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## A note on propagation of singularities of semiconcave functions of two variables

LUDEK ZAJICEK

*Abstract.* P. Albano and P. Cannarsa proved in 1999 that, under some applicable conditions, singularities of semiconcave functions in  $\mathbb{R}^n$  propagate along Lipschitz arcs. Further regularity properties of these arcs were proved by P. Cannarsa and Y. Yu in 2009. We prove that, for  $n = 2$ , these arcs are very regular: they can be found in the form (in a suitable Cartesian coordinate system)  $\psi(x) = (x, y_1(x) - y_2(x))$ ,  $x \in [0, \alpha]$ , where  $y_1, y_2$  are convex and Lipschitz on  $[0, \alpha]$ . In other words: singularities propagate along arcs with finite turn.

*Keywords:* semiconcave functions, singularities

*Classification:* Primary 26B25; Secondary 35A21

### 1. Introduction

Let  $u$  be a function defined on an open set  $\Omega \subset \mathbb{R}^n$  which is locally (linearly) semiconcave; i.e.,  $u$  is locally representable in the form  $u(x) = g(x) + K\|x\|^2$ , where  $g$  is concave (cf. [3]).

Let  $\Sigma(u)$  be the singular set of  $u$ , i.e.

$$\Sigma(u) = \{x \in \Omega : u \text{ is not differentiable at } x\}.$$

It is clear that in many questions concerning  $\Sigma(u)$  we can suppose that  $u$  is concave (or convex), since the results for semiconcave functions then easily follow. But it is reasonable to formulate theorems for semiconcave functions, since these functions are important in a number of applications (see [3]).

It is well-known that  $\Sigma(u)$  is a rather small set: it can be covered by countably many Lipschitz DC hypersurfaces ([12]). (Note that for  $A \subset \mathbb{R}^n$  there exists a convex (resp. semiconcave) function  $u$  on  $\mathbb{R}^n$  such that  $A = \Sigma(u)$ , if and only if  $A$  is an  $F_\sigma$  set which can be covered by countably many Lipschitz DC hypersurfaces, see [8].)

The set  $\Sigma(u)$  can have isolated points, but P. Albano and P. Cannarsa [1] found applicable conditions ensuring that  $\Sigma(u)$  is in a sense big in each neighbourhood of a given  $x_0 \in \Sigma(u)$ . (The results of [1] can be found also in the book [3].) In particular, they proved that if  $\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset$  (see Preliminaries for the

definitions), then a Lipschitz arc  $\xi : [0, \tau] \rightarrow \Omega$  emanating from  $x_0$  is a subset of the singular set  $\Sigma(u)$ . The results of [1] were refined in [5]; in particular it is proved in [5, Corollary 4.3] that  $\xi$  has nonzero (right continuous) right derivative at all points.

The purpose of the present note is to show that in  $\mathbb{R}^2$  the results of [5] and methods from [12] and [10] easily imply that the restriction of  $\xi$  to an interval  $[0, \tau']$  has an equivalent parametrization of the form (in a suitable Cartesian coordinate system)  $\psi(x) = (x, y_1(x) - y_2(x))$ ,  $x \in [0, \alpha]$ , where  $y_1, y_2$  are convex and Lipschitz on  $[0, \alpha]$ . (This result is equivalent to the assertion that the restriction of  $\xi$  to an interval  $[0, \tau^*]$  has finite turn, cf. Remark 3.3). In particular,  $\xi$  has (left continuous) left halftangents at all points.

The question whether the results can be generalized to the case  $n > 2$  remains open.

