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GENERAL INTEGRATION AND EXTENSIONS II

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Abstract. This work is a continuation of the paper (Š. Schwabik: General integration and extensions I, Czechoslovak Math. J. 60 (2010), 961–981). Two new general extensions are introduced and studied in the class $\mathfrak{T}$ of general integrals. The new extensions lead to approximate description of the Kurzweil-Henstock integral based on the Lebesgue integral close to the results of S. Nakanishi presented in the paper (S. Nakanishi: A new definition of the Denjoy's special integral by the method of successive approximation, Math. Jap. 41 (1995), 217–230).

Keywords: abstract integration, extension of integral, Kurzweil-Henstock integration

MSC 2010: 26A39, 26A42

1. Introduction

This paper is closely related to [10] and [11]. We use concepts and results presented therein. In this introductory part we give a short account from [10] and [11] for the readers' convenience.

For a compact interval $E = [a,b]$, $-\infty < a < b < +\infty$ in $\mathbb{R}$ real functions $f\colon E \to \mathbb{R}$ will be studied.

For $M \subset E$ and a function $f\colon E \to \mathbb{R}$ we put

$$|f|_M = \sup\{|f(x)|;\ x \in M\}.$$

If $J \subset E$ is a closed interval in E, then we denote by $\operatorname{Sub}(J)$ the set of all closed subintervals of J.

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The editors learnt with great sadness that Professor Štefan Schwabik passed away on November 4, 2009. The galleys of this paper were therefore not proofread by the author, and the responsibility for any typesetting inaccuracies lies solely with the editors.

If $I \in \mathrm{Sub}(E)$ and $A \subset E$ is closed then denote by $\mathrm{Comp}(I, A)$ *the set of all (maximal and non-empty) connected components* of the set $I \setminus A$.

A *functional S in E* is a mapping from a set of functions on E into $\mathbb{R}$, i.e. S is a set of pairs (f, γ) (f being a function $f\colon E \to \mathbb{R}$ and $\gamma \in \mathbb{R}$ the value of the functional S) and it is assumed that γ is uniquely determined by f. We write $\gamma = S(f)$. $\mathrm{Dom}(S)$ is the set of all f for which the functional S is defined. Denote by $C(E)$ the set of all continuous real-valued functions on E.

1.1. The Saks class $\mathfrak{S}$ of integrals

Definition 1.1. A functional S in E is called *additive* if the following two conditions hold:

A) $0 \in \mathrm{Dom}(S)$ and $S(0) = 0$,

B) if $c \in [a, b] = E$ and $I_1 = [a, c]$, $I_2 = [c, b]$, then $f \in \mathrm{Dom}(S)$ if and only if $f \cdot \chi(I_1), f \cdot \chi(I_2) \in \mathrm{Dom}(S)$ and

$$S(f) = S(f, I_1) + S(f, I_2).$$

($\chi(M)$ denotes the characteristic function of a set $M \subset E$ and $S(f, M) = S(f \cdot \chi(M))$ for $f \cdot \chi(M) \in \mathrm{Dom}(S)$.)

Definition 1.2. If S is an additive functional in E and $f \in \mathrm{Dom}(S)$, then a function $F\colon E \to \mathbb{R}$ is called an *S-primitive* to f provided

$$F[I] = S(f, I)$$

holds for every $I \in \mathrm{Sub}(E)$. For $I = [c, d] \in \mathrm{Sub}(E)$ the interval function $F[I]$ is given by $F[I] = F(d) - F(c)$.

An S-primitive function to $f \in \mathrm{Dom}(S)$ always exists (e.g. $F(x) = S(f, [a, x])$ for $x \in E = [a, b]$ is an S-primitive to f) and it is determined uniquely up to a constant.

In [11] the following concept of a general integral was introduced.

Definition 1.3. An additive functional S in E is called an *integral* in E if all S-primitive functions to $f \in \mathrm{Dom}(S)$ are continuous in E.

Denote the *set of all integrals in E* by $\mathfrak{S}$.

If $S \in \mathfrak{S}$ and $f \in \mathrm{Dom}(S)$, then f is called *S-integrable*.

If $S \in \mathfrak{S}$ and $M \subset E$, then a function f is said to be *S-integrable on M* if $f \cdot \chi(M) \in \mathrm{Dom}(S)$.

This concept coincides with the concept of S. Saks [9, VIII, § 4], the changes are insignificant as was shown in [11].

1.2. Ordering and extension of integrals

Definition 1.4. If $T, S \in \mathfrak{S}$ then T *includes* S (we write $S \sqsubset T$) provided $\mathrm{Dom}(S) \subset \mathrm{Dom}(T)$ and for $f \in \mathrm{Dom}(S)$ and every $I \in \mathrm{Sub}(E)$ the equality $T(f, I) = S(f, I)$ is satisfied ($f \cdot \chi(I) \in \mathrm{Dom}(S)$ holds by B) in Definition 1.1).

The concept of $S \sqsubset T$ for $S, T \in \mathfrak{S}$ in the above definition follows the setting given in the book of S. Saks [9, VIII, § 4], see also [4].

By definition it can be checked easily that the following holds:

If $R, S, T \in \mathfrak{S}$, then $R \sqsubset R$ (reflexivity); if $R \sqsubset S$ and $S \sqsubset T$ then $R \sqsubset T$ (transitivity), if $S \sqsubset T$ and $T \sqsubset S$ then $T = S$ (antisymmetry).

In other words, the binary relation $\sqsubset$ on $\mathfrak{S}$ is an order and $(\mathfrak{S}, \sqsubset)$ is an ordered set.

Definition 1.5. A mapping $Q\colon \mathfrak{S} \to \mathfrak{S}$ defined on $\mathrm{Dom}(Q) \subset \mathfrak{S}$ is called an *extension* if for every $S \in \mathrm{Dom}(Q)$ we have $S \sqsubset Q(S)$, $Q(S) \in \mathrm{Dom}(Q)$ and, moreover, if $S_1, S_2 \in \mathrm{Dom}(Q) \subset \mathfrak{S}$ with $S_1 \sqsubset S_2$, then $Q(S_1) \sqsubset Q(S_2)$.

The extension Q is called *effective* if $Q^2 = Q$, i.e. if $Q(Q(S)) = Q(S)$ for every $S \in \mathrm{Dom}(Q)$.

An integral S is called *invariant with respect to an extension* Q if $S \in \mathrm{Dom}(Q)$ and $Q(S) \sqsubset S$, i.e. $Q(S) = S$.

In [11] two classical and well known extensions, namely the Cauchy and Harnack extensions, were studied. Let us recall their definition.

First of all we need the following concept.

Definition 1.6. If f is a function on E and $S \in \mathfrak{S}$, then $x \in E$ is called an *S-regular point of f* if there is an $I \in \mathrm{Sub}(E)$ such that $x \in \mathrm{Int}(I)$ (the interior of I) and $f \cdot \chi(I) \in \mathrm{Dom}(S)$.

The set of all S-regular points of f is denoted by $\varrho(f, S)$.

The complement $\sigma(f, S) = E \setminus \varrho(f, S)$ of $\varrho(f, S)$ in E is called the set of *S-singular points of the function f.*

If $I \in \mathrm{Sub}(E)$ contains endpoints of E, then we consider them as points belonging to $\mathrm{Int}(I)$.

The set $\sigma(f, S)$ is closed because $\varrho(f, S)$ is evidently open by definition. Moreover, $\sigma(f, S) = \emptyset$ if and only if $f \in \mathrm{Dom}(S)$. (See also [2, 9.1 Theorem].)

