Abhijit Banerjee; Sujoy Majumder

On the uniqueness of a power of a meromorphic function sharing a small function with the power of its derivative

Commentationes Mathematicae Universitatis Carolinae, Vol. 51 (2010), No. 4, 565--576

Persistent URL: http://dml.cz/dmlcz/140838

On the uniqueness of a power of a meromorphic function sharing a small function with the power of its derivative

Abhijit Banerjee, Sujoy Majumder

Abstract. In the paper we discuss the uniqueness of the n-th power of a meromorphic function sharing a small function with the power of its k-th derivative and improve and supplement a result of Zhang-Lü [Complex Var. Elliptic Equ. **53** (2008), no. 9, 857–867]. We also rectify one recent result obtained by Chen and Zhang in [Kyungpook Math. J. **50** (2010), no. 1, 71–80] dealing with a question posed by T.D. Zhang and W.R. Lü in [Complex Var. Elliptic Equ. **53** (2008), no. 9, 857–867].

Keywords: meromorphic function, derivative, small function, weighted sharing

Classification: 30D35

1. Introduction. Definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [5]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$, as $r \longrightarrow \infty$ and $r \notin E$.

Let f and g be two non-constant meromorphic functions and let a be a complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

A meromorphic function a is said to be a small function of f provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \longrightarrow \infty$, $r \notin E$.

In 1979 Mues and Steinmetz proved the following theorem.

Theorem A ([11]). *Let f be a non-constant entire function. If f and $f^{'}$ share two distinct values a, b IM then $f^{'} \equiv f$.*

Considering the uniqueness problem of an entire function sharing one value with its derivative, the following result was proved in [3].

Theorem B ([3]). *Let f be a non-constant entire function. If f and f' share the value 1 CM and if $N(r,0;f')=S(r,f)$ then $\frac{f'-1}{f-1}$ is a nonzero constant.*

Later Yang [12], Zhang [15], Yu [14] worked on the uniqueness of meromorphic functions and their derivative which share one value. To state the next results we require the following definition known as weighted sharing of values which measure how close a shared value is to be shared IM or to be shared CM.

Definition 1.1 ([6], [7]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f)=E_k(a;g)$, we say that f,g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

If a is a small function we define that f and g share a IM or a CM or with weight l according as $f-a$ and $g-a$ share $(0,0)$ or $(0,\infty)$ or $(0,l)$ respectively.

Though we use the standard notations and definitions of the value distribution theory available in [5], we explain some definitions and notations which are used in the paper.

Definition 1.2 ([9]). Let p be a positive integer and $a \in \mathbb{C}\cup\{\infty\}$.

(i) $N(r,a;f \mid\geq p)$ $(\overline{N}(r,a;f \mid\geq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r,a;f \mid\leq p)$ $(\overline{N}(r,a;f \mid\leq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.3 ([13]). For $a \in \mathbb{C}\cup\{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f \mid\geq 2)+\ldots+\overline{N}(r,a;f \mid\geq p)$. Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

Definition 1.4 ([16]). For a positive integer p and $a \in \mathbb{C}\cup\{\infty\}$ we put

$$\delta_p(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{N_p(r,a;f)}{T(r,f)}\,.$$

Clearly $0 \leq \delta(a;f) \leq \delta_p(a;f) \leq \delta_{p-1}(a;f) \leq \ldots \leq \delta_2(a;f) \leq \delta_1(a;f)=\Theta(a;f)$.

Definition 1.5 ([1])**.** Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p>q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p=q=1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p=q\geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

Definition 1.6 ([6], [7])**.** Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

With the notion of weighted sharing of values Lahiri-Sarkar [9] improved the result of Zhang [15]. In [16] Zhang extended the result of Lahiri and Sarkar and replaced the concept of value sharing by small function sharing.

Recently in [18] Zhang and Lü considered the uniqueness of the n-th power of a meromorphic function sharing a small function with its k-th derivative and proved the following theorem.

