d}u_r^{(l)} + \mathrm{d}L : 0 \leq j \leq m_i - 1; i = 1, \ldots, p; 0 \leq l \leq s_1 - 1; r = 1, \ldots, m\}$, $\bar{y} = (\mathrm{d}y_1, \mathrm{d}\dot{y}_1, \ldots, \mathrm{d}y_1^{(m_1-1)}, \mathrm{d}y_2, \mathrm{d}\dot{y}_2, \ldots, \mathrm{d}y_2^{(m_2-1)}, \ldots, \mathrm{d}y_p, \mathrm{d}\dot{y}_p, \ldots, \mathrm{d}y_p^{(m_p-1)})^T$, $\bar{u} = (\mathrm{d}u_1, \mathrm{d}\dot{u}_1, \ldots, \mathrm{d}u_1^{(s_1-1)}, \mathrm{d}u_2, \mathrm{d}\dot{u}_2, \ldots, \mathrm{d}u_2^{(s_1-1)}, \ldots, \mathrm{d}u_m, \ldots, \mathrm{d}u_m^{(s_1-1)})^T$. By the standard form one can compute the matrices A and B such that $\dot{\bar{y}} \equiv A\bar{y} + B\bar{u}(\mathrm{mod}\ \mathrm{d}L)$. Denote $\mathrm{d}u = (\mathrm{d}u_1, \ldots, \mathrm{d}u_m)^T$. Let V be the subspace of H_0 which is generated by the equivalent classes of the components of the vector $\bar{y} + C\bar{u}$, where C is a matrix such that the following equality holds for some matrices S and F

$$\dot{\overbrace{\bar{y} + C\bar{u}}} \equiv S(\bar{y} + C\bar{u}) + F\mathrm{d}u(\mathrm{mod}\ \mathrm{d}L). \tag{21}$$

One can find some (C, S, F) which satisfies (21) and may not be unique. In fact, substituting $\dot{\bar{y}} \equiv A\bar{y} + B\bar{u}(\mathrm{mod}\ \mathrm{d}L)$ into the above equality, one has

$$\dot{\bar{y}} + \dot{C}\bar{u} + C\dot{\bar{u}} \equiv A\bar{y} + (B + \dot{C})\bar{u} + C\dot{\bar{u}} \equiv S\bar{y} + SC\bar{u} + F\mathrm{d}u(\mathrm{mod}\ \mathrm{d}L).$$

Denote $N_1 = m_1 + \cdots + m_p, N_2 = s_1 m$, then $\bar{y}$ is an N_1-dimensional vector, A is of the size $N_1 \times N_1$, B and C are of the size $N_1 \times N_2$. Let $C = (C_1, \ldots, C_{N_2}), B = (B_1, \ldots, B_{N_2}), S = A$, $F = (F_1, \ldots, F_m)$. Then $(B+\dot{C})\bar{u}+C\dot{\bar{u}} \equiv AC\bar{u}+F\mathrm{d}u (\mathrm{mod}\; \mathrm{d}L)$, or equivalently

$$\sum_{i=1}^{m}\sum_{j=1}^{s_1}(B_{(i-1)s_1+j} + \dot{C}_{(i-1)s_1+j} - AC_{(i-1)s_1+j})\mathrm{d}u_i^{(j-1)} + \sum_{i=1}^{m}\sum_{j=1}^{s_1} C_{(i-1)s_1+j}\mathrm{d}u_i^{(j)} - \sum_{i=1}^{m} F_i \mathrm{d}u_i \equiv 0 (\mathrm{mod}\ \mathrm{d}L).$$

The above equality holds if

$$(B_{(i-1)s_1+1} + \dot{C}_{(i-1)s_1+1} - AC_{(i-1)s_1+1} - F_i)\mathrm{d}u_i + \sum_{j=2}^{s_1}(B_{(i-1)s_1+j} + \dot{C}_{(i-1)s_1+j} - AC_{(i-1)s_1+j} + C_{(i-1)s_1+j-1})\mathrm{d}u_i^{(j-1)} + C_{(i-1)s_1+s_1}\mathrm{d}u_i^{(s_1)} \equiv 0 (\mathrm{mod}\ \mathrm{d}L)$$

holds for each $i = 1, \ldots, m$. The new equalities hold if all the coefficients vanish, that is, the following equalities hold.

$$C_{(i-1)s_1+s_1} = 0; \tag{22}$$

$$C_{(i-1)s_1+j-1} = -B_{(i-1)s_1+j} - \dot{C}_{(i-1)s_1+j} + AC_{(i-1)s_1+j}, j = 2, \ldots, s_1; \tag{23}$$

$$F_i = B_{(i-1)s_1+1} + \dot{C}_{(i-1)s_1+1} - AC_{(i-1)s_1+1}, i = 1, \ldots, m. \tag{24}$$

The equalities in (22 − 24) give the procedure for determining the matrices C and F. Therefore, one can use (22 − 24) to construct the subspace V such that (21) holds, that is, $\dot{V} \subseteq V + \overline{\mathcal{U}^0}$. It is now shown that this subspace is just $V_{\max}(L)$.

Proposition 6.1. The above V equals $V_{\max}(H_0, L)$, and $\dim V = m_1 + m_2 + \cdots + m_p$.

Proof. For convenience denote $V_{\max}(H_0, L)$ by V', then $V' \cap \mathcal{U} = 0$ must hold. In fact, if there exists some nonzero $w_1 + \mathrm{d}L \in V' \cap \mathcal{U}$, then let the highest order of $\mathrm{d}u$ in w_1 be k. By $\dot{V}' \subseteq V' + \overline{\mathcal{U}^0}$ and computing the derivative of w_1 one obtains a nonzero element $w_2 + \mathrm{d}L \in V' \cap (\overline{\mathcal{U}^{k+1}} \backslash \overline{\mathcal{U}^k})$. By induction one has $w_i + \mathrm{d}L \in V' \cap (\overline{\mathcal{U}^{k+i-1}} \backslash \overline{\mathcal{U}^{k+i-2}}), i \geq 2$. A similar proof as Lemma 4.1 leads to a contradiction. Therefore, $V' \cap \overline{\mathcal{U}} = 0$.

