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GENERALIZED DERIVATIONS ASSOCIATED WITH HOCHSCHILD 2-COCYCLES ON SOME ALGEBRAS

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Abstract. We investigate a new type of generalized derivations associated with Hochschild 2-cocycles which was introduced by A. Nakajima. We show that every generalized Jordan derivation of this type from CSL algebras or von Neumann algebras into themselves is a generalized derivation under some reasonable conditions. We also study generalized derivable mappings at zero point associated with Hochschild 2-cocycles on CSL algebras.

Keywords: CSL algebra, generalized derivation, generalized Jordan derivation, Hochschild 2-cocycle

MSC 2010: 47B47, 47L35

1. Introduction

In [12], Nakajima introduces a new type of generalized derivations associated with Hochschild 2-cocycles. The generalized derivations contain left multipliers, (α,β)-derivations and another type of generalized derivations discussed in [1], [7], [13]. In [12], Nakajima shows that under certain conditions, every generalized Jordan derivation is a generalized derivation. This result improves the results in [1], [6]. In [8], the first author and Pan consider the usual generalized derivable mappings of CSL algebras at zero point and prove that these mappings are usual generalized derivations. In this paper we study which algebras $\mathcal{A}$ have the following property: Every generalized Jordan derivation on them is a generalized derivation. We also consider generalized derivable mappings at zero point on CSL algebras.

Let $\mathcal{A}$ be an algebra over the complex field $\mathbb{C}$, and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. For a bilinear mapping $\alpha\colon \mathcal{A}\times\mathcal{A}\to\mathcal{M}$, α is said to be a *Hochschild 2-cocycle* if

$$x\alpha(y,z)-\alpha(xy,z)+\alpha(x,yz)-\alpha(x,y)z=0 \quad \text{for any } x,y,z\in\mathcal{A}.$$

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A linear mapping $\delta\colon \mathcal{A} \to \mathcal{M}$ is called a *generalized derivation* if there is a Hochschild 2-cocycle α such that

$$\delta(xy) = \delta(x)y + x\delta(y) + \alpha(x, y) \quad \text{for any } x, y \in \mathcal{A},$$

and δ is called a *generalized Jordan derivation* if

$$\delta(x^2) = \delta(x)x + x\delta(x) + \alpha(x, x) \quad \text{for any } x \in \mathcal{A}.$$

We denote it by (δ, α). If $\alpha = 0$, then they are the usual derivations and the Jordan derivations, respectively. By examples in [12], we know that the usual generalized derivations defined in [1], [7], [13], left centralizers and (α, β)-derivations are generalized derivations in the above sense.

In this paper, let H be a complex separable Hilbert space and let $B(H)$ be the set of all bounded operators on H. For convenience we disregard the distinction between a closed subspace and the orthogonal projection onto it. If e, f are in H, then the operator $x \mapsto f(x)e = (x, f)e$ is denoted by $e \otimes f$. A *subspace lattice* on H is a collection $\mathcal{L}$ of subspaces of H with (0), H in $\mathcal{L}$ and such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\bigcap M_r$ and $\bigvee M_r$ belong to $\mathcal{L}$, where $\bigvee M_r$ denotes the closed linear span of $\{M_r\}$. A totally ordered subspace lattice is called a *nest*. For a subspace lattice $\mathcal{L}$, we define $\operatorname{alg}\mathcal{L}$ by

$$\operatorname{alg}\mathcal{L} = \{T \in B(H)\colon\ TN \subseteq N \text{ for any } N \in \mathcal{L}\}.$$

A subspace lattice $\mathcal{L}$ is called a *commutative subspace lattice* (CSL) if it consists of mutually commuting projections. If $\mathcal{L}$ is a commutative subspace lattice, then $\operatorname{alg}\mathcal{L}$ is called a *CSL algebra*.

The paper is organized as follows.

In Section 2, motivated by [4], we show that every generalized Jordan derivation of the above type from a von Neumann algebra $\mathcal{A}$ into any normed $\mathcal{A}$-bimodule $\mathcal{M}$ is a generalized derivation under some reasonable conditions.

In Section 3, we generalize some results of [11] to generalized Jordan derivations. We show that every generalized Jordan derivation of the above type from a CSL algebra into itself is a generalized derivation under certain conditions.

In Section 4, we consider generalized derivable mappings at zero point associated with Hochschild 2-cocyles on CSL algebras and prove that these mappings are generalized derivations.

The following lemma, due to Nakajima [12], will be used repeatedly.

Lemma 1.1 [12, Lemma 2]. *Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ an $\mathcal{A}$-bimodule. If $(f,\alpha)\colon \mathcal{A}\to\mathcal{M}$ is a generalized Jordan derivation associated with a Hochschild 2-cocycle α, then the following relations hold:*

(1) $f(xy+yx)=f(x)y+xf(y)+\alpha(x,y)+f(y)x+yf(x)+\alpha(y,x)$,

(2) $f(xyx)=f(x)yx+xf(y)x+xyf(x)+x\alpha(y,x)+\alpha(x,yx)$,

(3) $f(xyz+zyx)=f(x)yz+xf(y)z+xyf(z)+x\alpha(y,z)+\alpha(x,yz)+f(z)yx+zf(y)x+zyf(x)+z\alpha(y,x)+\alpha(z,yx)$.

2. Generalized Jordan derivations associated with Hochschild 2-cocycles on von Neumann algebras

In what follows, we denote $h(x,y)=\delta(xy)-\delta(x)y-x\delta(y)-\alpha(x,y)$.

Lemma 2.1. *Let (δ,α) be a generalized Jordan derivation of an algebra $\mathcal{A}$ into a 2-torsion free $\mathcal{A}$-bimodule $\mathcal{M}$. If $\mathcal{A}$ has a unit element 1 and $h(x,y)1=0$, then $h(x,y)=0$.*

Proof. By Lemma 1.1 (2), for every $a\in\mathcal{A}$,

$$\delta(a)=\delta(1a1)=\delta(1)a+1\delta(a)1+a\delta(1)+1\alpha(a,1)+\alpha(1,a).$$

Thus

$$\delta(xy)=\delta(1)xy+1\delta(xy)1+xy\delta(1)+1\alpha(xy,1)+\alpha(1,xy).$$

By the assumption,

$$\delta(xy)1=\delta(x)y+x\delta(y)1+\alpha(x,y)1.$$

So

$$\begin{aligned}\delta(xy)=\delta(1)xy&+1\delta(x)y+x\delta(y)1+1\alpha(x,y)1\\&+xy\delta(1)+1\alpha(xy,1)+\alpha(1,xy).\end{aligned}\tag{2.1}$$

Similarly,

$$\begin{aligned}&\delta(x)y+x\delta(y)+\alpha(x,y)\\&\quad=(\delta(1)x+1\delta(x)1+x\delta(1)+1\alpha(x,1)+\alpha(1,x))y\\&\qquad+x(\delta(1)y+1\delta(y)1+y\delta(1)+1\alpha(y,1)+\alpha(1,y))+\alpha(x,y)\\&\quad=\delta(1)xy+1\delta(x)y+x\delta(1)y+1\alpha(x,1)y+\alpha(1,x)y+x\delta(1)y\\&\qquad+x\delta(y)1+xy\delta(1)+x\alpha(y,1)+x\alpha(1,y)+\alpha(x,y).\end{aligned}\tag{2.2}$$

Since

$$x\delta(1)y = x(\delta(1)1 + 1\delta(1) + \alpha(1,1))y = 2x\delta(1)y + x\alpha(1,1)y,$$

it follows that

$$x\delta(1)y = -x\alpha(1,1)y. \tag{2.3}$$

By (2.1), (2.2) and (2.3), we have

$$\begin{aligned} h(x,y) &= 1\alpha(x,y)1 + 1\alpha(xy,1) + \alpha(1,xy) + 2x\alpha(1,1)y - 1\alpha(x,1)y \\ &\quad - \alpha(1,x)y - x\alpha(y,1) - x\alpha(1,y) - \alpha(x,y) \\ &= 2x\alpha(1,1)y - 1\alpha(x,1)y - x\alpha(1,y) + (\alpha(x,y)1 \\ &\quad + \alpha(xy,1) - x\alpha(y,1) - \alpha(x,y)) \\ &= 2x\alpha(1,1)y - \alpha(x,1)y - x\alpha(1,y) \\ &= x(1\alpha(1,y) - \alpha(1,y)) + (-\alpha(x,1) + \alpha(x,1)1)y = 0. \end{aligned}$$

This completes the proof. □

Lemma 2.2. *Let (δ, α) be a generalized Jordan derivation of an algebra $\mathcal{A}$ into a 2-torsion free and 3-torsion free $\mathcal{A}$-bimodule $\mathcal{M}$. Then*

$$h(d,c)\mathcal{A}[a,b]\mathcal{A}[a,b]\mathcal{A}[a,b]\mathcal{A}[a,b] = 0$$

for any $a, b, c, d \in \mathcal{A}$.