Theorem C. *Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose f^n-a and $f^{(k)}-a$ share $(0,l)$. If $l=\infty$ and*

$$(3+k)\,\Theta(\infty;f)+2\,\Theta(0;f)+\delta_{2+k}(0;f)>6+k-n \tag{1.1}$$

or $l=0$ and

$$(6+2k)\,\Theta(\infty;f)+4\,\Theta(0;f)+2\delta_{2+k}(0;f)>12+2k-n \tag{1.2}$$

then $f^n\equiv f^{(k)}$.

In the same paper Zhang and Lü [18] raised the following question: *What will happen if f^n and $[f^{(k)}]^m$ share a small function*?

To answer the above question recently Chen and Zhang [4] obtained the following result.

Theorem D. *Let $k(\geq 1)$, $n(\geq 1)$, $m(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose f^n-a and $[f^{(k)}]^m-a$ share $(0,l)$. If $l=\infty$ and*

$$(3+k)\,\Theta(\infty;f)+\delta_2(0;f)+2\delta_{2+k}(0;f)>5+k-n \tag{1.3}$$

or $l=0$ and

$$(6+2k)\,\Theta(\infty;f)+3\,\Theta(0;f)+2\delta_{2+k}(0;f)>11+2k-n \tag{1.4}$$

then $f^n \equiv [f^{(k)}]^m$.

Theorem D seems to be fine but there are some mistakes in the proof. For example in the proof of Theorem 2, Subcase 1.1 in [4] the estimations of the zeros of the counting function of $G^{'}$ is not correct when $m \geq 2$. The same things happen after the equation (4.1) in Subcase 1.2. As a result (1.3)–(1.4) is not correct. So it will be a natural inquisition to find the correct inequalities for which the conclusion of Theorem D holds good. Here we are taking up this problem. We also improve and supplement Theorem C by relaxing the nature of sharing and at the same weakening inequality (1.2). The following theorems are the main results of the paper.

Theorem 1.1. *Let* $k(\geq 1)$, $n(\geq 1)$ *be integers and* f *be a non-constant meromorphic function. Also let* $a(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f. *Suppose* $f^n - a$ *and* $f^{(k)} - a$ *share* $(0, l)$. *If* $l = 2$ *and*

$$(3+k)\ \Theta(\infty; f) + 2\ \Theta(0; f) + \delta_{2+k}(0; f) > 6 + k - n \tag{1.5}$$

or $l = 1$ *and*

$$\left(\frac{7}{2} + k\right)\ \Theta(\infty; f) + \frac{5}{2}\ \Theta(0; f) + \delta_{2+k}(0; f) > 7 + k - n \tag{1.6}$$

or $l = 0$ *and*

$$(6+2k)\ \Theta(\infty; f) + 4\ \Theta(0; f) + \delta_{1+k}(0; f) + \delta_{2+k}(0; f) > 12 + 2k - n \tag{1.7}$$

then $f^n \equiv f^{(k)}$.

Theorem 1.2. *Let* $k(\geq 1)$, $n(\geq 1)$, $m(\geq 2)$ *be integers and* f *be a non-constant meromorphic function. Also let* $a(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f. *Suppose* $f^n - a$ *and* $[f^{(k)}]^m - a$ *share* $(0, l)$. *If* $l = 2$ *and*

$$(3+2k)\ \Theta(\infty; f) + 2\ \Theta(0; f) + 2\delta_{1+k}(0; f) > 7 + 2k - n \tag{1.8}$$

or $l = 1$ *and*

$$\left(\frac{7}{2} + 2k\right)\ \Theta(\infty; f) + \frac{5}{2}\ \Theta(0; f) + 2\delta_{1+k}(0; f) > 8 + 2k - n \tag{1.9}$$

or $l = 0$ *and*

$$(6+3k)\ \Theta(\infty; f) + 4\ \Theta(0; f) + 3\delta_{1+k}(0; f) > 13 + 3k - n \tag{1.10}$$

then $f^n \equiv [f^{(k)}]^m$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1 ([17])**.** *Let f be a non-constant meromorphic function and p and k two positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \le T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f),$$
$$N_p\left(r,0;f^{(k)}\right) \le N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.2 ([2])**.** *Let f, g share $(1,0)$. Then*

$$\overline{N}_L(r,1;f) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.3 ([10])**.** *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \ne 0$ and $b_m \ne 0$. Then

$$T(r,R(f)) = dT(r,f) + S(r,f),$$

where $d = \max\{n,m\}$.