The maximality of V' implies that $V \subseteq V'$. If $V \neq V'$, then $\dim V < \dim V'$. Note that for V' one can use elementary operations to transform a basis of V' into echelon form. Any nonzero element of the echelon form does not belong to $\overline{\mathcal{U}}$ since $V' \cap \overline{\mathcal{U}} = 0$, therefore it contains some term $\mathrm{d}y_i^{(j)} + \mathrm{d}L$. However, V has a basis which consists of the equivalents classes of the elements in the vector $\bar{y} + C\bar{u}$, and it is the greatest subspace in H_0 such that each nonzero element in the echelon form contains some term $\mathrm{d}y_i^{(j)} + \mathrm{d}L$. Thus, $\dim V' \leq \dim V$, which contradicts $\dim V < \dim V'$. This contradiction shows that $V = V'$.

The result $\dim V = m_1 + m_2 + \cdots + m_p$ comes from the clear fact that the equivalent classes of the components of $\bar{y} + C\bar{u}$ are linearly independent. □

For any given (6), one can use the following steps to check the realizability and compute the realization and minimal realization when it is realizable.

1. Compute the ideals $\mathcal{I}, \overline{\mathcal{I}}$ and their standard forms on some contractible open set U_1 which contains the operating point P_0. If they are not standard, then there is no realization on U_1, and one can consider a smaller open set. Otherwise, go to the next step.

2. Compute the ideal L, which satisfies the conditions in Theorem 4.4 or equals L_0 and satisfies the conditions in Theorem 4.8, and its standard form on U_1. Then compute $V_{\max}(L)$ by (22–24).

3. Check by the Frobenius Theorem [5] if $V_{\max}(L)$ is integrable. Check also if $\mathrm{d}y_1 + \mathrm{d}L, \ldots, \mathrm{d}y_p + \mathrm{d}L \in V_{\max}(L)$. If both are true, then one obtains a minimal realization on U_1 by letting $x_1 = \xi_1, \ldots, x_n = \xi_n$ and computing $f(\cdot, u)$ and $h(\cdot)$, where $V_{\max}(L) = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_1 + \mathrm{d}L, \ldots, \mathrm{d}\xi_n + \mathrm{d}L\}$ and the functions f, h are obtained by Poincaré Lemma.

Example 3.3. (continued) Now compute the system (15) from the input-output equation $\ddot{y} - \dot{u}(3\dot{y} - 4u)/(2\dot{y} - 3u) = 0$ in Example 3.3 on the open set U_0. Let L be the differential ideal generated by $\ddot{y} - \dot{u}(3\dot{y} - 4u)/(2\dot{y} - 3u)$. It is clear that $m = 1, p = 1, m_1 = 2, s_1 = 2, N_1 = 2, N_2 = 2$, $\bar{y} = (\mathrm{d}y, \mathrm{d}\dot{y})^T, \bar{u} = (\mathrm{d}u, \mathrm{d}\dot{u})^T$. Computing differentials one has

$$\begin{aligned} \dot{\bar{y}} &\equiv \begin{pmatrix} \mathrm{d}\dot{y} \\ \alpha_1 \mathrm{d}\dot{y} + \alpha_2 \mathrm{d}u + \alpha_3 \mathrm{d}\dot{u} \end{pmatrix} \\ &\equiv A\bar{y} + B\bar{u} (\mathrm{mod}\ \mathrm{d}L), \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & \alpha_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ \alpha_2 & \alpha_3 \end{pmatrix},$$

$$\alpha_1 = \frac{-\dot{u}u}{(2\dot{y} - 3u)^2}, \alpha_2 = \frac{\dot{u}\dot{y}}{(2\dot{y} - 3u)^2}, \alpha_3 = \frac{3\dot{y} - 4u}{2\dot{y} - 3u}.$$

By (22–24) one has

$$C_2 = 0, C_1 = AC_2 - B_2 - \dot{C}_2 = -B_2 = \begin{pmatrix} 0 \\ -\alpha_3 \end{pmatrix},$$

$$\bar{y} + C\bar{u} \equiv \begin{pmatrix} \mathrm{d}y \\ \mathrm{d}\dot{y} - \alpha_3 \mathrm{d}u \end{pmatrix} (\mathrm{mod}\ \mathrm{d}L).$$

Therefore, $V_{\max} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y + \mathrm{d}L, \mathrm{d}\dot{y} - \alpha_3 \mathrm{d}u + \mathrm{d}L\}$. Let $w_1 = \mathrm{d}y, w_2 = \mathrm{d}\dot{y} - \alpha_3 \mathrm{d}u$, then

$$\mathrm{d}\alpha_3 = \frac{-u}{(2\dot{y} - 3u)^2}\mathrm{d}\dot{y} + \frac{\dot{y}}{(2\dot{y} - 3u)^2}\mathrm{d}u, \mathrm{d}w_2 = -\mathrm{d}\alpha_3 \wedge \mathrm{d}u = \frac{-u}{(2\dot{y} - 3u)^2}\mathrm{d}\dot{y} \wedge \mathrm{d}u.$$

Thus, $\mathrm{d}w_2 \wedge w_2 = 0$, w_2 is closed by Poincaré Lemma. In fact, $(2\dot{y} - 3u)w_2 = \mathrm{d}(\dot{y}^2 - 3u\dot{y} + 2u^2)$. Hence $V_{\max} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y + \mathrm{d}L, \mathrm{d}(\dot{y}^2 - 3u\dot{y} + 2u^2) + \mathrm{d}L\}$ and one can let $x_1 = y, x_2 = \dot{y}^2 - 3u\dot{y} + 2u^2$. Now $y = x_1$, $\dot{x}_2 = 0$, $\dot{x}_1 = \dot{y} = (3u \pm \sqrt{u^2 + 4x_2})/2$. Suppose $\dot{x}_1 = \dot{y} = (3u - \sqrt{u^2 + 4x_2})/2$, then $\dot{\xi}_1 - f_1(\xi, u) = (2\dot{y} - 3u) > 0$ on U_0. Note that $2\dot{y} - 3u$ is an invertible element in R_1 and it belongs to J, hence the corresponding $J = R_1$ and can not be contained in $\overline{\mathcal{I}} = \langle y, u\rangle$. In this case it is not a realization. Thus, $\dot{x}_1 = \dot{y} = (3u + \sqrt{u^2 + 4x_2})/2$, and one obtains finally the system (15).