Definition 1.7. For $S \in \mathfrak{S}$ denote by S_C the set of all pairs (f, γ), where f is a function on E and $\gamma \in \mathbb{R}$, such that $\sigma(f, S)$ is a finite set for which there is a function $F \in C(E)$ such that $\gamma = F[E] = F(b) - F(a)$ and for every $I \subset \varrho(f, S)$ we have $f \cdot \chi(I) \in \mathrm{Dom}(S)$ and $F[I] = S(f, I)$.

For $I \in \mathrm{Sub}(E)$ put $S_C(f, I) = F[I]$.

The set $\{(S, S_C);\ S \in \mathfrak{S},\ S_C \text{ exists}\}$ is denoted by P_C.

It is easy to see that $S_C \in \mathfrak{S}$ and the map $P_C\colon \mathfrak{S} \to \mathfrak{S}$ is the *Cauchy extension.*

Definition 1.8. For $S \in \mathfrak{S}$ denote by S_H the set of all pairs (f, γ), where f is a function on E and $\gamma \in \mathbb{R}$, for which $f \cdot \chi(\sigma(f, S)) \in \mathrm{Dom}(S)$, $f \cdot \chi(U_j) \in \mathrm{Dom}(S)$ for $j \in \Gamma$, where $\{U_j;\ j \in \Gamma\} = \mathrm{Comp}(E, \sigma(f, S))$, and for which there is a function $F \in C(E)$ such that $\gamma = F[E] = F(b) - F(a)$,

$$\sum_{U \in \mathrm{Comp}(E, \sigma(f,S))} \omega(F, \overline{U}) = \sum_{j \in \Gamma} \omega(F, \overline{U}_j) < \infty$$

and

$$F[I] = S(f, I \cap \sigma(f, S)) + \sum_{j \in \Gamma} S(f, I \cap \overline{U_j})$$

for any $I \in \mathrm{Sub}(E)$. ($\omega(F, \overline{U})$ is the oscillation of F over the interval $\overline{U}$.)

The set $\{(S, S_H);\ S \in \mathfrak{S},\ S_H \text{ exists}\}$ is denoted by P_H.

As before, P_H is a map $\mathfrak{S} \to \mathfrak{S}$. Let us call it the Harnack extension.

1.3. Divisions

A *division* is a finite system $D = \{I_j;\ j \in \Gamma\}$ of intervals, where $\mathrm{Int}(I_j) \cap I_k = \emptyset$ for $j \neq k$, $\Gamma \subset \mathbb{N}$ is finite.

For a given set $M \subset E$ the division D is called a *division in* M if $M \supset \bigcup_{j \in \Gamma} I_j$, D is a *division of* M if $M = \bigcup_{j \in \Gamma} I_j$ and the division D *covers* M if $M \subset \bigcup_{j \in \Gamma} I_j$.

A map τ from $\mathrm{Sub}(E)$ into E is called a *tag* if $\tau(I) \in I$ for $I \in \mathrm{Dom}(\tau)$.

A *tagged system* is a pair (D, τ), where $D = \{I_j;\ j \in \Gamma\}$ is a division and τ is a tag defined on the range of D, i.e. for all I_j, $j \in \Gamma$. In this case we write τ_j instead of $\tau(I_j)$.

The tagged system (D, τ) is called *M-tagged* for some set $M \subset E$ if $\tau_j \in M$ for $j \in \Gamma$.

A *gauge* is any function on E with values in the set $\mathbb{R}^+$ of positive reals. $\Delta(E)$ is the set of all gauges.

If $\delta \in \Delta(E)$, then a tagged system (D, τ), where $D = \{I_j;\ j \in \Gamma\}$, is called *$\delta$-fine* if $|I_j| < \delta(\tau_j)$ for $j \in \Gamma$.

1.4. The Kurzweil-Henstock integral

Definition 1.9. K denotes the set of all pairs (f,γ), where f is a function on E and $\gamma \in \mathbb{R}$, such that for any $\varepsilon > 0$ there exists a gauge δ such that

$$\left|\sum_{j\in\Gamma} f(\tau_j)|I_j| - \gamma\right| < \varepsilon$$

for any δ-fine division $(\{I_j\,;\ j \in \Gamma\}, \tau)$ of the interval E.

The value $\gamma \in \mathbb{R}$ is called the *Kurzweil-Henstock integral* of f over E and it will be denoted by $K(f)$ or $(K)\int_E f$.

It is well known that the Kurzweil-Henstock integral is equivalent to the Perron (= narrow Denjoy) integral (see e.g. [3]). Its role is essential in this paper. The definition in the present form appeared in [10], [11]; some properties of the Kurzweil-Henstock integral given in those writings will be used in the sequel.

1.5. The variational measure W

The *oscillation* $\omega(F,I)$ of $F \in C(E)$ on an interval $I \in \mathrm{Sub}(E)$ is

$$\omega(F,I) = \sup\{|F(x) - F(y)|\,;\ x,y \in I\} = \sup\{|F[J]|\,;\ J \in \mathrm{Sub}(I)\}.$$

Definition 1.10. For $F \in C(E)$ and a division $D = \{I_j\,;\ j \in \Gamma\}$ set

$$\Omega(F,D) = \sum_{j\in\Gamma} \omega(F,I_j).$$

If $F \in C(E)$ and $M \subset E$ then for any $\delta \in \Delta(E)$ put

$$W_\delta(F,M) = \sup\{\Omega(F,D)\,;\ D \,\text{is}\, \delta\text{-fine, } M\text{-tagged}\}$$

and define

$$W_F(M) = \inf\{W_\delta(F,M)\,;\ \delta \in \Delta(E)\}.$$

W_F is the full variational measure generated by the interval functions $\omega(F,I)$ for $I \in \mathrm{Sub}(E)$ (see [10], [12]).

The basic properties of the function W_F are summarized in the following statement (see Theorem 3.10 in [10]).

Theorem 1.11. *Let $F, F_j \in C(E)$ and $M, M_j \subset E$, $j \in \mathbb{N}$. Then*

(i) $0 \leqslant W_F(M_1) \leqslant W_F(M_2)$ *if* $M_1 \subset M_2$,

(ii) $W_F\Big(\bigcup_{j\in\Phi} M_j\Big) \leqslant \sum_{j\in\Phi} W_F(M_j)$ *if Φ is at most countable,*

(iii) $W(\alpha F, I) = |\alpha| W_F(I)$ *for* $\alpha \in \mathbb{R}$,

(iv) $W_{\sum_{j\in\Phi} F_j}(M) \leqslant \sum_{j\in\Phi} W_{F_j}(M)$ *if Φ is finite.*

Denote by $C^*(E)$ the set of all continuous functions on E which are of negligible variation on sets of Lebesgue measure zero (see e.g. Definition 4.1.1 in [7] for this concept). Functions belonging to $C^*(E)$ are also called the functions satisfying the *strong Luzin condition.*

Denote by $\mu(M)$ the Lebesgue measure of $M \subset E$.

Using Lemma 2.9 from [10] it can be stated that

$$C^*(E) = \{F \in C(E);\ W_F(N) = 0 \text{ whenever } \mu(N) = 0\}.$$

A nice descriptive characterization of the Kurzweil-Henstock integral was presented by Bongiorno, Di Piazza and Skvortsov in [1, Theorem 3].