P r o o f. By [12, Lemma 4], we have

$$h(x,y)z[x,y] + [x,y]zh(x,y) = 0 \tag{2.4}$$

for any $x, y, z \in \mathcal{A}$. Replacing z by $x[a,b]y$, we obtain

$$[a,b]x[a,b]yh(a,b) = -h(a,b)x[a,b]y[a,b].$$

Using (2.4) twice yields

$$[a,b]x([a,b]yh(a,b)) = -[a,b]xh(a,b)y[a,b] = h(a,b)x[a,b]y[a,b].$$

Thus $2h(a,b)x[a,b]y[a,b] = 0$. Since $\mathcal{M}$ is 2-torsion free, it follows that

$$h(a,b)x[a,b]y[a,b] = 0. \tag{2.5}$$

In (2.5), replacing b by $b+nc$, we arrive at

$$\begin{aligned}0 &= h(a,b+nc)x[a,b+nc]y[a,b+nc]\\ &= h(a,b)x[a,b]y[a,b] + n(h(a,b)x[a,b]y[a,c] + h(a,b)x[a,c]y[a,b]\\ &\quad + h(a,c)x[a,b]y[a,b]) + n^2(h(a,b)x[a,c]y[a,c] + h(a,c)x[a,b]y[a,c]\\ &\quad + h(a,c)x[a,c]y[a,b]) + n^3 h(a,c)x[a,c]y[a,c]\\ &= n(h(a,b)x[a,b]y[a,c] + h(a,b)x[a,c]y[a,b] + h(a,c)x[a,b]y[a,b])\\ &\quad + n^2(h(a,b)x[a,c]y[a,c] + h(a,c)x[a,b]y[a,c] + h(a,c)x[a,c]y[a,b]).\end{aligned}$$

Let $n=1$ and then $n=-1$; comparing the two relations, we obtain

$$h(a,b)x[a,b]y[a,c] + h(a,b)x[a,c]y[a,b] + h(a,c)x[a,b]y[a,b] = 0. \tag{2.6}$$

Multiplying both sides of (2.6) by $z[a,b]$, and using (2.5), we obtain that

$$h(a,c)x[a,b]y[a,b]z[a,b] = 0. \tag{2.7}$$

Replacing a by $a+nd$ in (2.7), taking successively $n=1$, $n=-1$ and $n=2$ and comparing these equations, we obtain

$$\begin{aligned}&h(d,c)x[a,b]y[a,c]z[a,b] + h(a,c)x[d,b]y[a,b]z[a,b]\\ &\qquad + h(a,c)x[a,b]y[d,b]z[a,b] + h(a,c)x[a,b]y[a,b]z[d,b] = 0.\end{aligned} \tag{2.8}$$

Multiplying both sides of (2.8) by $w[a,b]$, we conclude from (2.7) that

$$h(d,c)x[a,b]y[a,b]z[a,b]w[a,b] = 0.$$

This completes the proof. □

Let $I([\mathcal{A},\mathcal{A}]^n)$ denote the ideal of $\mathcal{A}$ generated by all $[a,b]^n$, $a,b\in\mathcal{A}$.

Lemma 2.3 [4, Lemma 4]**.** *Let $\mathcal{A}$ be a C^*-algebra with a unit element. Then $I([\mathcal{A},\mathcal{A}]^n)=\mathcal{A}$ for each positive integer n if and only if there are no multiplicative linear functionals on $\mathcal{A}$. In particular, a von Neumann algebra with no abelian central summands has this property.*

Let $\mathcal{A}$ be a *-algebra, $P_{\mathcal{A}}$ the set of all projections in $\mathcal{A}$, and $H_{\mathcal{A}}$ the set of all self-adjoint elements in $\mathcal{A}$. By $D_{\mathcal{A}}$ we denote the set of those elements in $\mathcal{A}$ which can be represented as finite real-linear combinations of mutually orthogonal projections. Thus we have $P_{\mathcal{A}}\subseteq D_{\mathcal{A}}\subseteq H_{\mathcal{A}}$. If $\mathcal{A}$ is a von Neumann algebra, $D_{\mathcal{A}}$ is norm dense in $H_{\mathcal{A}}$.

Theorem 2.4. *Let $\mathcal{A}$ be a von Neumann algebra and let $\mathcal{M}$ be a normed $\mathcal{A}$-bimodule. If (δ, α) is a generalized Jordan derivation such that δ is norm continuous and α is norm continuous in the first component, then (δ, α) is a generalized derivation.*

Proof. By Lemma 2.2,

$$h(x,y)I([\mathcal{A},\mathcal{A}]^4) = 0 \quad \text{for any } x, y \in \mathcal{A}. \tag{2.9}$$

Let p be a central projection in $\mathcal{A}$ such that $p\mathcal{A}$ is of type I_1 and $(1-p)\mathcal{A}$ contains no abelian central summands. By Lemma 2.3, $1 - p \in I([\mathcal{A},\mathcal{A}]^4)$. It follows from (2.9) that

$$h(x,y)(1-p) = 0 \quad \text{for any } x, y \in \mathcal{A}. \tag{2.10}$$

In the following, we prove that $h(x,y)p = 0$. Let $x \in \mathcal{A}$, $q \in P_{\mathcal{A}} \cap p\mathcal{A}$. By Lemma 1.1 (2),

$$\delta(qx) = \delta(qxq) = \delta(q)xq + q\delta(x)q + qx\delta(q) + q\alpha(x,q) + \alpha(q,xq).$$

Since $\delta(q) = \delta(q^2) = \delta(q)q + q\delta(q) + \alpha(q,q)$, it follows that $q\delta(q)q = -q\alpha(q,q)q$. Thus

$$qx\delta(q)q = xq\delta(q)q = -xq\alpha(q,q)q.$$

So

$$\begin{aligned}
\delta(qx)q &= \delta(q)xq^2 + q\delta(x)q^2 + qx\delta(q)q + q\alpha(x,q)q + \alpha(q,xq)q \\
&= \delta(q)xq + q\delta(x)q - (\alpha(xq,q) - \alpha(xq,q) + \alpha(xq,q)q)q \\
&\quad + (\alpha(qx,q) + \alpha(q,x)q)q \\
&= \delta(q)xq + q\delta(x)q + \alpha(q,x)q.
\end{aligned}$$

Hence $h(q,x)q = 0$ for any $x \in \mathcal{A}$, $q \in P_{\mathcal{A}} \cap p\mathcal{A}$. Replacing q by $p - q$, we obtain $h(p,x)p - h(p,x)q - h(q,x)p + h(q,x)q = 0$. Since $h(q,x)q = 0$, $h(p,x)p = 0$ and $q \in P_{\mathcal{A}} \cap p\mathcal{A}$, it follows that $h(p,x)q = h(p,x)pq = 0$. Thus

$$h(q,x)p = -h(p,x)q = 0. \tag{2.11}$$

By (2.10) and (2.11) we have that $h(q,x)1 = 0$. It follows from Lemma 2.1 that $h(q,x) = 0$. Thus

$$\delta(qx) = \delta(q)x + q\delta(x) + \alpha(q,x)$$

for any $x \in \mathcal{A}$, $q \in P_{\mathcal{A}} \cap p\mathcal{A}$. Hence if $x \in \mathcal{A}$, $u \in p\mathcal{A}$ and u is a linear combination of projections in $p\mathcal{A}$, then

$$\delta(ux) = \delta(u)x + u\delta(x) + \alpha(u, x).$$

Since the set of all such elements is norm dense in $p\mathcal{A}$, δ is norm continuous and α is norm continuous in the first component, it follows that

$$\delta(cx) = \delta(c)x + c\delta(x) + \alpha(c, x)$$

for any $x \in \mathcal{A}$, $c \in p\mathcal{A}$. By Lemma 1.1 (1),

$$\delta(xc) = \delta(x)c + x\delta(c) + \alpha(x, c)$$

for any $x \in \mathcal{A}$, $c \in p\mathcal{A}$. For any $x, y \in \mathcal{A}$, since $yp \in p\mathcal{A}$, we have

$$\begin{aligned}\delta(xyp) &= \delta(xy)p + xy\delta(p) + \alpha(xy, p),\\ \delta(xyp) &= \delta(x)yp + x\delta(yp) + \alpha(x, yp)\\ &= \delta(x)yp + x(\delta(y)p + y\delta(p) + \alpha(y, p)) + \alpha(x, yp)\\ &= \delta(x)yp + x\delta(y)p + xy\delta(p) + x\alpha(y, p) + \alpha(x, yp).\end{aligned}$$

Thus

$$(\delta(xy) - \delta(x)y - x\delta(y) - \alpha(x, y))p = x\alpha(y, p) + \alpha(x, yp) - \alpha(xy, p) - \alpha(x, y)p = 0.$$

Hence

$$h(x, y)p = 0. \tag{2.12}$$

It follows from (2.10) and (2.12) that $h(x, y)1 = 0$. By Lemma 2.1, $h(x, y) = 0$. $\square$

3. Generalized Jordan derivations associated with Hochschild 2-cocycles on CSL algebras

For a CSL $\mathcal{L}$ on H, let Q_1 be the projection onto the closure of the linear span $\{PAP^{\perp}(H)\colon\ P \in \mathcal{L},\ A \in \operatorname{alg}\mathcal{L}\}$ and let Q_2 be the projection onto the closure of the linear span $\{P^{\perp}A^*P(H)\colon\ P \in \mathcal{L},\ A \in \operatorname{alg}\mathcal{L}\}$. Then

$$Q_1 \in \mathcal{L} \quad \text{and} \quad Q_2 \in \mathcal{L}^{\perp}.$$

Lemma 3.1. *Let $\mathcal{L}$ be a CSL on H and let (δ, α) be a generalized Jordan derivation from $\operatorname{alg}\mathcal{L}$ into itself. If $Q_1(H) \vee Q_2(H) = H$, then (δ, α) is a generalized derivation.*