Example 6.2. (Moog et al. [40]) Let $\Phi = \ddot{y} - \dot{y}u - y\dot{u} = 0$, U_0 be the whole Euclidian space, and the origin be the operating point. Then $\mathcal{I} = \langle \ddot{y} - \dot{y}u - y\dot{u}\rangle$, $\overline{\mathcal{I}} = \langle \dot{y} - yu\rangle$. For $\mathcal{I}$ one has $H_0(\mathcal{I}) = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y + \mathrm{d}\mathcal{I}, \mathrm{d}\dot{y} + \mathrm{d}\mathcal{I}, \mathrm{d}u + \mathrm{d}\mathcal{I}, \mathrm{d}\dot{u} + \mathrm{d}\mathcal{I}\}$, $\bar{y} = (\mathrm{d}y, \mathrm{d}\dot{y})^T, \bar{u} = (\mathrm{d}u, \mathrm{d}\dot{u})^T$. It is easy to compute

$$\dot{\bar{y}} \equiv \begin{pmatrix} \mathrm{d}\dot{y} \\ u\mathrm{d}\dot{y} + \dot{u}\mathrm{d}y + \dot{y}\mathrm{d}u + y\mathrm{d}\dot{u} \end{pmatrix} \equiv A\bar{y} + B\bar{u} (\mathrm{mod}\ \mathrm{d}\mathcal{I}),$$

where

$$A = \begin{pmatrix} 0 & 1 \\ \dot{u} & u \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ \dot{y} & y \end{pmatrix}.$$

It follows from (22 − 24) and the Frobenius Theorem that $V_{\max} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y + \mathrm{d}\mathcal{I}, \mathrm{d}\dot{y} - y\mathrm{d}u + \mathrm{d}\mathcal{I}\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y + \mathrm{d}\mathcal{I}, \mathrm{d}(\dot{y} - yu) + \mathrm{d}\mathcal{I}\}$ which is integrable. Let $x_1 = y, x_2 = \dot{y} - yu$, then one has a system

$$\begin{cases} \dot{x}_1 &= x_2 + ux_1, \\ \dot{x}_2 &= 0, \\ y &= x_1. \end{cases}$$

The corresponding ideal $J = s(J) = \mathcal{I}$, therefore it is a realization of $\Phi = 0$. However, it is not a minimal realization. Note that the ideal $\overline{\mathcal{I}}$ has a generator $\dot{y} - yu$, and let $H_0(\overline{\mathcal{I}}) = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y + \mathrm{d}\overline{\mathcal{I}}, \mathrm{d}u + \mathrm{d}\overline{\mathcal{I}}\}$, then C is a 1×1 matrix. It is easy to show that $C = 0$ and $V_{\max} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y + \mathrm{d}\mathcal{I}\}$. Now the following minimal realization is obtained

$$\begin{cases} \dot{x} &= ux, \\ y &= x. \end{cases}$$

The following linear system is not realizable since the transfer function matrix is not strictly proper. The corresponding $V_{\max}$ is computed to check how the conditions of Theorem 4.8 are violated.

Example 6.3. Consider the system

$$\begin{cases} \Phi_1 &= \dot{y}_1 - u_1 - \ddot{y}_2 = 0, \\ \Phi_2 &= \dot{y}_2 - u_2 = 0, \end{cases}$$

with U_0 the whole Euclidean space, and the origin the operating point, then $I_\Phi = \langle \dot{y}_1 - u_1 - \ddot{y}_2, \dot{y}_2 - u_2\rangle$. Now compute the standard form of I_Φ. The standard form of I_Φ is

$$\begin{cases} \dot{y}_1 - u_1 - \dot{u}_2, \\ \dot{y}_2 - u_2. \end{cases}$$

Then it is straightforward to check $\mathcal{I} = \overline{\mathcal{I}}$. By the standard form one has

$$H_0(\overline{\mathcal{I}}) = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1 + \mathrm{d}\overline{\mathcal{I}}, \mathrm{d}y_2 + \mathrm{d}\overline{\mathcal{I}}, \mathrm{d}u_1 + \mathrm{d}\overline{\mathcal{I}}, \mathrm{d}u_2 + \mathrm{d}\overline{\mathcal{I}}, \mathrm{d}\dot{u}_1 + \mathrm{d}\overline{\mathcal{I}}, \mathrm{d}\dot{u}_2 + \mathrm{d}\overline{\mathcal{I}}\}.$$

One can compute that $V_{\max} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1 - \mathrm{d}u_2 + \mathrm{d}\overline{\mathcal{I}}, \mathrm{d}y_2 + \mathrm{d}\overline{\mathcal{I}}\}$. Clearly $\mathrm{d}y_1 + \mathrm{d}\overline{\mathcal{I}} \notin V_{\max}$, and the system is not realizable by Theorem 4.8.

The following MIMO system is realizable and it shows the general procedure to obtain a realization.