3. Proofs of the theorems

Proof of Theorem 1.2: Let $F = \frac{f^n}{a}$ and $G = \frac{[f^{(k)}]^m}{a}$. Then $F - 1 = \frac{f^n - a}{a}$ and $G - 1 = \frac{[f^{(k)}]^m - a}{a}$. Since $f^n - a$ and $[f^{(k)}]^m - a$ share $(0,l)$ it follows that F, G share $(1,l)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1. Let $H \not\equiv 0$.

Subcase 1.1. $l \ge 1$.
From (2.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different (iii) those poles of F and G whose multiplicities are different, (iv) zeros of F' (G') which are not the zeros of $F(F-1)$ ($G(G-1)$).

Since H has only simple poles we get

$$\begin{aligned}N(r,\infty;H) \le\ & \overline{N}(r,\infty;F)+\overline{N}(r,1;F\mid\ge l+1)+\overline{N}(r,0;F\mid\ge 2)\\ &+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})\\ &+\overline{N}(r,0;a)+\overline{N}(r,\infty;a),\end{aligned}\tag{3.1}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is defined similarly. Let z_0 be a simple zero of $F-1$ but $a(z_0)\ne 0,\infty$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$\begin{aligned}N(r,1;F\mid=1) &\le N(r,0;H)+N(r,\infty;a)+N(r,0;a)\\ &\le N(r,\infty;H)+S(r,f).\end{aligned}\tag{3.2}$$

By the second fundamental theorem we see that

$$\begin{aligned}T(r,F)+T(r,G) \le\ & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)\\ &+\overline{N}(r,1;F)+\overline{N}(r,1;G)-N_0(r,0;F^{'})-N_0(r,0;G^{'})\\ &+S(r,F)+S(r,G).\end{aligned}\tag{3.3}$$

Using (3.1) and (3.2) we get

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) &\le N(r,1;F\mid=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\ &\quad+\overline{N}^{(2}_E(r,1;F)+\overline{N}(r,1;G)+S(r,f)\\ &\le \overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}(r,\infty;F)\\ &\quad+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}^{(2}_E(r,1;F)\\ &\quad+\overline{N}(r,1;G)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})\\ &\quad+S(r,f).\end{aligned}\tag{3.4}$$

While $l\ge 2$ we obtain

$$2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}^{(2}_E(r,1;F)+\overline{N}(r,1;G)\le N(r,1;G)+S(r,f).\tag{3.5}$$

So noting that $N_2(r,0;G)\le 2\overline{N}(r,0;f^{(k)})+S(r,f)$ using (3.4) and (3.5) in (3.3) and in view of Lemma 2.1 we obtain

$$T(r,F)\le (3+2k)\ \overline{N}(r,\infty;F)+2\overline{N}(r,0;f)+2N_{1+k}(r,0;f)+S(r,f),$$

that is

$$(3+2k)\Theta(\infty;f)+2\Theta(0;f)+2\delta_{1+k}(0;f)\le 7+2k-n,$$

which contradicts (1.8).

While $l = 1$, (3.5) changes to

$$\begin{aligned}(3.6)\qquad & 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}(r,1;G) \\ & \leq N(r,1;G) + \overline{N}_L(r,1;F) + S(r,f).\end{aligned}$$

Noting that

$$\overline{N}_L(r,1;F) \leq \frac{1}{2}N(r,0;F' \mid F \neq 0) \leq \frac{1}{2}(\overline{N}(r,0;F) + \overline{N}(r,\infty;F)),$$

using (3.4) and (3.6) in (3.3) and in view of Lemma 2.1 we have

$$T(r,F) \leq \left(\frac{7}{2} + 2k\right)\,\overline{N}(r,\infty;F) + \frac{5}{2}\overline{N}(r,0;f) + 2N_{1+k}(r,0;f) + S(r,f),$$

that is

$$\left(\frac{7}{2} + 2k\right)\Theta(\infty;f) + \frac{5}{2}\Theta(0;f) + 2\delta_{1+k}(0;f) \leq 8 + 2k - n,$$

which contradicts (1.9).