P r o o f. For every $E \in \mathcal{L}$, since $\delta(E) = \delta(E^2) = \delta(E)E + E\delta(E) + \alpha(E, E)$, we have that

$$E\delta(E)E = -E\alpha(E, E)E, \quad E^{\perp}\delta(E)E^{\perp} = E^{\perp}\alpha(E, E)E^{\perp}.$$

So

$$\delta(E) = E\delta(E)E^{\perp} + E^{\perp}\alpha(E, E)E^{\perp} - E\alpha(E, E)E.$$

For every $T \in \operatorname{alg}\mathcal{L}$, by Lemma 1.1 (1),

$$\begin{aligned}\delta(ETE^{\perp}) &= \delta(EETE^{\perp} + ETE^{\perp}E) \\ &= -\alpha(E, E)ETE^{\perp} + E\delta(ETE^{\perp}) + \alpha(E, ETE^{\perp}) + \delta(ETE^{\perp})E \\ &\quad + ETE^{\perp}\alpha(E, E) + \alpha(ETE^{\perp}, E).\end{aligned}$$

Since

$$ETE^{\perp}\alpha(E, E) + \alpha(ETE^{\perp}, E) - \alpha(ETE^{\perp}, E)E = 0$$

and

$$E\alpha(E, ETE^{\perp}) - \alpha(E, ETE^{\perp}) + \alpha(E, ETE^{\perp}) - \alpha(E, E)ETE^{\perp} = 0,$$

we have that

$$\delta(ETE^{\perp}) = E^{\perp}\alpha(E, ETE^{\perp}) + E\delta(ETE^{\perp}) + \delta(ETE^{\perp})E + \alpha(ETE^{\perp}, E)E.$$

Thus

$$\begin{aligned}E\delta(ETE^{\perp})E &= -E\alpha(ETE^{\perp}, E)E, \\ E^{\perp}\delta(ETE^{\perp})E^{\perp} &= E^{\perp}\alpha(E, ETE^{\perp})E^{\perp}.\end{aligned}$$

Hence

$$\delta(ETE^{\perp}) = E\delta(ETE^{\perp})E^{\perp} + E^{\perp}\alpha(E, ETE^{\perp})E^{\perp} - E\alpha(ETE^{\perp}, E)E.$$

By Lemma 1.1 (2),

$$\delta(ETE) = \delta(E)TE + E\delta(T)E + ET\delta(E) + E\alpha(T, E) + \alpha(E, TE).$$

So

$$E^{\perp}\delta(ETE) = E^{\perp}\alpha(E, ETE).$$

For any $S, T \in \operatorname{alg}\mathcal{L}$, we have

$$\begin{aligned}
\delta(SETE^{\perp}) &= \delta(ESEETE^{\perp} + ETE^{\perp}ESE) \\
&= \delta(ESE)ETE^{\perp} + ESE\delta(ETE^{\perp}) + \alpha(ESE, ETE^{\perp}) \\
&\quad + \delta(ETE^{\perp})ESE + ETE^{\perp}\delta(ESE) + \alpha(ETE^{\perp}, ESE) \\
&= \delta(S)ETE^{\perp} + S\delta(ETE^{\perp}) \\
&\quad + [-\alpha(E,E)SETE^{\perp} + \alpha(E, SE)ETE^{\perp}] \\
&\quad + [-S\alpha(E,E)ETE^{\perp} + \alpha(S,E)ETE^{\perp} \\
&\quad - SE^{\perp}\alpha(E, ETE^{\perp})E^{\perp} + \alpha(SE, ETE^{\perp})] \\
&\quad + [-\alpha(ETE^{\perp}, E)ESE + ETE^{\perp}\alpha(E, ESE^{\perp}) \\
&\quad + \alpha(ETE^{\perp}, ESE)].
\end{aligned} \tag{3.1}$$

(a) By virtue of

$$(E\alpha(E, SE) - \alpha(E, SE) + \alpha(E, ESE) - \alpha(E,E)SE)ETE^{\perp} = 0,$$

it follows that

$$-\alpha(E,E)SETE^{\perp} + \alpha(E, SE)ETE^{\perp} = 0.$$

(b) Since $E\alpha(ETE^{\perp}, E) - \alpha(ETE^{\perp}, E) - \alpha(E, ETE^{\perp})E = 0$, it follows that $S\alpha(E, ETE^{\perp})E = SE\alpha(E, ETE^{\perp})E = 0$. Further,

$$(SE^{\perp}\alpha(E, ETE^{\perp}) + \alpha(SE^{\perp}, ETE^{\perp}) - \alpha(SE^{\perp}, E)ETE^{\perp})E^{\perp} = 0,$$

implies that

$$\begin{aligned}
&-S\alpha(E,E)ETE^{\perp} + \alpha(S,E)ETE^{\perp} - SE^{\perp}\alpha(E, ETE^{\perp})E^{\perp} \\
&\qquad + \alpha(SE, ETE^{\perp}) \\
&\quad = -S\alpha(E,E)ETE^{\perp} + \alpha(SE, E)ETE^{\perp} \\
&\qquad + \alpha(SE^{\perp}, ETE^{\perp})E^{\perp} + \alpha(SE, ETE^{\perp}) \\
&\quad = (\alpha(S,E) - \alpha(S,E)E)ETE^{\perp} + \alpha(S, ETE^{\perp})E^{\perp} + \alpha(SE, ETE^{\perp})E \\
&\quad = \alpha(S, ETE^{\perp})E^{\perp} + S\alpha(E, ETE^{\perp})E + \alpha(S, ETE^{\perp})E \\
&\quad = \alpha(S, ETE^{\perp}) + S\alpha(E, ETE^{\perp})E = \alpha(S, ETE^{\perp}).
\end{aligned}$$

(c) Since

$$\begin{aligned}
ETE^{\perp}\alpha(E, ESE) - \alpha(ETE^{\perp}E, ESE) + \alpha(ETE^{\perp}, ESE)& \\
- \alpha(ETE^{\perp}, E)ESE = 0&,
\end{aligned}$$

we have

$$ETE^{\perp}\alpha(E, ESE) + \alpha(ETE^{\perp}, ESE) - \alpha(ETE^{\perp}, E)ESE = 0.$$

By (a), (b), (c) and (3.1),

$$\delta(SETE^{\perp}) = \delta(S)ETE^{\perp} + S\delta(ETE^{\perp}) + \alpha(S, ETE^{\perp}). \tag{3.2}$$

Similarly, we have

$$\delta(ESE^{\perp}T) = \delta(ESE^{\perp})T + ESE^{\perp}\delta(T) + \alpha(ESE^{\perp}, T). \tag{3.3}$$

For any $A, B, T \in \operatorname{alg}\mathcal{L}$ we have

$$\begin{aligned}\delta(ABETE^{\perp}) &= \delta(AB)ETE^{\perp} + AB\delta(ETE^{\perp}) + \alpha(AB, ETE^{\perp}),\\ \delta(ABETE^{\perp}) &= \delta(A)BETE^{\perp} + AB\delta(ETE^{\perp}) + A\delta(B)ETE^{\perp}\\ &\quad + A\alpha(B, ETE^{\perp}) + \alpha(A, BETE^{\perp}).\end{aligned}$$

So

$$\begin{aligned}&\delta(AB)ETE^{\perp} - \delta(A)BETE^{\perp} - A\delta(B)ETE^{\perp}\\ &\qquad + \alpha(AB, ETE^{\perp}) - \alpha(A, BETE^{\perp}) - A\alpha(B, ETE^{\perp}) = 0.\end{aligned}$$

Since α is a Hochschild 2-cocycle, we conclude that

$$(\delta(AB) - \delta(A)B - A\delta(B) - \alpha(A, B))ETE^{\perp} = 0.$$

So

$$(\delta(AB) - \delta(A)B - A\delta(B) - \alpha(A, B))Q_1 = 0. \tag{3.4}$$

For $A^* \in \operatorname{alg}\mathcal{L}^{\perp}$ define $\delta^*(A^*) = (\delta(A))^*$ and $\alpha^*(A^*, B^*) = (\alpha(B, A))^*$. If $A^* \in \operatorname{alg}\mathcal{L}^{\perp}$, then

$$\begin{aligned}\delta^*(A^{*2}) &= (\delta(A^2))^* = (\delta(A)A + A\delta(A) + \alpha(A, A))^*\\ &= A^*(\delta(A))^* + (\delta(A))^*A^* + (\alpha(A, A))^*\\ &= A^*\delta^*(A^*)^* + \delta^*(A^*)A^* + \alpha^*(A^*, A^*).\end{aligned}$$

Proceeding similarly to the proof of (3.4), we can show that

$$Q_2(\delta(AB) - \delta(A)B - A\delta(B) - \alpha(A, B)) = 0. \tag{3.5}$$

Define $\Delta(T) = \delta(T) - (T\delta(Q_1) - \delta(Q_1)T)$ for any $T \in \operatorname{alg}\mathcal{L}$. Then (Δ, α) is also a generalized Jordan derivation. Since $\delta(Q_1) = Q_1\delta(Q_1)Q_1^\perp + Q_1^\perp\alpha(Q_1, Q_1)Q_1^\perp - Q_1\alpha(Q_1, Q_1)Q_1$, we have

$$\begin{aligned}\Delta(Q_1) &= \delta(Q_1) - (Q_1\delta(Q_1) - \delta(Q_1)Q_1)\\ &= Q_1^\perp\alpha(Q_1, Q_1)Q_1^\perp - Q_1\alpha(Q_1, Q_1)Q_1.\end{aligned}$$