Theorem 1.12. *A function $F\colon E \to \mathbb{R}$ is a K-primitive function to some $f\colon E \to \mathbb{R}$ if and only if $F \in C^*(E)$.*

According to the above mentioned property of $C^*(E)$, this says that a function $f\colon E \to \mathbb{R}$ *is Kurzweil-Henstock integrable if and only if for the K-primitive F to f we have $W_F(N) = 0$ for any $N \subset E$ with $\mu(N) = 0$.*

1.6. The subclass $\mathfrak{T} \subset \mathfrak{S}$

Definition 1.13. $\mathfrak{T}$ denotes the set of all integrals $S \in \mathfrak{S}$ fulfilling the following conditions (1.1)–(1.5) ($N, A \subset E$, $\mu(A)$ is the Lebesgue measure of a set A, f is a function on E and F is an S-primitive function to f):

$$\text{If } \mu(N) = 0, \text{ then } f \cdot \chi(N) \in \mathrm{Dom}(S) \text{ and } S(f, N) = 0. \tag{1.1}$$

$$\text{If } f \in \mathrm{Dom}(S), \text{ then } F \in C^*(E). \tag{1.2}$$

(For $C^*(E)$ see its definition in part 1.5).

$$\text{If } f \in \mathrm{Dom}(S), \text{ then } f \text{ is measurable.} \tag{1.3}$$

There exists $\lambda < \infty$ such that

$$W_F(A) \leqslant \lambda |f|_A \tag{1.4}$$

if $f \in \operatorname{Dom}(S)$ and A is a closed set ($W_F(\cdot)$ is the full variational measure from Definition 1.10).

If $f, g \in \operatorname{Dom}(S)$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in \operatorname{Dom}(S)$ and

$$S(\alpha f + \beta g) = \alpha S(f) + \beta S(g). \tag{1.5}$$

If $T, S \in \mathfrak{S}$, $S \sqsubset T$ while $T \in \mathfrak{T}$, then also $S \in \mathfrak{T}$.

In Theorem 2.8 of [11] it was stated that the Kurzweil-Henstock integral K belongs to the class $\mathfrak{T}$.

Let us mention the following essential fact. With regard to the requirement (1.2) and according to Theorem 1.12 we have

$$S \in \mathfrak{T} \Longrightarrow S \sqsubset K, \tag{1.6}$$

where K is the Kurzweil-Henstock integral.

2. Some new extensions

The subclass $\mathfrak{T}$ of integrals given by Definition 1.13 will be dealt with in the sequel.

2.1. The extension Q_X

Definition 2.1. For $S \in \mathfrak{T}$ denote by S_X the set of all (f, γ) for which there exist $F \in C^*(E)$, measurable sets $N_1, N_2 \subset E$ with $\mu(N_1) = \mu(N_2) = 0$, a sequence (f_j) in $\operatorname{Dom}(S)$, $j \in \mathbb{N}$ and a sequence (M_k), $k \in \mathbb{N}$ of measurable subsets of E such that $\gamma = F[E]$ and

$$f(x) = \lim_{j\to\infty} f_j(x) \quad \text{for } x \in E \setminus N_1, \tag{2.1}$$

$$M_k \nearrow E \setminus N_2, \tag{2.2}$$

$$\text{if } k \in \mathbb{N} \text{ then } W_{F-F_j}(M_k) \to 0 \quad \text{for } j \to \infty, \tag{2.3}$$

F_j being an S-primitive to f_j.

The set $\{(S, S_X);\ S \in \mathfrak{T},\ S_X \text{ exists}\}$ is denoted by Q_X.

Q_X is a mapping from $\mathfrak{T}$ to the set of functionals in E defined by $Q_X(S) = S_X$ for $S \in \operatorname{Dom}(Q_X)$.

The following characterization (or equivalent definition) of S_X will be useful.

Lemma 2.2. *Let f be a function on E, $\gamma \in \mathbb{R}$ and $S \in \mathfrak{T}$. Then $(f, \gamma) \in S_X$ if and only if there exist $F \in C^*(E)$, a measurable set $N \subset E$ with $\mu(N) = 0$, a sequence (f_j) in* $\operatorname{Dom}(S)$, *$j \in \mathbb{N}$ and a sequence (A_k) of closed subsets of E such that $\gamma = F[E]$ and*

$$A_k \nearrow E \setminus N \quad \text{for } k \to \infty, \tag{2.4}$$

$$\text{if } k \in \mathbb{N}, \text{ then } |f - f_j|_{A_k} \to 0 \quad \text{for } j \to \infty, \tag{2.5}$$

$$\text{if } k \in \mathbb{N}, \text{ then } W_{F-F_j}(A_k) \to 0 \quad \text{for } j \to \infty \tag{2.6}$$

hold, where F_j is an S-primitive to f_j.

Proof. Assume that $(f, \gamma) \in S_X$, i.e. that (2.1)–(2.3) hold.

Since (2.1) holds and f_j are measurable (cf. (1.3) in Definition 1.13), by Egoroff's theorem (see e.g. Proposition 2.9 in [11] or Theorem 2.13 in [3]) there exists a subsequence (g_j) of (f_j) and a sequence (B_k) of closed sets such that $B_k \nearrow E \setminus N_3$ for $k \to \infty$ where $N_3 \subset E$ with $\mu(N_3) = 0$ and

$$|f - g_j|_{B_k} \to 0 \quad \text{for } j \to \infty$$

for any $k \in \mathbb{N}$.

Further, by (2.2), there is a sequence (C_k) of closed sets $C_k \subset M_k$ for $k \in \mathbb{N}$ and $C_k \nearrow E \setminus N_4$, $k \to \infty$ where $\mu(N_4) = 0$.

Then (2.4)–(2.6) is satisfied for $A_k = B_k \cap C_k$, $N = N_3 \cup N_4$ and $f_j = g_j$.

The other implication is straightforward. $\square$

Our effort is now oriented to showing that the functional S_X in E (see the Introduction) is an integral.

A quadruple $(F, (f_j), (A_k), N)$ having the properties given in Lemma 2.2 will be called *S_X-determining for f* if (2.4)–(2.6) hold.

Lemma 2.3. *Let $S \in \mathfrak{T}$, let f be a function on E and let*

$$(F, (f_j), (A_k), N_1), \quad (G, (g_j), (B_k), N_2)$$

be two S_X-determining quadruples for f.

Then there exists a constant $c \in \mathbb{R}$ such that

$$F(x) = G(x) + c \quad \text{for } x \in E. \tag{2.7}$$

Proof. Let F_j, G_j be S-primitives to f_j, g_j, respectively, $j \in \mathbb{N}$.

Let us set $C_k = A_k \cap B_k$ for $k \in \mathbb{N}$ and $N = N_1 \cup N_2$. Using the properties of the variational measure $W_F(\cdot)$ (see (i) and (iv) from Theorem 1.11) we have

$$\begin{aligned} W_{F-G}(C_k) &\leqslant W_{F-F_j}(C_k) + W_{F_j-G_j}(C_k) + W_{G-G_j}(C_k) \\ &\leqslant W_{F-F_j}(A_k) + W_{F_j-G_j}(C_k) + W_{G-G_j}(B_k) \end{aligned}$$

for any $j, k \in \mathbb{N}$.