Subcase 1.2. $l = 0$.
In this case F and G share $(1,0)$ except the zeros and poles of $a(z)$. It is easy to see that

$$N_E^{1)}(r,1;F) = N_E^{1)}(r,1;G) + S(r,f)$$

$$\overline{N}_E^{(2}(r,1;F) = \overline{N}_E^{(2}(r,1;G) + S(r,f)$$

and

$$(3.7)\qquad N_E^{1)}(r,1;F) \leq N(r,\infty;H) + S(r,f).$$

Here in view of Lemma 2.2, (3.4) changes into

$$\begin{aligned}(3.8)\qquad & \overline{N}(r,1;F) + \overline{N}(r,1;G) \\ & \leq N_E^{1)}(r,1;F) + \overline{N}_E^{(2}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) \\ & \quad + \overline{N}(r,1;G) + S(r,f) \\ & \leq \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}(r,\infty;F) + \overline{N}_E^{(2}(r,1;F) \\ & \quad + 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + \overline{N}_0(r,0;F') \\ & \quad + \overline{N}_0(r,0;G') + S(r,f) \\ & \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + 2\,\overline{N}_L(r,1;F) \\ & \quad + \overline{N}_L(r,1;G) + N(r,1;G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) \\ & \leq 4\,\overline{N}(r,\infty;F) + N_2(r,0;F) + \overline{N}(r,0;F) + N_2(r,0;G) + T(r,G) \\ & \quad + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f).\end{aligned}$$

So using (3.8) in (3.3) we get in view of Lemma 2.1 that

$$T(r,F) \leq (6+3k)\,\overline{N}(r,\infty;F)+N_2(r,0;f^n)+2\overline{N}(r,0;f)+3N_{1+k}(r,0;f)+S(r,f),$$

that is

$$(6+3k)\Theta(\infty;f)+4\Theta(0;f)+3\delta_{1+k}(0;f) \leq 13+3k-n.$$

This contradicts (1.10).

Case 2. Let $H \equiv 0$.
By integration we get from (2.1)

$$\frac{1}{F-1} \equiv \frac{C}{G-1}+D, \tag{3.9}$$

where C, D are constants and $C \neq 0$. We first show that $D=0$. Suppose that there exist a pole z_0 of f with multiplicity p which is not a pole or a zero of $a(z)$. Then z_0 is the pole of F with multiplicity np and the pole of G with multiplicity $m(p+k)$. We assume that $np \neq m(p+k)$, since otherwise we know from (3.9) that $D=0$ and we are done.

Subcase 2.1. Suppose $D \neq 0$.
Since $np \neq m(p+k)$, we get a contradiction from (3.9). So,

$$N(r,\infty;f) \leq N(r,0;a)+N(r,\infty;a)=S(r,f),$$

and hence $\Theta(\infty;f)=1$. Also it is clear that $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)=S(r,f)$.
From (1.8)–(1.10) we know respectively

$$2\Theta(0;f)+2\delta_{1+k}(0;f) > 4-n, \tag{3.10}$$

$$\frac{5}{2}\Theta(0;f)+2\delta_{1+k}(0;f) > \frac{9}{2}-n \tag{3.11}$$

and

$$4\Theta(0;f)+3\delta_{1+k}(0;f) > 7-n. \tag{3.12}$$

Since $D \neq 0$, from (3.9) we get

$$-\frac{D\left(F-1-\frac{1}{D}\right)}{F-1} \equiv C\frac{1}{G-1}\,.$$

So

$$\overline{N}\left(r,1+\frac{1}{D};F\right)=\overline{N}(r,\infty;G)=S(r,f).$$

Subcase 2.1.1. $D \neq -1$.
Using the second fundamental theorem for F we get

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1+\frac{1}{D};F) \\ &\leq \overline{N}(r,0;F) + S(r,f), \end{aligned}$$

that is

$$nT(r,f) \leq \overline{N}(r,0;f) + S(r,f).$$

If $n > 1$ we have a contradiction from above. So we have $n = 1$ and so $\Theta(0;f) = 0$, which contradicts (3.10)–(3.12).