For any $T \in \operatorname{alg}\mathcal{L}$, by virtue of $Q_1 \in \mathcal{L}$,

$$\begin{aligned}\Delta(TQ_1) = \Delta(T)Q_1 &- \alpha(Q_1, Q_1)TQ_1 + Q_1TQ_1^\perp\alpha(Q_1, Q_1)Q_1^\perp\\ &- T\alpha(Q_1, Q_1)Q_1 + Q_1\alpha(T, Q_1) + \alpha(Q_1, TQ_1).\end{aligned}$$

So

$$\begin{aligned}\Delta(TQ_1)Q_1^\perp &= Q_1TQ_1^\perp\alpha(Q_1, Q_1)Q_1^\perp + Q_1\alpha(T, Q_1)Q_1^\perp + \alpha(Q_1, TQ_1)Q_1^\perp\\ &= Q_1TQ_1^\perp\alpha(Q_1, Q_1)Q_1^\perp + (\alpha(Q_1T, Q_1) + \alpha(Q_1, T)Q_1)Q_1^\perp\\ &= \alpha(Q_1TQ_1, Q_1)Q_1^\perp = \alpha(TQ_1, Q_1)Q_1^\perp.\end{aligned} \tag{3.6}$$

Therefore,

$$\begin{aligned}&(\Delta(ABQ_1) - \Delta(A)BQ_1 - A\Delta(BQ_1) - \alpha(A, BQ_1))Q_1^\perp\\ &\quad= \alpha(ABQ_1, Q_1)Q_1^\perp - A\alpha(BQ_1, Q_1)Q_1^\perp - \alpha(A, BQ_1)Q_1^\perp\\ &\quad= -\alpha(A, BQ_1)Q_1Q_1^\perp = 0.\end{aligned} \tag{3.7}$$

By (3.4) and (3.7),

$$\Delta(ABQ_1) = \Delta(A)BQ_1 + A\Delta(BQ_1) + \alpha(A, BQ_1). \tag{3.8}$$

Since $\Delta(Q_1^\perp) = \Delta(Q_1^\perp)Q_1^\perp + Q_1^\perp\Delta(Q_1^\perp) + \alpha(Q_1^\perp, Q_1^\perp)$, we have

$$Q_1\Delta(Q_1^\perp)Q_1 = Q_1\alpha(Q_1^\perp, Q_1^\perp)Q_1$$

and

$$Q_1^\perp\Delta(Q_1^\perp)Q_1^\perp = -Q_1^\perp\alpha(Q_1^\perp, Q_1^\perp)Q_1^\perp.$$

Thus

$$\Delta(Q_1^\perp) = Q_1\Delta(Q_1^\perp)Q_1^\perp + Q_1\alpha(Q_1^\perp, Q_1^\perp)Q_1 - Q_1^\perp\alpha(Q_1^\perp, Q_1^\perp)Q_1^\perp.$$

By Lemma 1.1 (2),

$$Q_1\Delta(Q_1^\perp T) = Q_1\Delta(Q_1^\perp TQ_1^\perp) = Q_1\Delta(Q_1^\perp)TQ_1^\perp + Q_1\alpha(Q_1^\perp, TQ_1^\perp). \tag{3.9}$$

For any $A, B \in \operatorname{alg} \mathcal{L}$ we have

$$
\begin{aligned}
(3.10) \qquad & Q_1(\Delta(Q_1^\perp AB) - \Delta(Q_1^\perp A)B - Q_1^\perp A\Delta(B) - \alpha(Q_1^\perp A, B)) \\
&= Q_1\alpha(Q_1^\perp, AQ_1^\perp BQ_1^\perp) \\
&\quad - Q_1(Q_1^\perp \alpha(AQ_1^\perp, B) + \alpha(Q_1^\perp, AQ_1^\perp B)) \\
&= -Q_1\alpha(Q_1^\perp, AQ_1^\perp BQ_1) = -Q_1\alpha(Q_1^\perp, 0) = 0.
\end{aligned}
$$

Since $Q_1(H) \vee Q_2(H) = H$, it follows from (3.5) and (3.10) that

$$
(3.11) \qquad \Delta(Q_1^\perp AB) = \Delta(Q_1^\perp A)B + Q_1^\perp A\Delta(B) + \alpha(Q_1^\perp A, B).
$$

Also, by (3.2) and (3.3),

$$
\begin{aligned}
(3.12) \qquad & \Delta(AQ_1BQ_1^\perp) = \Delta(A)Q_1BQ_1^\perp + A\Delta(Q_1BQ_1^\perp) + \alpha(A, Q_1BQ_1^\perp), \\
(3.13) \qquad & \Delta(Q_1AQ_1^\perp B) = \Delta(Q_1AQ_1^\perp)B + Q_1AQ_1^\perp\Delta(B) + \alpha(Q_1AQ_1^\perp, B).
\end{aligned}
$$

Let $h(A, B) = \Delta(AB) - \Delta(A)B - A\Delta(B) - \alpha(A, B)$, $A, B \in \operatorname{alg} \mathcal{L}$. It follows from (3.8), (3.11), (3.12), and (3.13) that

$$
h(A, BQ_1) = h(Q_1^\perp A, B) = h(A, Q_1BQ_1^\perp) = h(Q_1AQ_1^\perp, B) = 0.
$$

Thus

$$
(3.14) \qquad h(A, Q_1B) = h(AQ_1^\perp, B) = 0.
$$

By (3.6), (3.9) and (3.14) we have

$$
\begin{aligned}
h(A, B) &= h(A, Q_1^\perp B) + h(A, Q_1B) = h(AQ_1, Q_1^\perp B) + h(AQ_1^\perp, Q_1^\perp B) \\
&= -\alpha(AQ_1, Q_1)Q_1^\perp B - AQ_1\Delta(Q_1^\perp)BQ_1^\perp - AQ_1\alpha(Q_1^\perp, BQ_1^\perp) \\
&\quad - \alpha(AQ_1, Q_1^\perp B).
\end{aligned}
$$

Since

$$
AQ_1\alpha(Q_1^\perp, BQ_1^\perp) + \alpha(AQ_1, Q_1^\perp BQ_1^\perp) - \alpha(AQ_1, Q_1^\perp)BQ_1^\perp = 0,
$$

we conclude that

$$
\begin{aligned}
-AQ_1\alpha(Q_1^\perp, BQ_1^\perp) - \alpha(AQ_1, Q_1^\perp BQ_1^\perp) - \alpha(AQ_1, Q_1)Q_1^\perp B \\
&= -\alpha(AQ_1, Q_1^\perp)BQ_1^\perp - \alpha(AQ_1, Q_1)Q_1^\perp BQ_1^\perp \\
&= -(\alpha(AQ_1, Q_1^\perp) + \alpha(AQ_1, Q_1)Q_1^\perp)BQ_1^\perp \\
&= -(AQ_1\alpha(Q_1, Q_1^\perp) + \alpha(AQ_1, Q_1Q_1^\perp))BQ_1^\perp \\
&= -AQ_1\alpha(Q_1, Q_1^\perp)BQ_1^\perp.
\end{aligned}
$$

Thus

$$\begin{aligned} h(A,B) &= h(AQ_1, Q_1^{\perp}B) \\ &= -AQ_1\Delta(Q_1^{\perp})BQ_1^{\perp} - AQ_1\alpha(Q_1,Q_1^{\perp})BQ_1^{\perp} \\ &= -A(Q_1\Delta(Q_1^{\perp}) + Q_1\alpha(Q_1,Q_1^{\perp}))BQ_1^{\perp}. \end{aligned}$$

Since $\Delta(I) = \delta(I) = -\alpha(I,I)$, we have

$$\Delta(Q_1^{\perp}) = \Delta(I) - \Delta(Q_1) = -\alpha(I,I) - Q_1^{\perp}\alpha(Q_1,Q_1)Q_1^{\perp} + Q_1\alpha(Q_1,Q_1)Q_1.$$

Thus $Q_1\Delta(Q_1^{\perp}) = -Q_1\alpha(I,I) + Q_1\alpha(Q_1,Q_1)Q_1$. Since $Q_1\alpha(I,I) - \alpha(Q_1,I) + \alpha(Q_1,I) - \alpha(Q_1,I) = 0$, it follows that

$$Q_1\alpha(I,I) = Q_1Q_1\alpha(I,I) = Q_1\alpha(Q_1,I).$$

So

$$Q_1\Delta(Q_1^{\perp}) = -Q_1\alpha(Q_1,I) + \alpha(Q_1,Q_1)Q_1.$$

Thus

$$\begin{aligned} h(A,B) &= -A(Q_1\alpha(Q_1,Q_1^{\perp}) - Q_1\alpha(Q_1,I) + \alpha(Q_1,Q_1)Q_1)BQ_1^{\perp} \\ &= A(Q_1\alpha(Q_1,Q_1) - \alpha(Q_1,Q_1)Q_1)BQ_1^{\perp} \\ &= A(\alpha(Q_1^2,Q_1) - \alpha(Q_1,Q_1^2))BQ_1^{\perp} = 0. \end{aligned}$$

Hence, (Δ,α) is a generalized derivation and (δ,α) is a generalized derivation. □

Proceeding similarly to the proof of Lemma 3.1, we can show the following result.