Since $S \in \mathfrak{T}$, (1.4) from Definition 1.13 yields

$$\begin{aligned} W_{F_j-G_j}(C_k) \leqslant \lambda |f_j - g_j|_{C_k} &\leqslant \lambda |f - f_j|_{C_k} + \lambda |f - g_j|_{C_k} \\ &\leqslant \lambda |f - f_j|_{A_k} + \lambda |f - g_j|_{B_k} \end{aligned}$$

and therefore

$$W_{F-G}(C_k) \leqslant W_{F-F_j}(A_k) + \lambda |f - f_j|_{A_k} + \lambda |f - g_j|_{B_k} + W_{G-G_j}(B_k).$$

By (2.5) and (2.6) the right-hand side of this inequality converges to 0 for $j \to \infty$ and therefore $W_{F-G}(C_k) = 0$ for $k \in \mathbb{N}$. Hence, by (ii) from Theorem 1.11 and by Lemma 2.13 in [10], we get

$$\begin{aligned} W_{F-G}(E) &\leqslant W_{F-G}(N) + W_{F-G}(E \setminus N) \\ &= W_{F-G}(N) + \lim_{k \to \infty} W_{F-G}(C_k) = 0 \end{aligned}$$

and this is equivalent to (2.7) because by Lemma 2.2 in [10] we have $W_{F-G}(E) = V(F - G, E) = 0$, $V(F - G, E)$ being the total variation of $F - G$ over E and $V(F - G, E) = 0$. $\square$

Lemma 2.4. *If $S \in \mathfrak{T}$ then $S_X \in \mathfrak{S}$, i.e. Q_X is a mapping from $\mathfrak{T}$ into $\mathfrak{S}$. Moreover, the S_X-primitive to $f \in \operatorname{Dom}(S_X)$ belongs to $C^*(E)$.*

P r o o f. It is clear that $0 \in \operatorname{Dom}(S_X)$ and $S_X(0) = 0$.

Assume that $c \in [a, b] = E$ and set $I_1 = [a, c]$, $I_2 = [c, b]$. If $f \in \operatorname{Dom}(S_X)$ and if $(F, (f_j), (A_k), N)$ is S_X-determining for f then it can be easily seen that $(G, (g_j), (A_k), N)$ with $G = (F - F(c)) \cdot \chi(I_1)$ and $g_j = f_j \cdot \chi(I_1)$ is S_X-determining for $f \cdot \chi(I_1)$, i.e. $f \cdot \chi(I_1) \in \operatorname{Dom}(S_X)$ and

$$S_X(f, I_1) = G[E] = F[I_1]. \tag{2.8}$$

Quite analogously it can be shown that $f \cdot \chi(I_2) \in \operatorname{Dom}(S_X)$.

On the other hand, let $f \cdot \chi(I_1), f \cdot \chi(I_2) \in \operatorname{Dom}(S_X)$ and let

$$(G, (g_j), (B_k), N_1), \quad (H, (h_j), (C_k), N_2)$$

be S_X-determining for $f \cdot \chi(I_1)$, $f \cdot \chi(I_2)$, respectively.

Then $(F, (f_j), (A_k), N)$ with $F = (G - G(c)) \cdot \chi(I_1) + (H - H(c)) \cdot \chi(I_2)$, $f_j = g_j \cdot \chi(I_1) + h_j \cdot \chi((c, b])$, $A_k = B_k \cap C_k$ and $N = N_1 \cup N_2$ is S_X-determining for f, i.e. $f \in \operatorname{Dom}(S_X)$.

This, in particular (2.8), shows that if $(F, (f_j), (A_k), N)$ is S_X-determining for f, then F is an S_X-primitive function to f and $F \in C^*(E)$. Therefore $S_X \in \mathfrak{S}$. $\square$

The next theorem is the main statement on the map Q_X.

Theorem 2.5. *Q_X is an extension which maps $\mathfrak{T}$ into $\mathfrak{T}$.*

P r o o f. It is easy to verify that $S \sqsubset S_X$ for $S \in \mathfrak{T}$ and that $S_X \sqsubset T_X$ whenever $S, T \in \mathfrak{T}$ and $S \sqsubset T$.

It remains to prove that if $S \in \mathfrak{T}$ then also $S_X \in \mathfrak{T}$.

The conditions (1.1), (1.3) are easy to check for S_X and (1.2) follows from Lemma 2.4.

Let $f \in \operatorname{Dom}(S_X)$ and let A be a closed subset of E. Further, let

$$(F, (f_j), (B_k), N)$$

be S_X-determining for f and let F_j be an S_X-primitive function to f_j for $j \in \mathbb{N}$.

For $k \in \mathbb{N}$ we then have (see (1.4) and Theorem 1.11)

$$\begin{aligned} W_F(A \cap B_k) &\leqslant W_{F-F_j}(A \cap B_k) + W_{F_j}(A \cap B_k) \\ &\leqslant W_{F-F_j}(A \cap B_k) + \lambda |f_j|_{A \cap B_k} \\ &\leqslant W_{F-F_j}(B_k) + \lambda |f - f_j|_{A \cap B_k} + \lambda |f|_{A \cap B_k} \\ &\leqslant W_{F-F_j}(B_k) + \lambda |f - f_j|_{B_k} + \lambda |f|_A \end{aligned}$$

for $j \in \mathbb{N}$. Hence, by (2.6) and (2.5),

$$W_F(A \cap B_k) \leqslant \lambda |f|_A.$$

Now we have

$$W_F(A) \leqslant W_F(A \cap N) + \lim_{k \to \infty} W_F(A \cap B_k) \leqslant \lambda |f|_A,$$

i.e. S_X fulfils (1.4) with the same λ as S.

Further, assume that $g, h \in \operatorname{Dom}(S_X)$ and that

$$(G,(g_j),(B_k),N_1), \quad (H,(h_j),(C_k),N_2)$$

are S_X- determining for g, h, respectively. Then it is easy to see that $(\alpha G+\beta H, (\alpha g_j+\beta h_j),(A_k),N)$ for $\alpha,\beta\in\mathbb{R}$ with $A_k=B_k\cap C_k$ and $N=N_1\cup N_2$ is S_X-determining for $\alpha g+\beta h$ and this yields the linearity of S_X required by (1.5) from Definition 1.13. $\square$

Theorem 2.6. *The extension Q_X is effective, i.e. $Q_X^2=Q_X$.*

Proof. Denote $S_{XX}=(S_X)_X$ and assume that $f\in\operatorname{Dom}(S_{XX})$. Let $(F,(f_j),(A_k),N)$ be S_{XX}-determining for f.

For $k\in\mathbb{N}$, $m\in\mathbb{N}$ let F_m be an S_X-primitive function to f_m and let

$$(F_m,(g_j^{(m)}),(B_k^{(m)}),N_m)$$

be S_X-determining for f_m.

It is straightforward that $\mu(B_j^{(j)})\geqslant\mu(E)-1/2^j$ may be supposed for $j\in\mathbb{N}$ and this yields $C_k\nearrow E\setminus M$ with $\mu(M)=0$, where

$$C_k=\bigcap_{j=k}^{\infty}B_j^{(j)}$$

for $k\in\mathbb{N}$. Indeed,

$$\mu(C_k)=\mu(E)-\mu(E\setminus C_k)\geqslant\mu(E)-\sum_{j=k}^{\infty}\mu(E\setminus B_j^{(j)})\geqslant\mu(E)-\frac{1}{2^{k-1}}$$

for $k\in\mathbb{N}$.

Further, it may be supposed that

$$|f_j-g_j^{(j)}|_{C_j}<\frac{1}{2^j}, \quad W_{F_j-G_j^{(j)}}(C_j)<\frac{1}{2^j}$$

for $j\in\mathbb{N}$, where $G_j^{(j)}$ is an S-primitive function to $g_j^{(j)}$.

It suffices to show that $(F,g_j^{(j)},(A_k\cap C_k),N\cup M)$ is S_X-determining for f.