## 2. Preliminaries

By  $B(x, r)$  we denote the open ball with center  $x$  and radius  $r$ . The scalar product of  $v, w \in \mathbb{R}^n$  is denoted by  $\langle v, w \rangle$ . If  $A \subset \mathbb{R}^n$ ,  $c \in \mathbb{R}$  and  $v \in \mathbb{R}^n$ , then we define the sets  $A + v$  and  $cA$  by the usual way and similarly set  $\langle v, A \rangle := \{\langle v, a \rangle : a \in A\}$ . The boundary and the convex hull of a set  $A \subset \mathbb{R}^n$  are denoted by  $\partial A$  and  $\text{conv } A$ , respectively. The (Fréchet) derivative  $Df(a)$  of a function  $f$  on  $\mathbb{R}^n$  at  $a \in \mathbb{R}^n$  is considered as an element of  $\mathbb{R}^n$ . The one-sided derivatives of a real or vector function  $\xi$  of one variable at  $x \in \mathbb{R}$  are denoted by  $\xi'_+(x)$  and  $\xi'_-(x)$ .

If  $f$  is a function defined on a subset of  $\mathbb{R}^n$ ,  $x \in \mathbb{R}^n$  and  $v \in \mathbb{R}^n$ , then we define the one-sided directional derivative as

$$f'_+(x, v) := \lim_{h \rightarrow 0^+} \frac{f(x + hv) - f(x)}{h}.$$

Let  $\Omega \subset \mathbb{R}^n$  be an open set and  $u$  a locally semiconcave function on  $\Omega$  (see Introduction). Then  $u$  is locally Lipschitz and so differentiable a.e. in  $\Omega$ . For  $x \in \Omega$ , we define (see [1] or [3, p. 54]) the set

$$D^*u(x) = \{p \in \mathbb{R}^n : \Omega \ni x_i \rightarrow x, Du(x_i) \rightarrow p\}$$

of all *reachable gradients* of  $u$  at  $x$  (note that  $D^*u(x)$  is also called limiting subdifferential, cf. [1, p. 725]).

The *superdifferential*  $D^+u(x)$  of  $u$  at  $x$  can be defined as the convex hull of  $D^*u(x)$  (see [1, p. 723], cf. [3, Theorem 3.3.6]).

Always  $D^*u(x) \subset \partial D^+u(x)$  (see [3, Proposition 3.3.4]). Note that the superdifferential  $D^+u(x) = \text{conv } D^*u(x)$  coincides with the Clarke's subdifferential  $\partial^C u(x)$  (since  $\partial^C u(x) = \text{conv } D^*u(x)$ , see, e.g., [4]).

Let  $u(x) = g(x) + K\|x\|^2$ , where  $g$  is concave, on a ball  $B(x_0, \delta) \subset \Omega$ . Set  $f := -g$ . Since  $D(K\|x\|^2) = 2Kx$ , we easily obtain that  $D^*u(x_0) = -D^*f(x_0) + 2Kx_0$ , and therefore

$$(2.1) \quad D^+u(x_0) = -\partial f(x_0) + 2Kx_0,$$

where  $\partial f$  is the classical subdifferential of the convex function  $f$ .

Recall that a function defined on an open convex subset of  $\mathbb{R}^n$  is a *DC function* if it is a difference of two convex functions. We will need the following simple lemma which is a special case of the “mixing lemma” [10, Lemma 4.8].

**Lemma 2.1.** *Let  $\varphi_1, \dots, \varphi_p$  be DC functions on  $\mathbb{R}$ , and let  $h$  be a continuous function on  $\mathbb{R}$  such that*

$$h(x) \in \{\varphi_1(x), \dots, \varphi_p(x)\} \quad \text{for each } x \in \mathbb{R}.$$

*Then  $h$  is DC on  $\mathbb{R}$ .*

We will need also the well-known fact that convex functions are semismooth (see [7, Proposition 3], cf. also [9, Proposition 2.3]). In other words:

**Lemma 2.2.** *Let  $f$  be a convex function on an open convex set  $C \subset \mathbb{R}^n$  and  $x_0 \in C$ . Let  $0 \neq q \in \mathbb{R}^n$ ,  $q_n \rightarrow q$ ,  $t_n \searrow 0$ , and  $z_n \in \partial f(x_n)$ , where  $x_n := x_0 + t_n q_n$ , be given. Then  $\langle q, z_n \rangle \rightarrow f'_+(x_0, q)$ . In particular,*

$$(2.2) \quad \text{diam}\langle q, \partial f(x_n) \rangle \rightarrow 0.$$

### 3. The result and its proof

The following result is an immediate consequence of [5, Corollary 4.3].