Lemma 3.2. *Suppose that $\mathcal{L}$ is a CSL on H. Let $G = Q_1(H) \vee Q_2(H)$. If (δ,α) is a generalized Jordan derivation from $G(\operatorname{alg}\mathcal{L})G$ into $\operatorname{alg}\mathcal{L}$, then (δ,α) is a generalized derivation.*

Theorem 3.3. *Suppose that $\mathcal{L}$ is a CSL on H. If (δ,α) is a generalized Jordan derivation from $\operatorname{alg}\mathcal{L}$ into $\operatorname{alg}\mathcal{L}$ such that δ is norm continuous and α is norm continuous in the first component, then (δ,α) is a generalized derivation.*

P r o o f. We divide the proof into two cases.

Case 1: $Q_1 \vee Q_2 = I$.

It follows from Lemma 3.1 that (δ,α) is a generalized derivation.

Case 2: $G = Q_1 \vee Q_2 \neq I$.

By [11, Lemma 1.1], we have that $G \in \mathcal{L} \cap \mathcal{L}^{\perp}$ and $(\operatorname{alg}\mathcal{L})G^{\perp} \subseteq \mathcal{L}'$. Hence $G^{\perp}(\operatorname{alg}\mathcal{L})G^{\perp}$ is a von Neumann algebra and

$$\operatorname{alg}\mathcal{L} = \operatorname{alg}(G\mathcal{L}G) \oplus \operatorname{alg}(G^{\perp}\mathcal{L}G^{\perp}).$$

Define $\delta_1 = \delta|_{\operatorname{alg} G\mathcal{L}G}$ and $\delta_2 = \delta|_{\operatorname{alg} G^\perp \mathcal{L} G^\perp}$. By Lemma 3.2, (δ_1, α) is a generalized derivation. Since δ is norm continuous, it follows that δ_2 is norm continuous. Since $G^\perp(\operatorname{alg}\mathcal{L})G^\perp$ is a von Neumann algebra and α is norm continuous in the first component, Theorem 2.4 yields that (δ_2, α) is a generalized derivation.

Thus for any $A, B \in \operatorname{alg}\mathcal{L}$,

$$\begin{aligned}
\delta(ABG) &= \delta(GAGGBG) \\
&= \delta(GAG)GBG + GAG\delta(GBG) + \alpha(GAG, GBG) \\
&= \delta(AG)BG + AG\delta(BG) + \alpha(AG, BG),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\delta(ABG^\perp) &= \delta(G^\perp AG^\perp G^\perp BG^\perp) \\
&= \delta(G^\perp AG^\perp)G^\perp BG^\perp + G^\perp AG^\perp \delta(G^\perp BG^\perp) \\
&\quad + \alpha(G^\perp AG^\perp, G^\perp BG^\perp) \\
&= \delta(AG^\perp)BG^\perp + AG^\perp\delta(BG^\perp) + \alpha(AG^\perp, BG^\perp).
\end{aligned} \tag{3.16}$$

By Lemma 1.1, for any $A \in \operatorname{alg}\mathcal{L}$,

$$\begin{aligned}
G^\perp\delta(GAG)G^\perp &= G^\perp(\delta(G)AG + G\delta(A)G + GA\delta(G) + G\alpha(A,G) + \alpha(G, AG))G^\perp \\
&= G^\perp\alpha(G, AG)G^\perp, \\
G\delta(G^\perp AG^\perp)G &= G(\delta(G^\perp)AG^\perp + G^\perp\delta(A)G^\perp + G^\perp A\delta(G^\perp) + G^\perp\alpha(A, G^\perp) \\
&\quad + \alpha(G^\perp, AG^\perp))G \\
&= G\alpha(G^\perp, AG^\perp)G.
\end{aligned}$$

Thus for any $A, B \in \operatorname{alg}\mathcal{L}$, it follows from $(\operatorname{alg}\mathcal{L})G^\perp \subseteq \mathcal{L}'$ that

$$\begin{aligned}
&\delta(AG)BG^\perp + AG\delta(BG^\perp) + \alpha(AG, BG^\perp) \\
&\quad = \delta(GAG)G^\perp BG^\perp + GAG\delta(G^\perp BG^\perp) + \alpha(GAG, G^\perp BG^\perp) \\
&\quad = \alpha(G, AG)G^\perp BG^\perp + \alpha(GAG, G^\perp)BG^\perp \\
&\quad = (G\alpha(AG, G^\perp) + \alpha(G, AGG^\perp))BG^\perp = 0,
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\delta(AG^\perp)BG + AG^\perp\delta(BG) + \alpha(AG^\perp, BG) \\
&\quad = \delta(G^\perp AG^\perp)GBG + G^\perp AG^\perp\delta(GBG) + \alpha(G^\perp AG^\perp, GBG) \\
&\quad = \alpha(G^\perp, AG^\perp)GBG + \alpha(G^\perp AG^\perp, G)BG \\
&\quad = G^\perp\alpha(AG^\perp, G)BG = 0.
\end{aligned} \tag{3.18}$$

By (3.15), (3.16), (3.17), and (3.18), we have that for any $A, B \in \operatorname{alg}\mathcal{L}$,

$$\delta(AB) = \delta(A)B + A\delta(B) + \alpha(A, B).$$

This completes the proof. □

In [15], Zhang showed that for a nest algebra $\mathcal{A}$, every $(\delta, 0)$ generalized Jordan derivation on $\mathcal{A}$ is a $(\delta, 0)$ derivation. In [11, Theorem 2.3], Lu proved that for a CSL algebra $\mathcal{A}$, Zhang's result is true.

4. Generalized derivable mappings at zero point associated with Hochschild 2-cocycles on CSL algebras

We call a linear mapping $\delta\colon \mathcal{A} \to \mathcal{M}$ a *generalized derivable mapping at zero point* if there is a Hochschild 2-cocycle $\alpha\colon \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ such that $\delta(A)B + A\delta(B) + \alpha(A, B) = 0$ for all $A, B \in \mathcal{A}$ with $AB = 0$.

Lemma 4.1. *Let $\mathcal{A}$ be a unital algebra and $\mathcal{M}$ a unital $\mathcal{A}$-bimodule. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ and α is a Hochschild 2-cocycle mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ such that $\delta(A)B + A\delta(B) + \alpha(A, B) = 0$ for all $A, B \in \mathcal{A}$ with $AB = 0$, then for any idempotent P in $\mathcal{A}$ and $A \in \mathcal{A}$,*

(i) $\delta(PA) = \delta(P)A + P\delta(A) + \alpha(P, A) - P(\delta(I) + \alpha(I, I))A$,

(ii) $\delta(AP) = \delta(A)P + A\delta(P) + \alpha(A, P) - A(\delta(I) + \alpha(I, I))P$,

(iii) $P\delta(P)P = P(\delta(I) + \alpha(I, I))P - P\alpha(P, P)P = P(\delta(I) + \alpha(I, I)) - P\alpha(P, P) = (\delta(I) + \alpha(I, I))P - \alpha(P, P)P$.

Proof. (iii) For any $P = P^2 \in \mathcal{A}$,

$$
\begin{aligned}
(4.1)\qquad 0 = \delta(P(I - P)) &= \delta(P)(I - P) + P\delta(I - P) + \alpha(P, I - P) \\
&= \delta(P) - \delta(P)P + P\delta(I) - P\delta(P) + \alpha(P, I - P), \\
(4.2)\qquad 0 = \delta((I - P)P) &= \delta(I - P)P + (I - P)\delta(P) + \alpha(I - P, P) \\
&= \delta(I)P - \delta(P)P + \delta(P) - P\delta(P) + \alpha(I - P, P).
\end{aligned}
$$

By (4.1) and (4.2) we have

$$
\begin{aligned}
P\delta(P)P &= P\delta(I) + P\alpha(P, I - P) = P(\delta(I) + \alpha(I, I)) - P\alpha(P, P), \\
P\delta(P)P &= P\delta(I)P + \alpha(P, I - P)P = P(\delta(I) + \alpha(I, I))P - \alpha(P, P)P, \\
P\delta(P)P &= \delta(I)P + \alpha(I - P, P)P = (\delta(I) + \alpha(I, I))P - \alpha(P, P)P.
\end{aligned}
$$

(4.1) minus (4.2) together with the above equations yields

$$P\delta(I) + \alpha(P, I) = \delta(I)P + \alpha(I, P).$$

(i) Using

$$\begin{aligned}0 &= \delta((I-P)PA) = \delta(I-P)PA + (I-P)\delta(PA) + \alpha(I-P, PA)\\ &= \delta(I)PA - \delta(P)PA + \delta(PA) - P\delta(PA) + \alpha(I, PA) - \alpha(P, PA),\\ 0 &= \delta(P(I-P)A) = \delta(P)(I-P)A + P\delta((I-P)A) + \alpha(P, (I-P)A)\\ &= \delta(P)A - \delta(P)PA + P\delta(A) - P\delta(PA) + \alpha(P, A) - \alpha(P, PA),\end{aligned}$$

we arrive at

$$\begin{aligned}\delta(PA) &= \delta(P)A + P\delta(A) + \alpha(P, A) - \delta(I)PA - \alpha(I, PA)\\ &= \delta(P)A + P\delta(A) + \alpha(P, A) - P\delta(I)A - \alpha(P, I)A\\ &\quad + \alpha(I, P)A - \alpha(I, PA).\end{aligned}$$

Since

$$\begin{gathered}P\alpha(I, I) - \alpha(P, I) + \alpha(P, I) - \alpha(P, I) = 0,\\ \alpha(P, A) - \alpha(P, A) + \alpha(I, PA) - \alpha(I, P)A = 0,\end{gathered}$$

it follows that

$$\delta(PA) = \delta(P)A + P\delta(A) + \alpha(P, A) - P\delta(I)A - P\alpha(I, I)A.$$