Subcase 2.1.2. $D = -1$.
Then

$$\frac{F}{F-1} \equiv C\frac{1}{G-1}. \tag{3.13}$$

Clearly we know from above that $\overline{N}(r,0;F) = \overline{N}(r,\infty;G) = S(r,f)$ and hence $\overline{N}(r,0;f) = S(r,f)$. If $C \neq -1$ we know from (3.13) that $\overline{N}(r,1+C;G) = \overline{N}(r,\infty;F) = S(r,f)$. So from Lemma 2.1 and the second fundamental theorem we get

$$\begin{aligned} mT(r,f^{(k)}) &= T(r,G) + S(r,f) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1+C;G) + S(r,f) \\ &\leq \overline{N}(r,0;f^{(k)}) + S(r,f) \\ &\leq T(r,f^{(k)}) - T(r,f) + N_{1+k}(r,0;f) + S(r,f), \end{aligned}$$

that is

$$(m-1)T\left(r,f^{(k)}\right) + T(r,f) \leq (k+1)\overline{N}(r,0;f) + S(r,f) = S(r,f),$$

which is absurd. So $C = -1$ and we get from (3.13) that $FG \equiv 1$, which ultimately yields $\left[\frac{f^{(k)}}{f}\right]^m = \frac{a^2}{f^{n+m}}$.

In view of the first fundamental theorem we get from above

$$\begin{aligned} (n+m)T(r,f) &\leq mT\left(r,\frac{f^{(k)}}{f}\right) + S(r,f) \\ &= mN\left(r,\infty;\frac{f^{(k)}}{f}\right) + S(r,f) \\ &\leq mk[\overline{N}(r,\infty;f) + \overline{N}(r,0;f)] + S(r,f) = S(r,f), \end{aligned}$$

which is impossible.

Subcase 2.2. $D = 0$ and so from (3.9) we get

$$G - 1 \equiv C\ (F - 1).$$

If $C \neq 1$, then

$$G \equiv C\ \left(F - 1 + \frac{1}{C}\right)$$

and

$$\overline{N}(r, 0; G) = \overline{N}\left(r, 1 - \frac{1}{C}; F\right).$$

By the second fundamental theorem and Lemma 2.1 for $p = 1$ and Lemma 2.3 we have

$$\begin{aligned}
&nT(r, f) + S(r, f) = T(r, F)\\
&\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}\left(r, 1 - \frac{1}{C}; F\right) + S(r, G)\\
&\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, f)\\
&\leq \overline{N}(r, 0; f) + N_{1+k}(r, 0; f) + (1 + k)\overline{N}(r, \infty; f) + S(r, f).
\end{aligned}$$

Hence

$$(1 + k)\Theta(\infty; f) + \Theta(0; f) + \delta_{1+k}(0; f) \leq 3 + k - n.$$

So, it follows that

$$\begin{aligned}
&(3 + 2k)\Theta(\infty; f) + 2\Theta(0; f) + 2\delta_{1+k}(0; f)\\
&\leq (2 + k)\Theta(\infty; f) + \Theta(0; f) + \delta_{1+k}(0; f) + (1 + k)\Theta(\infty; f)\\
&\quad + \Theta(0; f) + \delta_{1+k}(0; f)\\
&\leq 7 + 2k - n,
\end{aligned}$$

$$\left(\frac{7}{2} + 2k\right)\Theta(\infty; f) + \frac{5}{2}\ \Theta(0; f) + 2\delta_{1+k}(0; f) \leq 8 + 2k - n$$

and

$$(6 + 3k)\Theta(\infty; f) + 4\Theta(0; f) + 3\delta_{1+k}(0; f) \leq 13 + 2k - n.$$

This contradicts (1.8)–(1.10). Hence $C = 1$ and so $F \equiv G$, that is $f^n \equiv [f^{(k)}]^m$. This completes the proof of the theorem. □