**Theorem CY.** *Let  $u$  be a semiconcave function on an open set  $\Omega \subset \mathbb{R}^n$ ,  $x_0 \in \Sigma(u)$  be a singular point of  $u$  and*

$$\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset.$$

*Then there exist  $q \in \mathbb{R}^n$  with  $\|q\| = 1$ ,  $\tau > 0$ , and a Lipschitz curve  $\xi : [0, \tau] \rightarrow \Sigma(u)$  such that*

- (i)  $\xi'_+(0) = q$ ,
- (ii)  $\lim_{s \rightarrow 0^+} \xi'_+(s) = q$ , and
- (iii)  $\inf_{s \in [0, \tau]} \text{diam } D^+u(\xi(s)) > 0$ .

Note that it is proved in [5] also that  $\xi'_+(s)$  exists for each  $s \in [0, \tau)$  and  $\xi'_+$  is right continuous on  $[0, \tau)$ . Further note that the result without (ii) was proved already in [1].

Using Theorem CY and the method of the proof of the implicit function theorem for DC functions [10, Theorem 4.4], we easily prove the following result.

**Theorem 3.1.** *Let  $u$  be a semiconcave function on an open set  $\Omega \subset \mathbb{R}^2$ ,  $x_0 \in \Sigma(u)$  be a singular point of  $u$  and*

$$\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset.$$

*Then there exist a Cartesian coordinate system in  $\mathbb{R}^2$  given by an isometry  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $A(x_0) = (0, 0)$ , and convex Lipschitz functions  $y_1, y_2$  on some  $[0, \alpha]$  ( $\alpha > 0$ ) such that, denoting  $\psi(x) := (x, y_1(x) - y_2(x))$ ,  $x \in [0, \alpha]$ , we have  $\psi(0) = (0, 0)$  and  $A^{-1}(\psi([0, \alpha])) \subset \Sigma(u)$ .*

PROOF: Let  $\xi : [0, \tau] \rightarrow \Sigma(u)$  and  $q \in \mathbb{R}^2$  have properties from Theorem CY. We will proceed in four steps. In Steps 1–3 we will suppose that

$$(3.1) \quad x_0 = (0, 0) \quad \text{and} \quad q = (1, 0).$$

*Step 1.* Set  $e_2 := (0, 1)$ . Let  $u(x) = g(x) + K\|x\|^2$  for  $x \in B(x_0, \delta) \subset \Omega$ , where  $g$  is concave and Lipschitz with a constant  $L > 0$  on  $B(x_0, \delta)$ . Set  $f := -g$ . Applying (2.1) to any point  $x \in B(x_0, \delta)$ , we obtain  $D^+u(x) = -\partial f(x) + 2Kx$ ,  $x \in B(x_0, \delta)$ . So (iii) (of Theorem CY) easily implies that, for some  $0 < \tau_1 < \tau$ , we have that  $f(\xi(s)) \in B(x_0, \delta)$  and  $\partial f(\xi(s)) \subset B(0, L)$  for each  $s \in [0, \tau_1]$ , and

$$(3.2) \quad \inf_{s \in [0, \tau_1]} \text{diam } \partial f(\xi(s)) > 0.$$

We will show that there exists  $0 < \tau_2 < \tau_1$  such that

$$(3.3) \quad \delta := \inf_{s \in (0, \tau_2)} \text{diam } \langle e_2, \partial f(\xi(s)) \rangle > 0.$$

Suppose on the contrary that there exists a sequence  $(t_n)$  such that  $t_n \searrow 0$  and

$$(3.4) \quad \lim_{n \rightarrow \infty} \text{diam } \langle e_2, \partial f(\xi(t_n)) \rangle = 0.$$

Set  $q_n := \xi(t_n)/t_n$  and  $x_n := \xi(t_n) = t_n q_n$ . Since  $q_n \rightarrow q$  by (i), Lemma 2.2 gives that

$$(3.5) \quad \lim_{n \rightarrow \infty} \text{diam } \langle q, \partial f(\xi(t_n)) \rangle = 0.$$

Since (3.4) and (3.5) clearly imply  $\lim_{n \rightarrow \infty} \text{diam } \partial f(\xi(t_n)) = 0$ , we obtain a contradiction with (3.2).