(ii) By virtue of

$$\begin{aligned}0 = \delta(AP(I-P)) &= \delta(AP)(I-P) + AP\delta(I-P) + \alpha(AP, I-P)\\ &= \delta(AP) - \delta(AP)P + AP\delta(I) - AP\delta(P) + \alpha(AP, I)\\ &\quad - \alpha(AP, P),\\ 0 = \delta(A(I-P)P) &= \delta(A(I-P))P + A(I-P)\delta(P) + \alpha(A(I-P), P)\\ &= \delta(A)P - \delta(AP)P + A\delta(P) - AP\delta(P) + \alpha(A, P)\\ &\quad - \alpha(AP, P),\end{aligned}$$

we obtain

$$\begin{aligned}\delta(AP) &= \delta(A)P + A\delta(P) + \alpha(A, P) - AP\delta(I) - \alpha(AP, I)\\ &= \delta(A)P + A\delta(P) + \alpha(A, P) - A\delta(I)P - A\alpha(I, P)\\ &\quad + A\alpha(P, I) - \alpha(AP, I).\end{aligned}$$

Since

$$\begin{gathered}\alpha(I, P) - \alpha(I, P) + \alpha(I, P) - \alpha(I, I)P = 0,\\ A\alpha(P, I) - \alpha(AP, I) + \alpha(A, P) - \alpha(A, P) = 0,\end{gathered}$$

we conclude that

$$\delta(AP) = \delta(A)P + A\delta(P) + \alpha(A, P) - A\delta(I)P - A\alpha(I, I)P.$$

This completes the proof. □

Corollary 4.2. *Let δ, α, $\mathcal{A}$, $\mathcal{M}$ be as in Lemma* 4.1 *with $\delta(I) = -\alpha(I, I)$. Then for any idempotent P in $\mathcal{A}$ and any $A \in \mathcal{A}$,*

(i) $\delta(PA) = \delta(P)A + P\delta(A) + \alpha(P, A)$,

(ii) $\delta(AP) = \delta(A)P + A\delta(P) + \alpha(A, P)$,

(iii) $P\delta(P)P = -P\alpha(P, P)P$.

Corollary 4.3. *Let δ, α, $\mathcal{A}$, $\mathcal{M}$ be as in Lemma* 4.1 *with $\delta(I) = -\alpha(I, I)$. Suppose that $\mathcal{B}$ is the subalgebra of $\mathcal{A}$ generated by all idempotents in $\mathcal{A}$. Then for any $T \in \mathcal{B}$ and any $A \in \mathcal{A}$,*

(i) $\delta(TA) = \delta(T)A + T\delta(A) + \alpha(T, A)$,

(ii) $\delta(AT) = \delta(A)T + A\delta(T) + \alpha(A, T)$.

Proof. (i) We need to show that

$$\delta(P_1 \dots P_n A) = \delta(P_1 \dots P_n)A + P_1 \dots P_n \delta(A) + \alpha(P_1 \dots P_n, A). \tag{4.3}$$

If $n = 1$, then by Corollary 4.2, (4.3) is obvious.

Suppose that if $n = k$, then (4.3) is true. For $n = k + 1$,

$$\begin{aligned}
\delta(P_1 \dots P_k P_{k+1} A) &= \delta(P_1 \dots P_k) P_{k+1} A + P_1 \dots P_k \delta(P_{k+1} A) \\
&\quad + \alpha(P_1 \dots P_k, P_{k+1} A) \\
&= (\delta(P_1 \dots P_{k+1}) - P_1 \dots P_k \delta(P_{k+1}) - \alpha(P_1 \dots P_k, P_{k+1}))A \\
&\quad + P_1 \dots P_k(\delta(P_{k+1})A + P_{k+1}\delta(A) + \alpha(P_{k+1}, A)) \\
&\quad + \alpha(P_1 \dots P_k, P_{k+1} A) \\
&= \delta(P_1 \dots P_{k+1})A + P_1 \dots P_{k+1}\delta(A) + P_1 \dots P_k \alpha(P_{k+1}, A) \\
&\quad + \alpha(P_1 \dots P_k, P_{k+1} A) - \alpha(P_1 \dots P_k, P_{k+1})A \\
&= \delta(P_1 \dots P_{k+1})A + P_1 \dots P_{k+1}\delta(A) + \alpha(P_1 \dots P_{k+1}, A).
\end{aligned}$$

Similarly, we can prove (ii). □

We say that a subset $\mathcal{S}$ of $\mathcal{A}$ *separates $\mathcal{M}$ from the left*, if for any $T \in \mathcal{M}$, $\mathcal{S}T = \{0\}$ implies $T = 0$. Similarly, we say $\mathcal{S}$ *separates $\mathcal{M}$ from the right*, if for any $T \in \mathcal{M}$, $T\mathcal{S} = \{0\}$ implies $T = 0$. We say $\mathcal{S}$ *separates* $\mathcal{M}$ if $\mathcal{S}$ separates $\mathcal{M}$ both from the left and from the right.

Theorem 4.4. *Let δ, α, $\mathcal{A}$, $\mathcal{M}$ be as in Corollary 4.3. Suppose that $\mathcal{A}$ contains a left (right) ideal $\mathcal{I}$ that separates $\mathcal{M}$ from the right (from the left, respectively). If $\mathcal{I}$ is contained in the subalgebra of $\mathcal{A}$ generated by all idempotents in $\mathcal{A}$, then (δ, α) is a generalized derivation.*

Proof. We only prove the case that $\mathcal{I}$ is a left ideal of $\mathcal{A}$ and $\mathcal{I}$ separates $\mathcal{M}$ from the right; the other case is similar.

For any $A, B \in \mathcal{A}$ and $S \in \mathcal{I}$, Corollary 4.3 implies that

$$\begin{aligned}\delta(ABS) &= \delta((AB)S) = \delta(AB)S + AB\delta(S) + \alpha(AB, S),\\ \delta(ABS) &= \delta(A(BS)) = \delta(A)BS + A\delta(BS) + \alpha(A, BS)\\ &= \delta(A)BS + A\delta(B)S + AB\delta(S) + A\alpha(B, S) + \alpha(A, BS).\end{aligned}$$

So

$$\begin{aligned}(\delta(AB) - A\delta(B) - \delta(A)B - \alpha(A, B))S\\ = A\alpha(B, S) - \alpha(AB, S) + \alpha(A, BS) - \alpha(A, B)S = 0.\end{aligned} \tag{4.4}$$

Since $\mathcal{I}$ separates $\mathcal{M}$ from the right, it follows by (4.4) that

$$\delta(AB) - A\delta(B) - \delta(A)B - \alpha(A, B) = 0.$$

This completes the proof. □

Corollary 4.5. *Suppose that $\mathcal{A}$ is a unital Banach subalgebra of $B(H)$ such that either $\mathcal{A}$ contains $\{x \otimes f_0 \colon x \in H\}$, where $0 \neq f_0 \in H$, or $\{x_0 \otimes f \colon f \in H\}$, where $0 \neq x_0 \in H$. If $\delta \colon \mathcal{A} \to B(H)$ is a linear mapping and $\alpha \colon \mathcal{A} \times \mathcal{A} \to B(H)$ is a Hochschild 2-cocycle mapping such that $\delta(I) = -\alpha(I, I)$ and $\delta(A)B + A\delta(B) + \alpha(A, B) = 0$ for all $A, B \in \mathcal{A}$ with $AB = 0$, then (δ, α) is a generalized derivation.*

Remarks.

1. Suppose that $\mathcal{L}$ is a subspace lattice and $\mathcal{A} = \mathcal{M} = \operatorname{alg}\mathcal{L}$. If $\mathcal{L}$ satisfies one of the conditions
   (i) $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice on X,
   (ii) $\mathcal{L}$ satisfies $0_+ \neq 0$ and $H_- \neq H$,
   (iii) $\mathcal{L}$ is a completely distributive CSL on H,

then $\operatorname{alg}\mathcal{L}$ has an ideal $\mathcal{I}$ which is contained in a subalgebra of $\operatorname{alg}\mathcal{L}$ generated by its idempotents and $\mathcal{I}$ separates $\mathcal{M}$.

2. If $\mathcal{A}$ is a unital algebra, then by [5, Proposition 2.2], for $2 \leqslant n$, $M_n(\mathcal{A})$ is generated by its idempotents.