This follows from the fact that for $j\geqslant k$ the estimates

$$|f-g_j^{(j)}|_{A_k\cap C_k}\leqslant|f-f_j|_{A_k}+\frac{1}{2^j},$$
$$W_{F-G_j^{(j)}}(A_k\cap C_k)\leqslant W_{F-F_j}(A_k)+\frac{1}{2^j}$$

hold. $\square$

2.2. The extension Q_Z

Definition 2.7. If $S \in \mathfrak{T}$ then S_Z denotes the set of all pairs (f, γ) for which there exists a function $F \in C^*(E)$ and a sequence (A_k) of closed subsets of E such that $\gamma = F[E]$ and

$$A_k \nearrow E, \tag{2.9}$$

$$f_j = f \cdot \chi(A_j) \in \mathrm{Dom}(S) \quad \text{for } j \in \mathbb{N}, \tag{2.10}$$

$$W_{F-F_j}(A_k) = 0 \quad \text{for } j \geqslant k, \tag{2.11}$$

$$\text{if } k \in \mathbb{N}, \text{ then } \sum_{U \in \mathrm{Comp}(E, A_k)} \omega(F - F_j, \overline{U}) \to 0 \quad \text{for } j \to \infty \tag{2.12}$$

hold, where F_j is an S-primitive function to f_j and $\mathrm{Comp}(E, A_k)$ is the set of all maximal non-empty connected components of the set $E \setminus A_k$.

The set $\{(S, S_Z);\ S \in \mathfrak{T},\ S_Z \text{ exists}\}$ is denoted by Q_Z.

Comparing this definition with the characterization of S_X given in Lemma 2.2 we can easily see that if $S \in \mathfrak{T}$ then $S \sqsubset S_Z \sqsubset S_X$. The first inclusion is clear, (2.9) implies (2.4) with $N = \emptyset$, (2.10) implies (2.5) for $f_j = f \cdot \chi(A_j)$ and (2.11) implies (2.6). In Theorem 2.5 we have shown that $S_X \in \mathfrak{T}$. Hence by $S_Z \sqsubset S_X$ we have also $S_Z \in \mathfrak{T}$.

In other words, the following statement is valid.

Theorem 2.8. *Q_Z is an extension which maps $\mathfrak{T}$ into itself and*

$$Q_Z(S) \sqsubset Q_X(S) \tag{2.13}$$

for any $S \in \mathfrak{T}$.

The next assertion will be used directly for some characterization theorems using the Cauchy and Harnack extensions P_C and P_H presented in Section 4 of [11], cf. the subsection 1.2.

Theorem 2.9. *For any $S \in \mathfrak{T}$ the integral $Q_Z(S)$ is P_C-invariant, i.e.*

$$P_C(Q_Z(S)) \sqsubset Q_Z(S), \tag{2.14}$$

holds.

Proof. We have to show that if $S \in \mathfrak{T}$ then $(S_Z)_C \sqsubset S_Z$.

Assume that $f \in \mathrm{Dom}((S_Z)_C)$. Then $\sigma(f, S_Z)$ is finite by Definition 1.7 (of the Cauchy extension) and there is an $F \in C(E)$ such that $F[I] = S_Z(f, I)$ for every $I \in \mathrm{Sub}(E)$, $I \subset \varrho(f, S_Z)$.

Let us consider the special situation when $\sigma(f, S_Z) = b$, i.e. there is only one S_Z-singular point of f at the right endpoint of E. Then $f \cdot \chi([a, x]) \in \mathrm{Dom}(S_Z)$ for every $x < b$ and $F[[a, x]] = S_Z(f, [a, x])$ and therefore also $f \cdot \chi([a, x]) \in \mathrm{Dom}(S)$ for every $x < b$ and $F[[a, x]] = S(f, [a, x])$.

If $I \subset [a, b)$ then $f \cdot \chi(I) \in \mathrm{Dom}(S_Z)$ and because $S \subset S_Z$ we have also $f \cdot \chi(I) \in \mathrm{Dom}(S)$ by (2.10) and

$$F[I] = S_Z(f, I) = S(f, I).$$

This implies that $F \in C^*([a, c])$ for every $c \in [a, b)$.

Assume that $N \subset E$ is measurable, that $\mu(N) = 0$ and define

$$M_k = \Big[a, b - \frac{1}{k}(b-a)\Big] \cap N, \quad k \in \mathbb{N}.$$

Then M_k is measurable, $M_k \subset M_{k+1}$, $\mu(M_k) = 0$ for $k \in \mathbb{N}$ and $N = \bigcup\limits_{k=1}^{\infty} M_k$.

Since $M_k \subset \big[a, b - k^{-1}(b-a)\big]$, we have $W_F(M_k) = 0$ because $F \in C^*\big(\big[a, b - k^{-1}(b-a)\big]\big)$.

Hence by (ii) from Theorem 1.11 we have

$$0 \leqslant W_F(N) = W_F\Big(\bigcup_{k=1}^{\infty} M_k\Big) \leqslant \sum_{k=1}^{\infty} W_F(M_k) = 0$$

and $W_F(N) = 0$. By the property of $C^*(E)$ presented in the subsection 1.5 this means that $F \in C^*(E)$.

Define now

$$A_k = \Big[a, b - \frac{1}{k}(b-a)\Big] \cup \{b\}.$$

Evidently for $k \in \mathbb{N}$ the sets $A_k \subset E$ are closed, $A_k \subset A_{k+1}$, $A_k \nearrow E$ and

$$f_j = f \cdot \chi(A_j) = f \cdot \chi\Big(\Big[a, b - \frac{1}{j}(b-a)\Big]\Big) + f \cdot \chi(\{b\}) \in \mathrm{Dom}(S)$$

for every $j \in \mathbb{N}$.

Assume that F_j is an S-primitive function to f_j. Then $F - F_j$ is constant on $\big[a, b - j^{-1}(b-a)\big]$ and by Lemma 2.2 in [10] we get $W_{F-F_j}\big(\big[a, b - j^{-1}(b-a)\big]\big) = 0$. Evidently we also have $W_{F-F_j}(\{b\}) = 0$. Hence by (ii) from Theorem 1.11 we obtain

$$0 \leqslant W_{F-F_j}(A_j) \leqslant W_{F-F_j}\Big(\Big[a, b - \frac{1}{j}(b-a)\Big]\Big) + W_{F-F_j}(\{b\}) = 0,$$

i.e. $W_{F-F_j}(A_j) = 0$ for every $j \in \mathbb{N}$.

If $k \in \mathbb{N}$ is given then $A_k \subset A_j$ for $j \geqslant k$ and by (i) from Theorem 1.11 we get

$$W_{F-F_j}(A_k) \leqslant W_{F-F_j}(A_j) = 0,$$

i.e. (2.11) is satisfied.

Let us mention that in our situation $\mathrm{Comp}(E, A_k) = \big(b - k^{-1}(b-a), b\big) = V$ consists of only one element and $\overline{V} = \big[b - k^{-1}(b-a), b\big]$.

Assume that $j \geqslant k$; then $\overline{V} = \big[b - k^{-1}(b-a), b - j^{-1}(b-a)\big] \cup \big[b - j^{-1}(b-a), b\big]$.