Example 6.4. Consider the system

$$\begin{cases} \Phi_1 & := \dot{y}_1 - u_1 = 0, \\ \Phi_2 & := u_1\ddot{y}_2 - \dot{u}_1\dot{y}_2 - u_2u_1^2 = 0, \end{cases} \tag{25}$$

on the open set $U_0 := \{(y_1, \dot{y}_1,\ y_2, \dot{y}_2, \ddot{y}_2, u_1,\ \dot{u}_1,\ u_2) : u_1 > 0\}$ with an operating point $P_0 = (0,0,0,0,0,1,0,0)$. System (25) is easily transformed into the standard form:

$$\begin{cases} \Phi_1' & := \dot{y}_1 - u_1 = 0, \\ \Phi_2' & := \ddot{y}_2 - \frac{\dot{u}_1}{u_1}\dot{y}_2 - u_2u_1 = 0. \end{cases}$$

Let L be the differential idea generated by Φ_1' and Φ_2', then $\overline{y} = (\mathrm{d}y_1,\ \mathrm{d}y_2,\ \mathrm{d}\dot{y}_2)^T$, $\overline{u} = (\mathrm{d}u_1,\ \mathrm{d}\dot{u}_1,\ \mathrm{d}u_2, \mathrm{d}\dot{u}_2)^T$, $\dot{\overline{y}} \equiv A\overline{y} + B\overline{u} (\mathrm{mod}\ \mathrm{d}L)$,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{\dot{u}_1}{u_1} \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ u_2 - \frac{\dot{y}_2\dot{u}_1}{u_1^2} & \frac{\dot{y}_2}{u_1} & u_1 & 0 \end{pmatrix},$$

$$\overline{y} + C\overline{u} \equiv \overline{y} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\dot{y}_2}{u_1} & 0 & 0 & 0 \end{pmatrix} \overline{u} \equiv \begin{pmatrix} \mathrm{d}y_1 \\ \mathrm{d}y_2 \\ \mathrm{d}\dot{y}_2 - \frac{\dot{y}_2}{u_1}\mathrm{d}u_1 \end{pmatrix} (\mathrm{mod}\ \mathrm{d}L),$$

and $V_{\max} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1 + \mathrm{d}L, \mathrm{d}y_2 + \mathrm{d}L, \mathrm{d}\dot{y}_2 - \frac{\dot{y}_2}{u_1}\mathrm{d}u_1 + \mathrm{d}L\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1 + \mathrm{d}L, \mathrm{d}y_2 + \mathrm{d}L, u_1\mathrm{d}(\frac{\dot{y}_2}{u_1}) + \mathrm{d}L\}$ which is obviously integrable. Let $x_1 = y_1, x_2 = y_2, x_3 = \frac{\dot{y}_2}{u_1}$, then the following system of state equations is formulated

$$\begin{cases} \dot{x}_1 & = u_1, \quad \dot{x}_2 & = x_3u_1, \quad \dot{x}_3 & = u_2, \\ y_1 & = x_1, \quad y_2 & = x_2. \end{cases} \tag{26}$$

By Definition 3.1 system (26) is a realization of (25) on U_0. Furthermore, it follows from Theorem 5.3 and the observability and accessibility of (26) that it is also a minimal realization of (25) on U_0.

7. CONCLUSIONS

In this paper, the realization problem has been studied for multi-input multi-output nonlinear systems. A standard form differential ideal is defined for a given set of input-output differential equations. The corresponding notion of differential closure yields a natural definition of realization as stated in Definition 3.1. The Appendix proves that this new definition coincides completely with the linear theory. Definition 3.1 also generalizes the idea of transfer equivalence in [45] and [40] to multi-input multi-output systems. That is, if a system of state equations is a realization of a single input-output equation in the sense of [40], then it is also a realization in the sense of Definition 3.1. Criteria for the realizability have been developed in the main theorems. The definition of minimal realization is also presented which is consistent with the linear theory in the sense that it is minimal with respect to the state space dimension. By this definition, an observable and accessible realization is minimal but the converse is not true. An algorithm for the computation of realization and minimal realization is provided, and the examples have shown its effectiveness. The problem whether the minimal realization at a given operating point is unique, up to a diffeomorphism, remains open for further research.

APPENDIX

A. Operations in $R_1(U_0)$

For any two elements g_1, $g_2 \in R_1(U_0)$, one can assume that g_1 is real analytic on a contractible open $U_0' \subseteq \mathbb{R}^{n'}$ and g_2 is real analytic on a contractible open set $U_0'' \subseteq \mathbb{R}^{n''}$, and there are two projection mappings P_1 and P_2 which map U_0' and U_0'' onto U_0 respectively. Then there exists a contractible open set $U_0''' \subseteq \mathbb{R}^{n'''}$ and two projection mappings $P_3' : \mathbb{R}^{n'''} \to \mathbb{R}^{n'}$, $P_3'' : \mathbb{R}^{n'''} \to \mathbb{R}^{n''}$, such that

$$P_1(P_3'(U_0''')) = P_2(P_3''(U_0''')) = U_0, P_1(P_3'(x)) = P_2(P_3''(x)) \text{ for any } x \in U_0'''. \quad (27)$$

The triple (U_0''', P_3', P_3'') with the property (27) may not be unique; however, there does exist such a unique triple $(\overline{U_0'''}, \overline{P_3'}, \overline{P_3''})$ satisfying the following:

(i) $\overline{U_0'''}$ is open and contractible in $\mathbb{R}^{\overline{n'''}}$, $\overline{P_3'}$ is a mapping from $\mathbb{R}^{\overline{n'''}}$ to $\mathbb{R}^{n'}$, $\overline{P_3''}$ is a mapping from $\mathbb{R}^{\overline{n'''}}$ to $\mathbb{R}^{n''}$, $P_1(P_3'(U_0''')) = P_2(P_3''(U_0''')) = U_0$, and $P_1 \circ \overline{P_3'} = P_2 \circ \overline{P_3''}$;

(ii) For any (U_0''', P_3', P_3'') satisfying (27), there exists a mapping θ from $\mathbb{R}^{n'''}$ to $\mathbb{R}^{\overline{n'''}}$ such that
$$P_3' = \overline{P_3'} \circ \theta, P_3'' = \overline{P_3''} \circ \theta.$$

Proving the existence of $(\overline{U_0'''}, \overline{P_3'}, \overline{P_3''})$ is simple; actually one can define $\overline{U_0'''} = (U_0' \times U_0'')/\sim$ and $\overline{P_3'}([x_1, x_2]) = x_1, \overline{P_3''}([x_1, x_2]) = x_2$, where x_1 and x_2 are arbitrary points in U_0' and U_0'' respectively, $\sim$ is the equivalent relation defined by $\{(x_1, x_2) : P_1(x_1) - P_2(x_2) = 0, x_1 \in U_0', x_2 \in U_0''\}$, and $[x_1, x_2]$ denotes the equivalent class of $(x_1, x_2) \in U_0' \times U_0''$ with respect to the relation $\sim$. Note that $\overline{U_0'''}$ can be embedded into some Euclidean space $\mathbb{R}^{\overline{n'''}}$. By the universal property of Cartesian product, properties in the above (i) and (ii) hold. The uniqueness of $(\overline{U_0'''}, \overline{P_3'}, \overline{P_3''})$ follows directly from the property in (ii). This construction is also similar to the construction of fibred product of schemes in algebraic geometry (see Section 2, Chapter 2 of [25]).