*Step 2.* Let  $\xi = (\xi_1, \xi_2)$ . By (ii), we have  $\lim_{s \rightarrow 0+} (\xi_1)'_+(s) = 1$  and therefore there exists  $0 < \tau_3 < \tau_2$  such that  $1/2 \leq (\xi_1)'(s)$  for a.e.  $s \in (0, \tau_3)$ . So  $\xi_1$  is Lipschitz strictly increasing on  $[0, \tau_3]$  and  $(\xi_1)^{-1}$  is Lipschitz on  $[0, \alpha]$ , where  $\alpha := \xi_1(\tau_3)$ . Set  $d(x) := \xi_2 \circ (\xi_1)^{-1}(x)$ ,  $x \in [0, \alpha]$ . Then  $d$  is Lipschitz and  $\psi(x) := (x, d(x))$ ,  $x \in [0, \alpha]$ , is an equivalent parametrization of  $\xi|_{[0, \tau_3]}$ .

*Step 3.* Choose a partition  $\{-L = y_0 < y_1 < \dots < y_p = L\}$  of the interval  $[-L, L]$  such that  $\max\{y_i - y_{i-1}, i = 1, \dots, p\} < \delta/2$ . For each  $x \in (0, \alpha)$ , the set  $\langle e_2, \partial f(\psi(x)) \rangle \subset [-L, L]$  is a closed interval of length at least  $\delta$  and so we can choose  $i_x \in \{1, \dots, p\}$  such that

$$(3.6) \quad y_{i_x} \in \langle e_2, \partial f(\psi(x)) \rangle \quad \text{and} \quad y_{i_x-1} \in \langle e_2, \partial f(\psi(x)) \rangle.$$

For  $i \in \{1, \dots, p\}$ , set  $A_i := \{x \in (0, \alpha) : i_x = i\}$ . We will show that, for each  $i \in \{1, \dots, p\}$  with  $A_i \neq \emptyset$ , the function  $d|_{A_i}$  can be extended to a Lipschitz DC function  $\varphi_i$  on  $\mathbb{R}$ .

To this end, fix a such  $i$  and set

$$\omega_1(x) := f(x, d(x)) - y_i d(x) \quad \text{and} \quad \omega_2(x) := f(x, d(x)) - y_{i-1} d(x) \quad \text{for } x \in A_i.$$

Since  $\omega_1(x) - \omega_2(x) = (y_{i-1} - y_i)d(x)$ ,  $x \in A_i$ , it is sufficient to prove that  $\omega_j$  ( $j = 1, 2$ ) can be extended to a Lipschitz convex function  $c_j$  defined on  $\mathbb{R}$ .

For each  $x \in A_i$ , choose  $p_x \in \mathbb{R}$  such that  $(p_x, y_i) \in \partial f(x, d(x))$  and consider the affine function

$$a_x(t) := \omega_1(x) + p_x(t - x), \quad t \in \mathbb{R}.$$

Set

$$c_1(t) := \sup\{a_x(t) : x \in A_i\}, \quad t \in \mathbb{R}.$$

Since  $\omega_1$  is clearly bounded on  $A_i$  and  $|p_x| \leq L$  for  $x \in A_i$ , it is easy to see that  $c_1$  is a Lipschitz convex function on  $\mathbb{R}$ .