We have $F[I] = F_j[I]$ for every $I \subset \big[b - k^{-1}(b-a), b - j^{-1}(b-a)\big]$ and therefore $\omega\big(F - F_j, \big[b - k^{-1}(b-a), b - j^{-1}(b-a)\big]\big) = 0$. Further, on $\big[b - j^{-1}(b-a), b\big]$ the function $F - F_j$ equals $F\big(b - j^{-1}(b-a)\big)$ and therefore

$$\omega\Big(F - F_j, \Big[b - \frac{1}{j}(b-a), b\Big]\Big) = \omega\Big(F, \Big[b - \frac{1}{j}(b-a), b\Big]\Big).$$

Since F is continuous at the point b we get that for every $\varepsilon > 0$ there is a $j_0 \in \mathbb{N}$ such that for $j \geqslant j_0$ and $x \in \big[b - j^{-1}(b-a), b\big]$ we have $|F(x) - F(b)| < \varepsilon$. Hence

$$|F(x) - F(y)| \leqslant |F(x) - F(b)| + |F(y) - F(b)| < 2\varepsilon$$

for $x, y \in \big[b - j^{-1}(b-a), b\big]$ and

$$\omega\Big(F, \Big[b - \frac{1}{j}(b-a), b\Big]\Big) < 2\varepsilon$$

for $j \geqslant j_0$. This implies

$$\sum_{U \in \mathrm{Comp}(E, A_k)} \omega(F - F_j, \overline{U}) = \omega(F - F_j, \overline{V}) \to 0$$

for $j \to \infty$ and (2.12) holds.

Hence $f \in \mathrm{Dom}(S_Z)$ and (2.14) is proved.

The case $\sigma(f, S_Z) = a$ (only one S_Z-singular point of f at the left endpoint of E) can be treated similarly.

In the general situation of $f \in \mathrm{Dom}((S_Z)_C)$ the set $\sigma(f, S_Z)$ is finite and the set $\mathrm{Comp}(E, \sigma(f, S_Z))$ consists therefore of a finite set $\{U_j;\ j = 1, \ldots, k\}$ of intervals the endpoints of which belong to $\sigma(f, S_Z)$. Taking a point $c \in U \in \mathrm{Comp}(E, \sigma(f, S_Z))$ we get two intervals $[l(\overline{U}), c]$ and $[c, r(\overline{U})]$ having the left or right endpoint in $\sigma(f, S_Z)$; using the procedure described above we show that

$$\begin{aligned} f \cdot \chi(\overline{U}) &= f \cdot \chi([l(\overline{U}), r(\overline{U})]) \\ &= f \cdot \chi([l(\overline{U}), c]) + f \cdot \chi([c, r(\overline{U})]) \in \mathrm{Dom}(S_Z) \end{aligned}$$

and since $\{\overline{U};\ U \in \mathrm{Comp}(E, \sigma(f, S_Z))\}$ is a division of E we obtain immediately $f \in \mathrm{Dom}(S_Z)$. This means that (2.14) holds in general. $\square$

Lemma 2.10. *For $F \in C(E)$, $I \in \operatorname{Sub}(E)$ and any closed set $A \subset E$ the inequality*

$$\omega(F, I) \leqslant W_F(I \cap A) + \sum_{U \in \operatorname{Comp}(I,A)} \omega(F, \overline{U}) \tag{2.15}$$

holds.

P r o o f. Assume that $\operatorname{Comp}(I, A) = \{U_j ;\ j \in \Phi\}$. If $\Phi = \emptyset$, i.e. if $A = I$, then $W_F(I \cap A) = W(F, I) = V_F(I) = V(F, I)$ by Lemma 2.2 in [10] and (2.15) holds because evidently $\omega(F, I) \leqslant V(F, I)$.

Therefore we may suppose without loss of generality that $A \subset I$, i.e. $I \cap A = A$, and that $\Phi \neq \emptyset$.

Let $\varepsilon > 0$ be given and let $\delta \in \Delta(E)$ be such that

$$W_\delta(F, A) < W_F(A) + \varepsilon.$$

Define a gauge

$$\eta(x) = \begin{cases} \delta(x) & \text{for } x \in A, \\ \min\bigl\{\delta(x), \frac{1}{2}\operatorname{dist}(x, A)\bigr\} & \text{for } x \notin A. \end{cases}$$

Let further $(\{I_j,\ j \in \Gamma\}, \tau)$ be an η-fine division of I and set $\Gamma_1 = \{j \in \Gamma;\ \tau_j \in A\}$, $\Gamma_2 = \Gamma \setminus \Gamma_1$.

Then $(\{I_j,\ j \in \Gamma_1\}, \tau)$ is an η-fine A-tagged division which covers A and therefore any I_j for $j \in \Gamma_2$ is contained in some $\overline{U_k}$ by the choice of the gauge η.

Since $(\{I_j,\ j \in \Gamma_1\}, \tau)$ is evidently also a δ-fine A-tagged division (because $\eta \leqslant \delta$), we have

$$\sum_{j \in \Gamma_1} \omega(F, I_j) \leqslant W_\delta(F, A) < W_F(A) + \varepsilon = W_F(I \cap A) + \varepsilon.$$

Denote $B = \bigcup\limits_{j \in \Gamma_2} I_j$. The set B is closed. Let us set $\operatorname{Comp}(I, B) = \{V_j,\ j \in \Psi\}$; clearly Ψ is finite.

Then any of the finite number of maximal components V_j of $I \setminus B$ is contained in some $\overline{U_k}$ and any $\overline{U_k}$ contains at most one V_j.

Moreover, evidently

$$\sum_{V \in \operatorname{Comp}(I,B)} \omega(F, \overline{V}) \leqslant \sum_{U \in \operatorname{Comp}(I,A)} \omega(F, \overline{U}).$$

Further,

$$\begin{aligned}\omega(F,I) &\leqslant \sum_{j\in\Gamma_1}\omega(F,I_j)+\sum_{V\in\operatorname{Comp}(I,B)}\omega(F,\overline{V})\\ &\leqslant \sum_{j\in\Gamma_1}\omega(F,I_j)+\sum_{U\in\operatorname{Comp}(I,A)}\omega(F,\overline{U})\\ &< W_F(I\cap A)+\sum_{U\in\operatorname{Comp}(I,A)}\omega(F,\overline{U})+\varepsilon\end{aligned}$$

and the lemma is proved since $\varepsilon>0$ can be taken arbitrarily small. □

Theorem 2.11. *For any $S\in\mathfrak{T}$ the integral $Q_Z(S)$ is P_H-invariant, i.e.*

$$P_H(Q_Z(S))\sqsubset Q_Z(S) \tag{2.16}$$

holds.

Proof. For proving (2.16) assume that $S\in\mathfrak{T}$ and $f\in\operatorname{Dom}((S_Z)_H)$. By Definition 1.5 we have to show that $f\in\operatorname{Dom}(S_Z)$.

Theorems 2.8 and 2.5 yield $S_Z\in\mathfrak{T}$.

Definition 4.4 of the Harnack extension in [11] ensures that $f\cdot\chi(\sigma(f,S_Z))\in\operatorname{Dom}(S_Z)$ and $f\cdot\chi(U_j)\in\operatorname{Dom}(S_Z)$ for $j\in\Gamma$, where $\{U_j;\ j\in\Gamma\}=\operatorname{Comp}(E,\sigma(f,S_Z))$, and there is a function $F\in C(E)$ such that $F[E]=F(b)-F(a)$,

$$\sum_{U\in\operatorname{Comp}(E,\sigma(f,S_Z))}\omega(F,\overline{U})<\infty$$

and

$$F[I]=S_Z(f,I\cap\sigma(f,S_Z))+\sum_{j\in\Gamma}S_Z(f,I\cap\overline{U_j}) \tag{2.17}$$

for any $I\in\operatorname{Sub}(E)$.