Now one can define addition and multiplication of g_1 and g_2 on $\overline{U_0'''}$ formally as the usual operations of real functions on $\mathbb{R}^{\overline{n'''}}$, that is, for any $x_1 \in U_0', x_2 \in U_0''$, define

$$(g_1 + g_2)([x_1, x_2]) = g_1(x_1) + g_2(x_2), (g_1 g_2)([x_1, x_2]) = g_1(x_1) g_2(x_2).$$

By the properties in (i) and (ii), the above addition and multiplication are well defined. Then it is simple to check that $R_1(U_0)$ is an integral domain.

Note that any function in $R_1(U_0)$ has only a finite number of variables from the infinite set $\{y_i^{(j)}, u_r^{(l)} : j, l \geq 0; i = 1, \ldots, p; r = 1, \ldots, m\}$, therefore derivatives of any function in $R_1(U_0)$ still belongs to $R_1(U_0)$. Then by the usual differentiation operation, $R_1(U_0)$ becomes a differential ring.

B. Realization of Linear Systems

In the following it is proved that Definition 3.1 is consistent with the definition of realization of linear systems. Consider the following linear system

$$\dot{x} = Ax + Bu, \tag{28}$$
$$y = Cx. \tag{29}$$

in which $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. Let U_0 be the whole Euclidean space, and the operating point P_0 be the origin of U_0. Without loss of generality, assume that (C, A) is observable. Then the matrix $Q = (C^T, (CA)^T, \ldots, (CA^{n-1})^T)^T$ has rank n. Let P be the left inverse of Q, and define $\overline{y} = (y^T, \dot{y}^T, \ldots, (y^{(n-1)})^T)^T$, $\overline{u} = (u^T, \dot{u}^T, \ldots, (u^{(n-2)})^T)^T$, and

$$W = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ CA^{n-2}B & CA^{n-3}B & \cdots & CB \end{pmatrix}.$$

Then x can be solved as

$$x = \xi(y, u) = P\overline{y} - PW\overline{u}. \tag{30}$$

The corresponding ideal J defined in Section 2.3 is $J = \langle \dot{\xi} - A\xi - Bu, y - C\xi \rangle$.

For any system of linear equations of the form

$$\Phi_i := \sum_{k=0}^{k_0} F_k^i y^{(k)} - \sum_{j=0}^{s_0} H_j^i u^{(j)} = 0, i = 1, \ldots, p, \tag{31}$$

suppose the Laplace transform of the system is $M(s)Y(s) - N(s)U(s) = 0$, where $Y(s)$ (respectively, $U(s)$) is the Laplace transform of y (respectively, u), and the elements of the matrices $M(s), N(s)$ are polynomials in the variable s. Let $\mathbb{R}[s]$ be the ring of polynomials of s and $\mathbb{R}(s)$ its fraction field. Suppose (31) satisfies the Standard Form Hypothesis, then the matrix $M(s)$ is invertible over $\mathbb{R}(s)$ and the matrix $M(s)^{-1}N(s)$ is the so-called transfer matrix. The system (28−29) is an observable realization if $M(s)^{-1}N(s) = C(sI-A)^{-1}B$. Note also that the ideal I_Φ defined in Section 2.3 is $I_\Phi = \langle \Phi_1, \dots, \Phi_p \rangle$.

It is clear that for linear systems the set of all the possible input-output equations are linear equations in the derivatives of y and u. Thus, consider the following set

$$R_0 = \mathrm{span}_{\mathbb{R}}\{y_i^{(j)}, u_r^{(l)} : \; j \geq 0, l \geq 0, i = 1, \dots, p; r = 1, \dots, m\} \tag{32}$$

instead of R_1. The following results are needed to show that Definition 3.1 is consistent with the linear theory.

Lemma B.1 For any functions $\phi_1, \dots, \phi_r \in R_0$, let L be the differential ideal of R_1 which is generated by $\phi_1, \dots, \phi_r$. Then

$$\begin{aligned} s(L) \cap R_0 = L \cap R_0 = \; & \mathrm{span}_{\mathbb{R}}\{\phi_i^{(j)} : \; j \geq 0, i = 1, \dots, r\}; \\ \overline{s(L)} \cap R_0 = \overline{L} \cap R_0 = \; & \{g \in R_0 : \text{ there exist some real numbers } \lambda_0, \lambda_1, \\ & \dots, \lambda_l \text{ such that } \textstyle\sum_{j=0}^{l} \lambda_j g^{(j)} \in L\}. \end{aligned}$$

P r o o f. The equalities follow directly from the definitions of L, $s(L)$, $\overline{L}, \overline{s(L)}$, and the fact that R_0 contains no product of any $\phi_{i_1}^{(j_1)}$ and $\phi_{i_2}^{(j_2)}$. □

Denote by $\mathcal{L}(\cdot)$ the Laplace transform, $\mathcal{L}(L)$ the set of Laplace transforms of all the elements of L. For simplicity, it is assumed that the initial value is zero whenever a Laplace transform is applied to a function.