Proof of Theorem 1.1: Let $F = \frac{f^n}{a}$ and $G = \frac{f^{(k)}}{a}$. Then $F - 1 = \frac{f^n - a}{a}$ and $G - 1 = \frac{f^{(k)} - a}{a}$. Since $f^n - a$ and $f^{(k)} - a$ share $(0, l)$ it follows that F, G share $(1, l)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1. Let $H \not\equiv 0$.
First we note that in view of Lemma 2.1 here

$$N_2(r,0;G) = N_2\left(r,0;f^{(k)}\right) + S(r,f) \leq N_{2+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Now following the same procedure as adopted in the proof of Case 1 of Theorem 1.2 with $m = 1$ we can easily deduce a contradiction corresponding to (1.5)–(1.7).

Case 2. Let $H \equiv 0$.
Proceeding in the same way as done in the proof of Case 1.2 of Theorem 1.2 with $m = 1$ and using (1.5)–(1.7) instead of (1.8)–(1.10), we can easily prove $f^n = f^{(k)}$ and so we are omitting the details of the proof. □

References

[1] Alzahary T.C., Yi H.X., *Weighted value sharing and a question of I. Lahiri*, Complex Var. Theory Appl. **49** (2004), no. 15, 1063–1078.

[2] Banerjee A., *Meromorphic functions sharing one value*, Int. J. Math. Math. Sci. **22** (2005), 3587–3598.

[3] Brück R., *On entire functions which share one value CM with their first derivative*, Results Math. **30** (1996), 21–24.

[4] Chen A., Zhang G., *Unicity of meromorphic function and its derivative*, Kyungpook Math. J. **50** (2010), no. 1, 71–80.

[5] Hayman W.K., *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[6] Lahiri I., *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J. **161** (2001), 193–206.

[7] Lahiri I., *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl. **46** (2001), 241–253.

[8] Lahiri I., Banerjee A., *Weighted sharing of two sets*, Kyungpook Math. J. **46** (2006), no. 1, 79–87.

[9] Lahiri I., Sarkar A., *Uniqueness of meromorphic function and its derivative*, J. Inequal. Pure Appl. Math. **5** (2004), no. 1, Article 20, `http://jipam.vu.edu.au/`.

[10] Mohon'ko A.Z., *The Nevanlinna characteristics of some meromorphic functions*, Theory of Functions Funct. Anal. Appl. **14** (1971), 83–87.

[11] Mues E., Steinmetz N., *Meromorphe Funktionen die unit ihrer Ableitung Werte teilen*, Manuscripta Math. **29** (1979), 195–206.

[12] Yang L.Z., *Solution of a differential equation and its applications*, Kodai Math. J. **22** (1999), 458–464.

[13] Yi H.X., *On characteristic function of a meromorphic function and its derivative*, Indian J. Math. **33** (1991), no. 2, 119–133.

[14] Yu K.W., *On entire and meromorphic functions that share small functions with their derivatives*, J. Inequal. Pure Appl. Math. **4** (2003), no. 1, Article 21, `http://jipam.vu.edu.au/`.

[15] Zhang Q.C., *The uniqueness of meromorphic functions with their derivatives*, Kodai Math. J. **21** (1998), 179–184.

[16] Zhang Q.C., *Meromorphic function that shares one small function with its derivative*, J. Inequal. Pure Appl. Math. **6** (2005), no. 4, Article 116, `http://jipam.vu.edu.au/`.

[17] Zhang J.L., Yang L.Z., *Some results related to a conjecture of R. Bruck*, J. Inequal. Pure Appl. Math. **8** (2007), no. 1, Article 18.
[18] Zhang T.D., W.R. Lü W.R., *Notes on meromorphic function sharing one small function with its derivative*, Complex Var. Elliptic Equ. **53** (2008), no. 9, 857–867.

Department of Mathematics, West Bengal State University, Barasat, 24 Prgs. (North), Kolkata-700126, West Bengal, India

E-mail: abanerjee_kal@yahoo.co.in, abanerjee_kal@rediffmail.com

(*Received* April 20, 2010, *revised* July 15, 2010)