Since the integral is linear by definition, we have to show that $f-f\cdot\chi(\sigma(f,S_Z))\in\operatorname{Dom}(S_Z)$ because $f\cdot\chi(\sigma(f,S_Z))\in\operatorname{Dom}(S_Z)$. Without loss of generality we can assume that $f\cdot\chi(\sigma(f,S_Z))=0$.

The set $\sigma(f,S_Z)$ is closed. Assume that for

$$\{U_j;\ j\in\Gamma\}=\operatorname{Comp}(E,\sigma(f,S_Z))$$

we have $\Gamma=\mathbb{N}$. The case when Γ is finite is easy.

Denoting $A=\sigma(f,S_Z)$ we can reformulate the properties given above as follows.

There are a closed set $A \subset E$, a countable system $\{U_j;\ j \in \mathbb{N}\} = \operatorname{Comp}(E, A)$ and functions $F \in C(E)$, $F_j \in C(E)$, $j \in \mathbb{N}$ such that

$$f \cdot \chi(A) = 0, \quad f_j = f \cdot \chi(U_j) \in (S_Z), \quad j \in \mathbb{N},$$
$$\sum_{j=1}^{\infty} \omega(F, \overline{U_j}) < \infty,$$

F is an $(S_Z)_H$ primitive to f, F_j are S_Z primitives to f_j, $j \in \mathbb{N}$. By Corollary 4.13 in [11] we have $F \in C^*(E)$ and $F_j \in C^*(E)$, $j \in \mathbb{N}$, because $S_Z \in \mathfrak{T}$.

By (2.17) we have

$$F(x) - F(y) = S_Z(f, [x, y]) = F_j(x) - F_j(y)$$

for $[x, y] \subset \overline{U_j}$, $j \in \mathbb{N}$. This yields

$$\omega(F, \overline{U_j}) = \omega(F_j, \overline{U_j}) \quad \text{for } j \in \mathbb{N} \tag{2.18}$$

and also

$$\omega(F - F_j, \overline{U_j}) = 0 \quad \text{for } j \in \mathbb{N},$$

i.e. $F - F_j$ is constant on $\overline{U_j}$ and

$$W_{F-F_j}(\overline{U_j}) = 0. \tag{2.19}$$

If $j \neq k$ then $f_j(x) = 0$ for $x \in \overline{U_k}$. Hence

$$F_j(x) - F_j(y) = S_Z(f_j, [x, y]) = 0$$

for $[x, y] \subset \overline{U_k}$. Therefore

$$\omega(F_j, \overline{U_k}) = 0, \quad \omega(F - F_j, \overline{U_k}) = \omega(F, \overline{U_k}) \quad \text{for } j \neq k.$$

By (2.18) we have

$$\sum_{j \in \mathbb{N}} \omega(F_j, \overline{U_j}) = \sum_{j \in \mathbb{N}} \omega(F, \overline{U_j}) = \sum_{U \in \operatorname{Comp}(E, A)} \omega(F, \overline{U}) < \infty.$$

This means that for any $\varepsilon > 0$ there is an $m \in \mathbb{N}$ such that

$$\sum_{j=m}^{\infty} \omega(F, \overline{U_j}) < \varepsilon. \tag{2.20}$$

Since $f_j \in \mathrm{Dom}(S_Z)$ for all $j \in \mathbb{N}$, Definition 2.7 of S_Z yields that there is a sequence of closed subsets $B_{j,k} \subset E$, $k \in \mathbb{N}$ such that

(a) $B_{j,k} \nearrow E$ for $k \to \infty$,

(b) $g_{j,i} = f_j \cdot \chi(B_{j,i}) = f \cdot \chi(U_j \cap B_{j,i}) \in \mathrm{Dom}(S)$ for $i \in \mathbb{N}$,

(c) $W_{F_j - G_{j,i}}(B_{j,k}) = 0$ for $i \geqslant k$,

(d) if $k \in \mathbb{N}$ then $\sum\limits_{U \in \mathrm{Comp}(E, B_{j,k})} \omega(F_j - G_{j,i}, \overline{U}) \to 0$ for $i \to \infty$

hold, where $G_{j,i} \in C^*(E)$ is an S-primitive function to $g_{j,i}$.

Let us reformulate property (d) as follows.

For every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$, $n_k > k$, $n_{k+1} > n_k$ such that for any $i \geqslant n_k$ the inequality

$$\sum_{U \in \mathrm{Comp}(E, B_{j,k})} \omega(F_j - G_{j,i}, \overline{U}) < \frac{1}{k^2} \tag{2.21}$$

holds.

Define now

$$C_k = A \cup \left(\bigcup_{j=1}^{k} (B_{j,n_k} \cap \overline{U_j}) \right)$$

for $k \in \mathbb{N}$.

The sets C_k are closed and $C_k \nearrow E$ for $k \to \infty$. Further, set

$$h_k = f \cdot \chi(C_k) = \sum_{j=1}^{k} g_{j,n_k} \in \mathrm{Dom}(S) \quad \text{for} \;\; k \in \mathbb{N}$$

(cf. (b)) and put

$$H_k = \sum_{j=1}^{k} G_{j,n_k} \in C^*(E).$$

Note that $H_k = G_{j,n_k}$ on $\overline{U_j}$.

It remains to show that

$$W_{F - H_k}(C_l) = 0 \quad \text{for} \;\; k \geqslant l \tag{2.22}$$

and

$$\sum_{U \in \mathrm{Comp}(E, C_l)} \omega(F - H_k, \overline{U}) \to 0 \quad \text{for} \;\; k \to \infty. \tag{2.23}$$

By (ii) from Theorem 1.11 we have

$$W_{F-H_k}(C_l) \leqslant W_{F-H_k}(A) + \sum_{j=1}^{l} W_{F-H_k}(B_{j,n_l} \cap \overline{U_j}).$$

By Lemma 4.12 in [11] we have $W_F(A) = 0$. Since $g_{j,n_k} \in \mathrm{Dom}(S)$ and $g_{j,n_k} = 0$ on A, Lemma 2.10 from [11] implies

$$W_{G_{j,n_k}}(A) \leqslant \lambda |g_{j,n_k}|_A = 0$$

because $S \in \mathfrak{T}$.

Therefore

$$W_{F-H_k}(A) \leqslant W_F(A) + \sum_{j=1}^{k} W_{G_{j,n_k}}(A) = 0.$$

Further, by (iv) from Theorem 1.11, we get

$$\begin{aligned} &W_{F-H_k}(B_{j,n_l} \cap \overline{U_j}) \\ &\qquad \leqslant W_{F-G_{j,n_k}}(B_{j,n_l} \cap \overline{U_j}) + \sum_{m-1, m \neq j}^{k} W_{G_{m,n_k}}(B_{j,n_l} \cap \overline{U_j}). \end{aligned}$$

We have $W_{G_{m,n_k}}(B_{j,n_l} \cap \overline{U_j}) = 0$ for $m \neq j$ and

$$W_{F-G_{j,n_k}}(B_{j,n_l} \cap \overline{U_j}) \leqslant W_{F-F_j}(\overline{U_j}) + W_{F-G_{j,n_k}}(B_{j,n_l}) = 0$$

by (2.19) and (c). Hence (2.22) holds.