Lemma B.2 With the notations above one has

$$\begin{aligned} & \mathcal{L}(R_0 \cap \overline{I_\Phi}) \\ = \; & \{\mathcal{L}(g) : \; g \in R_0, \text{and there exist polynomials } \lambda(s), a_1(s), \dots, a_p(s) \in \mathbb{R}[s] \text{ such} \\ & \text{that } \lambda(s)\mathcal{L}(g) = (a_1(s), \dots, a_p(s))(M(s)Y(s), -N(s)U(s))\} \\ \cong \; & \{\beta(s) : \; \beta(s) \text{ is a row vector in } \mathbb{R}[s]^{p+m} \text{ such that there exist a polynomial} \\ & \lambda(s) \in \mathbb{R}[s] \text{ and a row vector } \alpha(s) \text{ in } \mathbb{R}[s]^p \text{ which satisfy } \lambda(s)\beta(s) \\ & = \alpha(s)(M(s), -N(s))\} (\text{isomorphic as } \mathbb{R}\text{-vector spaces}), \mathcal{L}(R_0 \cap \overline{J}) \\ = \; & \{\mathcal{L}(g) : \; g \in R_0, \text{and there exist a row vector } \alpha(s) \in \mathbb{R}[s]^p \text{ and a polynomial} \\ & \lambda(s) \in \mathbb{R}[s] \text{such that } \lambda(s)\mathcal{L}(g) = \alpha(s)G(s)(Y(s)^T, U(s)^T)^T\} \\ \cong \; & \{\beta(s) : \; \beta(s) \text{ is a row vector in } \mathbb{R}[s]^{p+m} \text{ such that there exist a polynomial} \\ & \lambda(s) \in \mathbb{R}[s] \text{ and a row vector } \alpha(s) \text{ in } \mathbb{R}[s]^p \text{ which satisfy } \lambda(s)\beta(s) = \alpha(s)G(s)\} \\ & (\text{isomorphic as } \mathbb{R}\text{-vector spaces}), \end{aligned}$$

where $G(s)$ is defined as

$$G(s) = \begin{pmatrix} G_1(s) & G_2(s) \\ G_3(s) & G_4(s) \end{pmatrix},$$

and $G_1(s) = I_p - CP(I_p, sI_p, \ldots, s^{n-1}I_p)^T$, $G_2(s) = CPW(I_m, sI_m, \ldots, s^{n-2}I_m)^T$, $G_3(s) = P(sI_p, s^2I_p, \ldots, s^nI_p)^T - AP(I_p, sI_p, \ldots, s^{n-1}I_p)^T$, $G_4(s) = -PW(sI_m, s^2I_m, \ldots, s^{n-1}I_m)^T + APW(I_m, sI_m, \ldots, s^{n-2}I_m)^T - B$. Here I_p is the $p \times p$ identity matrix.

Proof. The equality about $\mathcal{L}(R_0 \cap \overline{I_\Phi})$ follows from Lemma B.1, while the equality for $\mathcal{L}(R_0 \cap \overline{J})$ is due to Lemma B.1 and the following computation $\mathcal{L}(y - C\xi) = \mathcal{L}(y - CP\overline{y} + CPW\overline{u}) = G_1(s)Y(s) + G_2(s)U(s)$, $\mathcal{L}(\dot{\xi} - A\xi - Bu) = G_3(s)Y(s) + G_4(s)U(s)$. The two isomorphisms follow from the two equalities respectively. □

Lemma B.3 The equality $\mathcal{L}(R_0 \cap \overline{I_\Phi}) = \mathcal{L}(R_0 \cap \overline{J})$ holds if and only if $C(sI - A)^{-1}B = M(s)^{-1}N(s)$.

Proof. By Lemma B.2 one needs only to prove that $C(sI - A)^{-1}B = M(s)^{-1}N(s)$ holds if and only if the two row spaces spanned by the rows of $(M(s), -N(s))$ and $G(s)$ over $\mathbb{R}(s)$ respectively are equal. Let

$$P_1 = \begin{pmatrix} I_p, & C(sI_n - A)^{-1} \\ 0, & I_n \end{pmatrix}, P_2 = \begin{pmatrix} I_p, & 0 \\ -(sI_n - A)P(I_p, sI_p, \ldots, s^{n-1}I_p)^T, & I_n \end{pmatrix},$$

then P_1 and P_2 are elementary matrices over $\mathbb{R}(s)$ and

$$P_2P_1G(s) = \begin{pmatrix} I_p, & -C(sI_n - A)^{-1}B \\ 0, & P_3 \end{pmatrix},$$

where $P_3 = (sI_n - A)P(I_p, sI_p, \ldots, s^{n-1}I_p)^TC(sI_n - A)^{-1}B - (sI_n - A)PW(I_m, sI_m, \ldots, s^{n-2}I_m)^T - B$.

In the following we show that $P_3 = 0$. Suppose the matrix P is partitioned into the block form $P = (Z_1, Z_2, \ldots, Z_n)$ such that the condition $PQ = I_n$ can be expressed as $I_n = \sum_{i=1}^n Z_iCA^{i-1}$. Let $W_1 = \Big(0, C^T, (CA + sC)^T, \ldots, (CA^{n-2} + sCA^{n-3} + \cdots + s^{n-2}C)^T\Big)^T$, then $W(I_m, sI_m, \ldots, s^{n-2}I_m)^T = W_1(sI_n - A)(sI_n - A)^{-1}B$ and therefore

$$P_3 = (sI - A)\left(P(I_p, sI_p, \ldots, s^{n-1}I_p)^TC - PW_1(sI_n - A)\right)(sI_n - A)^{-1}B - B.$$