Theorem 4.6. *Suppose that $\mathcal{L}$ is a CSL on H. Let δ be a norm continuous linear mapping from $\operatorname{alg}\mathcal{L}$ into $B(H)$ and let α be a norm continuous Hochschild 2-cocycle mapping in the first component from $\operatorname{alg}\mathcal{L}\times\operatorname{alg}\mathcal{L}$ into $B(H)$ such that $\delta(I)=-\alpha(I,I)$ and $\delta(A)B+A\delta(B)+\alpha(A,B)=0$ for all $A,B\in\mathcal{A}$ with $AB=0$. Then (δ,α) is a generalized derivation.*

Proof. Define $\mathcal{I}=\operatorname{span}\{P(\operatorname{alg}\mathcal{L})P^{\perp}\colon P\in\mathcal{L}\}$. Then $\mathcal{I}$ is an ideal of $\operatorname{alg}\mathcal{L}$. Since $PTP^{\perp}=P-(P-PTP^{\perp})$ and $P-PTP^{\perp}$ is an idempotent in $\operatorname{alg}\mathcal{L}$, we have that every element in $\mathcal{I}$ is a linear combination of idempotents in $\operatorname{alg}\mathcal{L}$. Let Q_1 be the projection onto the closure of the linear span of $\{PTP^{\perp}H\colon P\in\mathcal{L},\ T\in\operatorname{alg}\mathcal{L}\}$ and let Q_2 be the projection onto the closure of the linear span of $\{P^{\perp}T^*PH\colon P\in\mathcal{L},\ T\in\operatorname{alg}\mathcal{L}\}$. Thus

$$Q_1\in\mathcal{L}\subseteq\mathcal{L}'=\operatorname{alg}\mathcal{L}\cap(\operatorname{alg}\mathcal{L})^*$$

and

$$Q_2\in\mathcal{L}^{\perp}\subseteq\mathcal{L}'=\operatorname{alg}\mathcal{L}\cap(\operatorname{alg}\mathcal{L})^*.$$

Let $Q=Q_1\vee Q_2$. By [11, Lemma 1.1], $Q\in\mathcal{L}\cap\mathcal{L}^{\perp}$ and $(\operatorname{alg}\mathcal{L})Q^{\perp}\subseteq\mathcal{L}'$. Hence $Q^{\perp}\operatorname{alg}\mathcal{L}Q^{\perp}$ is a von Neumann algebra and $\operatorname{alg}\mathcal{L}$ can be written as the direct sum $\operatorname{alg}\mathcal{L}=\operatorname{alg}(Q\mathcal{L}Q)\oplus\operatorname{alg}(Q^{\perp}\mathcal{L}Q^{\perp})$. Let $\delta=\delta_1+\delta_2$, where $\delta_1=\delta|_{\operatorname{alg}(Q\mathcal{L}Q)}$ and $\delta_2=\delta|_{\operatorname{alg}(Q^{\perp}\mathcal{L}Q^{\perp})}$.

Claim 1. *(δ_1,α) is a generalized derivation.*

Since $\mathcal{I}$ is an ideal of $\mathcal{A}$ generated by idempotents in $\mathcal{A}$, it follows from Corollary 4.3 that for any $A,B,T\in\operatorname{alg}\mathcal{L}$, $P\in\mathcal{L}$,

$$\begin{aligned}\delta(ABPTP^{\perp})&=\delta(AB)PTP^{\perp}+AB\delta(PTP^{\perp})+\alpha(AB,PTP^{\perp})\\ \delta(ABPTP^{\perp})&=\delta(A)BPTP^{\perp}+A\delta(BPTP^{\perp})+\alpha(A,BPTP^{\perp})\\ &=\delta(A)BPTP^{\perp}+A\delta(B)PTP^{\perp}+AB\delta(PTP^{\perp})\\ &\quad+A\alpha(B,PTP^{\perp})+\alpha(A,BPTP^{\perp}).\end{aligned}$$

So

$$\begin{aligned}&(\delta(AB)-\delta(A)B-A\delta(B)-\alpha(A,B))PTP^{\perp}\\ &\quad=A\alpha(B,PTP^{\perp})-\alpha(AB,PTP^{\perp})+\alpha(A,BPTP^{\perp})\\ &\qquad-\alpha(A,B)PTP^{\perp}=0.\end{aligned}$$

Thus

$$(\delta(AB)-\delta(A)B-A\delta(B)-\alpha(A,B))Q_1=0. \tag{4.5}$$

Define $\delta^*(A^*) = (\delta(A))^*$, $\alpha^*(A^*, B^*) = (\alpha(B, A))^*$ for any $A^*, B^* \in \operatorname{alg}\mathcal{L}^\perp$. If $A^*, B^* \in \operatorname{alg}\mathcal{L}^\perp$ and $A^*B^* = 0$, then $A, B \in \operatorname{alg}\mathcal{L}$ and $BA = 0$. Moreover,

$$\begin{aligned}\delta^*(A^*B^*) &= (\delta(BA))^* = (\delta(B)A + B\alpha(B, A))^* \\ &= A^*(\delta(B))^* + (\delta(A))^*B^* + (\alpha(B, A))^* \\ &= A^*\delta^*(B^*) + \delta^*(A^*)B^* + \alpha^*(A^*, B^*).\end{aligned}$$

Combining this with an argument similar to the proof of (4.5), we can obtain

$$Q_2(\delta(AB) - \delta(A)B - A\delta(B) - \alpha(A, B)) = 0. \tag{4.6}$$

By Corollary 4.3, $\delta(Q_1) = \delta(Q_1)Q_1 + Q_1\delta(Q_1) + \alpha(Q_1, Q_1)$. Thus

$$\begin{aligned}Q_1\delta(Q_1)Q_1 &= -Q_1\alpha(Q_1, Q_1)Q_1, \\ Q_1^\perp\delta(Q_1)Q_1^\perp &= Q_1^\perp\alpha(Q_1, Q_1)Q_1^\perp.\end{aligned}$$

Let $\Delta(A) = \delta(A) - (A\delta(Q_1) - \delta(Q_1)A)$ for any $A \in \operatorname{alg}\mathcal{L}$. If $AB = 0$, then

$$\begin{aligned}0 &= \delta(A)B + A\delta(B) + \alpha(A, B) \\ &= (\delta(A) - (A\delta(Q_1) - \delta(Q_1)A))B \\ &\quad + A(\delta(B) - (B\delta(Q_1) - \delta(Q_1)B)) + \alpha(A, B) \\ &= \Delta(A)B + A\Delta(B) + \alpha(A, B)\end{aligned}$$

and $\Delta(I) = \delta(I) = -\alpha(I, I)$. We also have

$$\begin{aligned}\Delta(Q_1) &= \delta(Q_1) - (Q_1\delta(Q_1) - \delta(Q_1)Q_1) \\ &= \delta(Q_1) - Q_1\delta(Q_1)Q_1^\perp \\ &= Q_1\delta(Q_1)Q_1 + Q_1^\perp\delta(Q_1)Q_1^\perp \\ &= -Q_1\alpha(Q_1, Q_1)Q_1 + Q_1^\perp\alpha(Q_1, Q_1)Q_1^\perp.\end{aligned}$$

By Corollary 4.3,

$$\begin{aligned}\Delta(AQ_1) &= \Delta(A)Q_1 + A\Delta(Q_1) + \alpha(A, Q_1), \\ &= \Delta(A)Q_1 - AQ_1\alpha(Q_1, Q_1)Q_1 + AQ_1^\perp\alpha(Q_1, Q_1)Q_1^\perp + \alpha(A, Q_1),\end{aligned}$$

$$\begin{aligned}\Delta(AQ_1)Q_1^\perp &= AQ_1^\perp\alpha(Q_1, Q_1)Q_1^\perp + \alpha(A, Q_1)Q_1^\perp \\ &= -\alpha(AQ_1^\perp, Q_1)Q_1^\perp + \alpha(A, Q_1)Q_1^\perp \\ &= \alpha(AQ_1, Q_1)Q_1^\perp.\end{aligned} \tag{4.7}$$

Thus

$$(4.8)\quad (\Delta(ABQ_1) - \Delta(A)BQ_1 - A\Delta(BQ_1) - \alpha(A, BQ_1))Q_1^{\perp}$$
$$= \alpha(ABQ_1, Q_1)Q_1^{\perp} - A\alpha(BQ_1, Q_1)Q_1^{\perp} - \alpha(A, BQ_1)Q_1^{\perp} = 0.$$

By (4.5),

$$(4.9)\quad (\Delta(ABQ_1) - \Delta(A)BQ_1 - A\Delta(BQ_1) - \alpha(A, BQ_1))Q_1 = 0.$$

It follows from (4.8) and (4.9) that

$$(4.10)\quad \Delta(ABQ_1) - \Delta(A)BQ_1 - A\Delta(BQ_1) - \alpha(A, BQ_1) = 0.$$

By Corollary 4.3, we have

$$\Delta(Q_1^{\perp}) = \Delta(Q_1^{\perp})Q_1^{\perp} + Q_1^{\perp}\Delta(Q_1^{\perp}) + \alpha(Q_1^{\perp}, Q_1^{\perp}).$$

So

$$Q_1^{\perp}\Delta(Q_1^{\perp})Q_1^{\perp} = -Q_1^{\perp}\alpha(Q_1^{\perp}, Q_1^{\perp})Q_1^{\perp},$$
$$Q_1\Delta(Q_1^{\perp})Q_1 = Q_1\alpha(Q_1^{\perp}, Q_1^{\perp})Q_1.$$

Thus

$$\Delta(Q_1^{\perp}) = Q_1\Delta(Q_1^{\perp})Q_1^{\perp} + Q_1\alpha(Q_1^{\perp}, Q_1^{\perp})Q_1 - Q_1^{\perp}\alpha(Q_1^{\perp}, Q_1^{\perp})Q_1^{\perp}.$$

Corollary 4.3 yields

$$\begin{aligned}\Delta(Q_1^{\perp}A) &= \Delta(Q_1^{\perp})A + Q_1^{\perp}\Delta(A) + \alpha(Q_1^{\perp}, A)\\ &= Q_1\Delta(Q_1^{\perp})Q_1^{\perp}A + Q_1\alpha(Q_1^{\perp}, Q_1^{\perp})Q_1A\\ &\quad - Q_1^{\perp}\alpha(Q_1^{\perp}, Q_1^{\perp})Q_1^{\perp}A + Q_1^{\perp}\Delta(A) + \alpha(Q_1^{\perp}, A).\end{aligned}$$