For showing (2.23) fix $l \in \mathbb{N}$. The components of the complement $E \setminus C_l$, i.e. of $\mathrm{Comp}(E, C_l)$ consist of U_j for $j > l$ and of $\mathrm{Comp}(\overline{U_j}, B_{n,n_l})$ for $j = 1, 2, \ldots, l$, i.e.

$$\mathrm{Comp}(E, C_l) = \{U_j, j > l\} \cup \bigcup_{j=1}^{l} \mathrm{Comp}(\overline{U_j}, B_{n,n_l}).$$

Let $\varepsilon > 0$ be given. Assume that $k > \max(l, m)$. (For $m \in \mathbb{N}$ see (2.20).) Then

$$\begin{aligned} (2.24) \quad \sum_{U \in \mathrm{Comp}(E, C_l)} \omega(F - H_k, \overline{U}) = &\sum_{j=l+1}^{k} \omega(F - H_k, \overline{U_j}) + \sum_{j=k+1}^{\infty} \omega(F - H_k, \overline{U_j}) \\ &+ \sum_{j=1}^{l} \sum_{U \in \mathrm{Comp}(\overline{U_j}, B_{n,n_l})} \omega(F - H_k, \overline{U}). \end{aligned}$$

If $k \geqslant j > l$ then

$$\begin{aligned}\omega(F - H_k, \overline{U_j}) &= \omega(F - G_{j,n_k}, \overline{U_j}) \\ &= \omega(F_j - G_{j,n_k}, \overline{U_j}) = \omega(F_j - G_{j,n_k}, E).\end{aligned}$$

Lemma 2.10, (c) and (2.18) give

$$\omega(F_j - G_{j,n_k}, E) \leqslant W_{F_j - G_{j,n_k}}(B_{j,k}) + \sum_{U \in \mathrm{Comp}(E, B_{j,k})} \omega(F_j - G_{j,n_k}, \overline{U}) \leqslant \frac{1}{k^2}$$

and consequently,

$$\sum_{j=l+1}^{k} \omega(F - H_k, \overline{U_j}) \leqslant \frac{1}{k}$$

is an estimate of the first term on the right-hand side of (2.24).

If $j > k$, then $h_k(x) = 0$ for $x \in U_j$, therefore H_k is constant on U_j and $\omega(F - H_k, \overline{U_j}) = \omega(F, \overline{U_j})$. Hence

$$\sum_{j=k+1}^{\infty} \omega(F - H_k, \overline{U_j}) < \sum_{j=m}^{\infty} \omega(F, \overline{U_j}) < \varepsilon$$

by (2.20) and this is the estimate of the second term on the right-hand side of (2.24).

Let us denote $\mathrm{Comp}(\overline{U_j}, B_{n,n_l}) = \{V_l\,;\ l \in \Gamma_{j,l}\}$ for $j = 1, 2, \ldots, l$. Then

$$\begin{aligned}\sum_{l \in \Gamma_{j,l}}^{k} \omega(F - H_k, \overline{V_j}) &= \sum_{l \in \Gamma_{j,l}}^{k} \omega(F - G_{j,n_k}, \overline{V_j}) \\ &\leqslant \sum_{U \in \mathrm{Comp}(E, B_{j,n_l})} \omega(F_j - G_{j,n_k}, \overline{U}),\end{aligned}$$

while the right-hand side goes to zero for $k \to \infty$ by (d).

Finally, we get

$$\sum_{U \in \mathrm{Comp}(E, C_l)} \omega(F - H_k, \overline{U}) < \frac{1}{k} + \varepsilon + \sum_{U \in \mathrm{Comp}(E, B_{j,n_l})} \omega(F_j - G_{j,n_k}, \overline{U})$$

and (2.23) is satisfied.

All these facts show that $f \in \mathrm{Dom}(S_Z)$ and (2.16) is proved. $\square$

3. Some consequences

By Theorem 2.5 we know that if $S \in \mathfrak{T}$ then $Q_X(S)$ is Kurzweil-Henstock integrable, i.e.

$$Q_X(S) \sqsubset K \tag{3.1}$$

(see (1.6)).

This together with Theorem 2.8 leads for $S \in \mathfrak{T}$ to

$$Q_Z(S) \sqsubset Q_X(S) \sqsubset K. \tag{3.2}$$

Further, Theorems 2.9 and 2.11 give for the Cauchy and the Harnack extension the following two relations:

$$P_C(Q_Z(S)) \sqsubset Q_Z(S) \sqsubset Q_X(S) \sqsubset K, \tag{3.3}$$

$$P_H(Q_Z(S)) \sqsubset Q_Z(S) \sqsubset Q_X(S) \sqsubset K. \tag{3.4}$$

This means that for a given $S \in \mathfrak{T}$ the extension $Q_Z(S)$ is P_C-invariant and P_H-invariant as well.

Since the Lebesgue integral L belongs to $\mathfrak{T}$, the relations given above can be used for $S = L$. First of all we have, by definition of an extension, the relation $L \sqsubset Q_Z(L)$.

In Theorem 4.10 in the paper [11] the following was shown:

Assume that $S \in \mathfrak{S}$, where $L \sqsubset S$ and $P_C(S) = P_H(S) = S$. Then $K \sqsubset S$.

The Kurzweil-Henstock integral K is contained in every integral which contains the Lebesgue integral L and which is P_C- and P_H-invariant.

Hence the before mentioned result quoted from [11] and (3.2) for $S = L$ give

$$K \sqsubset Q_Z(L) \sqsubset Q_X(L) \sqsubset K \tag{3.5}$$

and this means that

$$Q_Z(L) = Q_X(L) = K. \tag{3.6}$$

Let us consider the equality $Q_X(L) = K$ using the property of the extension Q_X presented in Lemma 2.2. We obtain the following statement.

Proposition 3.1. *A function is Kurzweil-Henstock integrable ($f \in \mathrm{Dom}(K)$) if and only if there exist $F \in C^*(E)$, a measurable set $N \subset E$ with $\mu(N) = 0$, a sequence (f_j) in $\mathrm{Dom}(L)$, $j \in \mathbb{N}$ and a sequence (A_k) of closed subsets of E such that*

$$A_k \nearrow E \setminus N \quad \text{for } k \to \infty, \tag{3.7}$$

$$\text{if } k \in \mathbb{N}, \text{ then } |f - f_j|_{A_k} \to 0 \quad \text{for } j \to \infty, \tag{3.8}$$

$$\text{if } k \in \mathbb{N}, \text{ then } W_{F-F_j}(A_k) \to 0 \quad \text{for } j \to \infty \tag{3.9}$$

hold, where F_j is an L-primitive to f_j.

Using this statement we obtain

Proposition 3.2. *Let $f_j \in \mathrm{Dom}(L)$, $j \in \mathbb{N}$ and*

$$\lim_{j \to \infty} f_j(x) = f(x) \quad \text{almost everywhere in } E.$$

Then there exists a sequence (A_k) of closed subsets of E and a subsequence (g_j) of (f_j) such that $A_k \nearrow E \setminus N$, where $\mu(N) = 0$ and for every $k \in \mathbb{N}$ we have

$$|f - g_j|_{A_k} \to 0 \quad \text{for } j \to \infty.$$

If $k \in \mathbb{N}$ and

$$W_{F-G_j}(A_k) \to 0 \quad \text{for } j \to \infty \tag{3.10}$$

where G_j is an L-primitive to g_j and $F \in C^*(E)$, then f is Kurzweil-Henstock integrable ($f \in \mathrm{Dom}(K)$).

The first part of the proposition is the Egoroff Theorem, the latter is a consequence of Proposition 3.1.

Taking into account the relation (3.6) and the definitions of the extensions Q_X and Q_Z applied to the Lebesgue integral L various descriptions of the Kurzweil-Henstock (= Denjoy special) integral can be presented in the flavour of similar results given by S. Nakanishi in [8], and also some convergence results for the Kurzweil-Henstock integral are easily derivable.

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