Now consider arbitrary  $x, t \in A_i$ ,  $x \neq t$ . Since  $(p_x, y_i) \in \partial f(x, d(x))$ , we have

$$f(t, d(t)) - f(x, d(x)) \geq p_x(t - x) + y_i(d(t) - d(x)),$$

and therefore

$$\omega_1(t) = f(t, d(t)) - y_i d(t) \geq f(x, d(x)) - y_i d(x) + p_x(t - x) = a_x(t).$$

Since  $a_t(t) = \omega_1(t)$ ,  $t \in A_i$ , we obtain that  $c_1$  extends  $\omega_1$ . Quite similarly we can find a convex Lipschitz extension  $c_2$  of  $\omega_2$ .

Since  $d(x) \in \{\varphi_1(x), \dots, \varphi_p(x)\}$  for each  $x \in (0, \alpha)$ , and  $d, \varphi_1, \dots, \varphi_p$  are continuous on  $[0, \alpha]$ , we can clearly find  $i_0, i_\alpha \in \{1, \dots, p\}$  such that  $d(0) = \varphi_{i_0}(0)$  and  $d(\alpha) = \varphi_{i_\alpha}(\alpha)$ .

Let  $h$  be the extension of  $d$  with  $h(x) = \varphi_{i_0}(x)$ ,  $x < 0$  and  $h(x) = \varphi_{i_\alpha}(x)$ ,  $x > \alpha$ . Then  $h$  is continuous on  $\mathbb{R}$  and  $h(x) \in \{\varphi_1(x), \dots, \varphi_p(x)\}$  for each  $x \in \mathbb{R}$ . Thus Lemma 2.1 implies that  $h$  is DC on  $\mathbb{R}$ , i.e.,  $h = \gamma_1 - \gamma_2$ , where  $\gamma_1$  and  $\gamma_2$  are convex on  $\mathbb{R}$ . Then  $y_j := \gamma_j|_{[0, \alpha]}$ ,  $j = 1, 2$ , are clearly convex Lipschitz functions, and  $\psi(x) = (x, y_1(x) - y_2(x))$ ,  $x \in [0, \alpha]$ .

*Step 4.* If (3.1) does not hold, we can choose a Cartesian system of coordinates given by an isometry  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $A(x_0) = (0, 0)$  and  $A(q) = (1, 0)$ . Applying steps 1-3 to  $u^* := u \circ A^{-1}$  and  $\xi^* := A \circ \xi$ , we obtain  $\psi$  of the demanded form with  $\psi([0, \alpha]) \subset \Sigma(u^*) = A(\Sigma(u))$ . □

*Remark 3.2.* Well-known elementary properties of convex functions on  $\mathbb{R}$  easily imply that the one-sided derivative  $\psi'_+$  ( $\psi'_-$ ) exists and is right (left) continuous on  $[0, \alpha)$  ( $(0, \alpha]$ ) and has finite variation on this interval. In other words,  $\psi$  has *bounded convexity* (see [11, Theorem 3.1] or [6, Lemma 5.5]). Further, since clearly  $|\psi'_+| \geq 1$ ,  $|\psi'_-| \geq 1$  we obtain that the curve  $\psi$  has *finite turn* (see [2, Theorem 5.4.2] or [6, Theorem 5.11]). So the curve  $\psi^* := A^{-1} \circ \psi$ , for which  $\psi^*([0, \alpha]) \subset \Sigma(u)$ , has also bounded convexity and finite turn.

*Remark 3.3.* The proof of Theorem 3.1 and Remark 3.2 show that, for the curve  $\xi : [0, \tau] \rightarrow \Sigma(u)$  from Theorem CY, there exists  $0 < \tau^* < \tau$  such that  $\xi|_{[0, \tau^]}$  has finite turn. In fact, this assertion “is not weaker” than Theorem 3.1, since it implies quickly Theorem 3.1 by standard methods.

*Remark 3.4.* We *did not show* that the curve  $\xi$  from Theorem CY has near 0 (left-continuous) left derivative  $\xi'_-$  at all points. However, the proof of Theorem 3.1 clearly implies that  $\xi$  has (left-continuous) left half-tangent on  $(0, \tau^*]$  for some  $0 < \tau^* < \tau$ .

We will not give detailed proofs of facts from Remarks 3.2–3.4, since they would be inadequately long, and these facts are not essential for the present short note.

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