Then we have that

$$\begin{aligned}(4.11)\quad & Q_1\Delta(Q_1^{\perp}A)\\ &= Q_1\Delta(Q_1^{\perp})Q_1^{\perp}A + Q_1\alpha(Q_1^{\perp}, Q_1^{\perp})Q_1A + Q_1\alpha(Q_1^{\perp}, A)\\ &= Q_1\Delta(Q_1^{\perp})Q_1^{\perp}A + Q_1\alpha(Q_1^{\perp}, A)\\ &\quad + Q_1(Q_1^{\perp}\alpha(Q_1^{\perp}, Q_1A) - \alpha(Q_1^{\perp}, Q_1A) + \alpha(Q_1^{\perp}, Q_1^{\perp}Q_1A))\\ &= Q_1\Delta(Q_1^{\perp})Q_1^{\perp}A + Q_1\alpha(Q_1^{\perp}, Q_1^{\perp}A).\end{aligned}$$

Thus

$$\begin{aligned}(4.12)\quad & Q_1(\Delta(Q_1^{\perp}AB) - \Delta(Q_1^{\perp}A)B - Q_1^{\perp}A\Delta(B) - \alpha(Q_1^{\perp}A, B))\\ &= Q_1\Delta(Q_1^{\perp})Q_1^{\perp}AB + Q_1\alpha(Q_1^{\perp}, Q_1^{\perp}AB)\\ &\quad - Q_1\Delta(Q_1^{\perp})Q_1^{\perp}AB - Q_1\alpha(Q_1^{\perp}, Q_1^{\perp}A)B - Q_1\alpha(Q_1^{\perp}A, B)\\ &= -Q_1Q_1^{\perp}\alpha(Q_1^{\perp}A, B) = 0.\end{aligned}$$

By (4.6),

$$Q_2(\Delta(Q_1^\perp AB) - \Delta(Q_1^\perp A)B - Q_1^\perp A\Delta(B) - \alpha(Q_1^\perp A, B)) = 0. \tag{4.13}$$

Since $Q = Q_1 \vee Q_2$, it follows from (4.12) and (4.13) that

$$Q(\Delta(Q_1^\perp AB) - \Delta(Q_1^\perp A)B - Q_1^\perp A\Delta(B) - \alpha(Q_1^\perp A, B)) = 0. \tag{4.14}$$

For any $A, B \in Q(\operatorname{alg}\mathcal{L})Q$, since $(Q_1^\perp Q)^2 = Q_1^\perp Q$, it follows from Corollary 4.3 that

$$\begin{aligned} &Q^\perp(\Delta(Q_1^\perp AB) - \Delta(Q_1^\perp A)B - Q_1^\perp A\Delta(B) - \alpha(Q_1^\perp A, B)) \\ &\quad = Q^\perp(\Delta(Q_1^\perp Q)AB + Q_1^\perp Q\Delta(AB) + \alpha(Q_1^\perp Q, AB)) \\ &\qquad - Q^\perp(\Delta(Q_1^\perp Q)A + Q_1^\perp Q\Delta(A) + \alpha(Q_1^\perp Q, A))B \\ &\qquad - Q^\perp\alpha(Q_1^\perp A, B) \\ &\quad = Q^\perp\alpha(Q_1^\perp Q, AB) - Q^\perp\alpha(Q_1^\perp Q, A)B - Q^\perp\alpha(Q_1^\perp A, B) \\ &\quad = Q^\perp Q_1^\perp Q\alpha(A, B) = 0. \end{aligned} \tag{4.15}$$

By (4.14) and (4.15) we have

$$\Delta(Q_1^\perp AB) - \Delta(Q_1^\perp A)B - Q_1^\perp A\Delta(B) - \alpha(Q_1^\perp A, B) = 0 \tag{4.16}$$

for any $A, B \in Q(\operatorname{alg}\mathcal{L})Q$.

Let $h(x, y) = \Delta(xy) - \Delta(x)y - x\Delta(y) - \alpha(x, y)$ for any $x, y \in \operatorname{alg}\mathcal{L}$. By (4.10), (4.16), and Corollary 4.3,

$$h(A, BQ_1) = h(A, Q_1BQ_1^\perp) = h(Q_1AQ_1^\perp, B) = 0, \quad A, B \in \operatorname{alg}\mathcal{L},$$
$$h(Q_1^\perp A, B) = 0, \quad A, B \in Q(\operatorname{alg}\mathcal{L})Q.$$

Thus

$$h(A, Q_1B) = h(AQ_1^\perp, B) = 0, \quad \text{for any } A, B \in Q(\operatorname{alg}\mathcal{L})Q.$$

For any $A, B \in Q(\operatorname{alg}\mathcal{L})Q$, it follows from (4.7) and (4.11) that

$$\begin{aligned} h(A, B) &= h(A, Q_1^\perp B) + h(A, Q_1B) = h(A, Q_1^\perp B) \\ &= h(AQ_1, Q_1^\perp B) + h(AQ_1^\perp, Q_1^\perp B) \\ &= h(AQ_1, Q_1^\perp B) \\ &= \Delta(AQ_1Q_1^\perp B) - \Delta(AQ_1)Q_1^\perp B - AQ_1\Delta(Q_1^\perp B) - \alpha(AQ_1, Q_1^\perp B) \\ &= -\alpha(AQ_1, Q_1)Q_1^\perp B - AQ_1\Delta(Q_1^\perp)Q_1^\perp B \\ &\quad - AQ_1\alpha(Q_1^\perp, Q_1^\perp B) - \alpha(AQ_1, Q_1^\perp B). \end{aligned}$$

Since

$$AQ_1\alpha(Q_1^\perp, Q_1^\perp B) + \alpha(AQ_1, Q_1^\perp B) - \alpha(AQ_1, Q_1^\perp)Q_1^\perp B = 0,$$

we conclude

$$\begin{aligned} h(A,B) =& -\alpha(AQ_1, I)Q_1^\perp B - AQ_1\Delta(Q_1^\perp)Q_1^\perp B \\ =& -AQ_1\alpha(I,I)Q_1^\perp B - AQ_1\Delta(Q_1^\perp)Q_1^\perp B \\ =& AQ_1\Delta(I)Q_1^\perp B - AQ_1\Delta(Q_1^\perp)Q_1^\perp B \\ =& AQ_1\Delta(Q_1)Q_1^\perp B \\ =& AQ_1(-Q_1\alpha(Q_1,Q_1)Q_1 + Q_1^\perp\alpha(Q_1,Q_1)Q_1^\perp)Q_1^\perp B = 0. \end{aligned}$$

Thus $\Delta(AB) = \Delta(A)B + A\Delta(B) + \alpha(A,B)$ for any $A, B \in Q(\operatorname{alg}\mathcal{L})Q$. Hence $\delta(AB) = \delta(A)B + A\delta(B) + \alpha(A,B)$ for any $A, B \in Q(\operatorname{alg}\mathcal{L})Q$.

Claim 2. (δ_2, α) *is a generalized derivation.*

By Corollary 4.3, $\delta(PB) = \delta(P)B + P\delta(B) + \alpha(P,B)$ for any $P = P^2 \in \operatorname{alg}\mathcal{L}$ and $B \in \operatorname{alg}\mathcal{L}$. In particular, $\delta(PB) = \delta(P)B + P\delta(B) + \alpha(P,B)$ for any $P = P^2 \in Q^\perp \operatorname{alg}\mathcal{L}Q^\perp$ and $B \in Q^\perp \operatorname{alg}\mathcal{L}Q^\perp$. Since $Q^\perp \operatorname{alg}\mathcal{L}Q^\perp$ is a von Neumann algebra and α is norm continuous in the first component, we have that (δ_2, α) is a generalized derivation.

Thus for any $A, B \in \operatorname{alg}\mathcal{L}$, Claims 1 and 2 yield

$$\begin{aligned} \delta(ABQ) &= \delta(QAQQBQ) \\ &= \delta(QAQ)QBQ + QAQ\delta(QBQ) + \alpha(QAQ, QBQ) \\ &= \delta(AQ)BQ + AQ\delta(BQ) + \alpha(AQ, BQ), \\ \delta(ABQ^\perp) &= \delta(Q^\perp AQ^\perp Q^\perp BQ^\perp) \\ &= \delta(Q^\perp AQ^\perp)Q^\perp BQ^\perp + Q^\perp AQ^\perp\delta(Q^\perp BQ^\perp \\ &\quad + \alpha(Q^\perp AQ^\perp, Q^\perp BQ^\perp) \\ &= \delta(AQ^\perp)BQ^\perp + AQ^\perp\delta(BQ^\perp) + \alpha(AQ^\perp, BQ^\perp). \end{aligned}$$

By the assumption,

$$\begin{aligned} 0 &= \delta(Q^\perp AQ^\perp)QBQ + Q^\perp AQ^\perp\delta(QBQ) + \alpha(Q^\perp AQ^\perp, QBQ) \\ &= \delta(AQ^\perp)BQ + AQ^\perp\delta(BQ) + \alpha(AQ^\perp, BQ), \\ 0 &= \delta(QAQ)Q^\perp BQ^\perp + QAQ\delta(Q^\perp BQ^\perp) + \alpha(QAQ, Q^\perp BQ^\perp) \\ &= \delta(AQ)BQ^\perp + AQ\delta(BQ^\perp) + \alpha(AQ, BQ^\perp). \end{aligned}$$

Hence $\delta(AB) = \delta(A)B + A\delta(B) + \alpha(A,B)$. □

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