To show that $P_3 = 0$ one needs only to prove $P_4 := P(I_p, sI_p, \ldots, s^{n-1}I_p)^TC - PW_1(sI_n - A) = I_n$. In fact,

$$\begin{aligned} P_4 = & \Big(\sum_{i=1}^n s^{i-1}Z_i\Big)C - \Big((Z_2C + Z_3CA + Z_4CA^2 + \cdots + Z_nCA^{n-2}) \\ & + s(Z_3C + Z_4CA + \cdots + Z_nCA^{n-3}) + \cdots + s^{n-2}Z_nC\Big)(sI_n - A) \\ = & \Big(\sum_{i=1}^n s^{i-1}Z_i\Big)C - \Big(-\sum_{i=2}^n Z_iCA^{i-1} + s\big(\sum_{i=2}^n Z_iCA^{i-2} - \sum_{i=3}^n Z_iCA^{i-3+1}\big) \\ & + \cdots + s^{n-2}\Big(\sum_{i=n-1}^n Z_iCA^{i-(n-1)} - \sum_{i=n} Z_iCA^{i-n+1}\Big) + s^{n-1}\sum_{i=n} Z_iCA^{i-n}\Big) \end{aligned}$$

$$= \Big(\sum_{i=1}^{n} s^{i-1}Z_i)C - (Z_1C - I_n + sZ_2C + s^2Z_3C + \cdots s^{n-2}Z_{n-1}C + s^{n-1}Z_nC\Big)$$
$$= I_n.$$

Therefore, $P_3 = 0$ and the row space of $G(s)$ over $\mathbb{R}(s)$, denoted by S_1, is spanned by the rows of $(I_p, -C(sI_n - A)^{-1}B)$. Note that $(M(s), -N(s)) = M(s)(I_p, -M(s)^{-1}N(s))$, hence the row space of $(M(s), -N(s))$, denoted by S_2, is spanned by the rows of $(I_p, M(s)^{-1}N(s))$. It follows that $S_1 = S_2$ if and only if $M(s)^{-1}N(s) = C(sI_n - A)^{-1}B$. □

Theorem B.4 With the notations above, $R_0 \cap \overline{\mathcal{I}} = R_0 \cap \overline{s(J)}$ if and only if $\overline{\mathcal{I}} = \overline{s(J)}$. The equation $C(sI - A)^{-1}B = M(s)^{-1}N(s)$ holds if and only if $R_0 \cap \overline{\mathcal{I}} = R_0 \cap \overline{s(J)}$.

In plain words, Definition 3.1 reduces to equality of transfer function matrices computed either from the input-output equations or from the state equations. Therefore, this theorem shows that Definition 3.1 is consistent with the linear theory (see the definition of realization for linear systems in [3]).

Proof of Theorem B.4 Since I_Φ and J are generated by the elements of R_0, it is clear that $R_0 \cap \overline{\mathcal{I}} = R_0 \cap \overline{s(J)}$ if and only if $\overline{\mathcal{I}} = \overline{s(J)}$. Note that $R_0 \cap \overline{I_\Phi} = R_0 \cap \overline{J}$ if and only if $\mathcal{L}(R_0 \cap \overline{I_\Phi}) = \mathcal{L}(R_0 \cap \overline{J})$, then the second result follows from Lemma B.1 and Lemma B.3. □

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REFERENCES

[1] E. Aranda-Bricaire, C. H. Moog, J.-B. Pomet: A linear algebraic framework for dynamic feedback linearization. IEEE Trans. Automat. Control *40* (1995), 127–132.

[2] Z. Bartosiewicz: A new setting for polynomial continuous-time systems, and a realization theorem. IMA J. Math. Control Inform. Theory *2* (1985), 71–80.

[3] F. M. Callier and C. A. Desoer: Linear System Theory. Springer, New York 1991.

[4] F. Celle and J. P. Gauthier: Realizations of nonlinear analytic input-output maps. Math. Systems Theory *19* (1987), 227–237.

[5] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick: Analysis, Manifolds and Physics, Part I: Basics. Elsevier Science Publishers, Amsterdam 1981.

[6] G. Conte, C. H. Moog, and A. M. Perdon: Nonlinear Control Systems. Lecture Notes in Control and Inform. Sci. *242*, Springer, New York 1990.

[7] G. Conte, A. M. Perdon, and C. H. Moog: The differential field associated to a general analytic nonlinear dynamical system. IEEE Trans. Automat. Control *38* (1993), 1120–1124.

[8] D. A. Cox, J. B. Little, and D. O'Shea: Ideals, varieties, and algorithms. Second edition. Springer, New York 1996.

[9] P. E. Crouch and F. Lamnabhi-Lagarrigue: State space realizations of nonlinear systems defined by input output differential equations. In: Analysis and Optimization Systems (A. Bensousan and J. L. Lions, eds.), Lecture Notes in Control and Inform. Sci. *111*, 138–149.

[10] P. E. Crouch and F. Lamnabhi-Lagarrigue: Realizations of input output differential equations. In: Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing II Proceeding MTNS-91, Mita Press 1992.

[11] P. E. Crouch, F. Lamnabhi-Lagarrigue, and D. Pinchon: A realization algorithm for input output systems. Internat. J. Control *62* (1995), 941–960.

[12] E. Delaleau and W. Respondek: Lowering the orders of derivatives of controls in generalized state space systems. J. Math. Systems Estim. Control *5* (1995), 1–27.

[13] M. C. Di Benedetto, J. Grizzle, and C. H. Moog: Rank invariants of nonlinear systems. SIAM J. Control Optim. *27* (1989), 658–672.

[14] S. Diop: A state elimination procedure for nonlinear systems. In: New Trends in Nonlinear Control Theory, (J. Decusse, M. Fliess, A. Isidori,D. Leborgne, eds.), Lecture Notes in Control and Inform. Sci. *122* (1989), 190–198.

[15] S. Diop and F. Fliess: Nonlinear observability, identification, and persistent trajectories. In: Proc. 30th CDC, Brighton 1991.

[16] M. Fliess: Realizations of nonlinear systems and abstract transitive Lie algebras. Bull. Amer. Math. Soc. (N. S.) *2* (1980), 444–446.

[17] M. Fliess: Some remarks on nonlinear invertibility and dynamic state feedback. In: Theory and Applications of Nonlinear Control Systems, also in: Proc. MTNS'85, (C. Byrnes and A. Lindquist, eds.), North Holland, Amsterdam 1986.

[18] M. Fliess: A note on the invertibility of nonlinear input output systems. Syst. Control Lett. *8* (1986), 147–151.

[19] M. Fliess: Automatique et corps différentiels. Forum Math. *1* (1986), 227–238.

[20] M. Fliess: Generalized controller canonical forms for linear and nonlinear dynamics. IEEE Trans. Autom. Control *35* (1990), 994–1001.

[21] M. Fliess and I. Kupka: Finiteness conditions for nonlinear input output differential systems. SIAM J. Control Optim. *21* (1983), 721–728.

[22] S. T. Glad: Nonlinear state space and input output descriptions using differential polynomials. In: New Trends in Nonlinear Control Theory, (J. Decusse, M. Fliess, A. Isidori and D. Leborgne, eds.), Lecture Notes in Control and Inform. Sci. *122* (1989), 182–189.

[23] M. Halas and M. Huba: Symbolic computation for nonlinear systems using quotients over skew polynomial ring. In: 14th Mediterranean Conference on Control and Automation, Ancona 2006.

[24] M. Halas: An algebraic framework generalizing the concept of transfer functions to nonlinear systems. Automatica *44* (2008), 1181–1190.

[25] R. Hartshorne: Algebraic Geometry. Springer, New York 1977.

[26] R. Hermann and A. J. Krener: Nonlinear controllability and observability. IEEE Trans. Automat. Control *22* (1977), 728–740.

[27] A. Isidori: Nonlinear Control Systems. Third edition. Springer, New York 1995.

[28] A. Isidori, P. D'Alessandro, and A. Ruberti: Realization and structure theory of bilinear dynamical systems. SIAM J. Control *13* (1974), 517–535.

[29] N. Jacobson: Basic Algebra I. W. H. Freeman and Company, San Francisco 1974.

[30] B. Jakubczyk: Existence and uniqueness of realizations of nonlinear systems. SIAM J. Control Optim. *18* (1980), 455–471.

[31] B. Jakubczyk: Construction of formal and analytic realizations of nonlinear systems. In: Feedback Control of Linear and Nonlinear Systems. Lecture Notes in Control and Inform. Sci. *39*, Springer 1982.

[32] B. Jakubczyk: Realization theory for nonlinear systems, three approaches. In: Alg. & Geom. Methods in Nonlin. Control. Theory. Springer 1986.

[33] I. Kaplansky: An Introduction to Differential Algebra. Hermann, Paris 1957.

[34] E. R. Kolchin: Differential Algebra and Algebraic Groups. Academic Press, New York 1973.

[35] S. Kobayashi and K. Nomizu: Foundations of Differential Geometry. Volume I. John Willey & Sons, New York 1963.

[36] U. Kotta, P. Kotta, S. Nomm, and M. Tonso: Irreducibility conditions for continuous-time multi-input multi-output nonlinear systems. In: Proc. 9th International Conference on Control, Automation, Robotics and Vision (ICARCV 2006). Singapore 2006.

[37] U. Kotta, A. S. I. Zinober, and P. Liu: Transfer equivalence and realization of nonlinear higher order input output difference equations. Automatica *37* (2001), 1771–1778.

[38] S. R. Kou, D. L. Elliot, and T. J. Tarn: Observability of nonlinear systems. Inform. Control *22* (1973), 89–99.

[39] A. J. Krener and W. Respondek: Nonlinear observers with linearizable error dynamics. SIAM J. Control Optim. *23* (1985), 197–216.

[40] C. H. Moog, Y. F. Zheng, and P. Liu: Input-output equivalence of nonlinear systems and their realizations. In: 15th IFAC World Congress on Automatic Control, IFAC, Barcelona 2002.

[41] H. Nijmeijer and A. van der Schaft: Nonlinear Dynamical Control Systems. Springer, New York 1990.

[42] O. Ore: Linear equations in non-commutative fields. Ann. Math. *32* (1931), 463–477.

[43] O. Ore: Theory of non-commutative polynomials. Ann. Math. *34* (1933), 80–508.

[44] J. F. Ritt: Differential Algebra. American Mathematical Society, Providence 1950.

[45] J. Rudolph: Viewing input-output system equivalence from differential algebra. J. Math. Systems Estim. Control *4* (1994), 353–383.

[46] A. J. van der Schaft: Observability and controllability for smooth nonlinear systems. SIAM J. Control Optim. *20* (1982), 338–354.

[47] A. J. van der Schaft: On realization of nonlinear systems described by higher-order differential equations. Math. Systems Theory *19* (1987), 239–275.

[48] A. J. van der Schaft: Transformations of nonlinear systems under external equivalence. In: New Trends in Nonlinear Control Theory, Lecture Notes in Control and Information Sciences *122*, Springer, New York 1989, pp. 33–43.

[49] A. J. van der Schaft: Representing a nonlinear state space system as a set of higher-order differential equations in the inputs and outputs. Syst. Control Lett. *12* (1989), 151–160.

[50] E. D. Sontag: Bilinear realizability is equivalent to existence of a singular affine differential i/o equation. Syst. Control Lett. *11* (1988), 190–198.

[51] H. S. Sussmann: Existence and uniqueness of minimal realizations of nonlinear systems. Math. Systems Theory *10* (1977), 263–284.

[52] Y. Wang and E. D. Sontag: Algebraic differential equations and rational control systems. SIAM J. Control Optim. *30* (1992), 1126–1149.

[53] Y. Wang and E. D. Sontag: Generating series and nonlinear systems: analytic aspects, local realizability and i/o representations. Forum Math. *4* (1992), 299–322.

[54] Y. Wang and E. D. Sontag: Orders of input/output differential equations and state-space dimensions. SIAM J. Control Optim. *33* (1995), 1102–1126.

[55] X. Xia, L. A. Márquez, P. Zagalak, and C. H. Moog: Analysis of nonlinear time-delay systems using modules over non-commutative rings. Automatica *38* (2002), 1549–1555.

[56] Y. Zheng and L. Cao: Transfer function description for nonlinear systems. J. East China Normal University (Natural Science) *2* (1995), 5–26.

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