Mathematica Bohemica

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On the rational recursive sequence $x_{n+1} = \dfrac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k}}$

Mathematica Bohemica, Vol. 135 (2010), No. 3, 319–336

Persistent URL: `http://dml.cz/dmlcz/140707`

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(Received April 17, 2009)

Abstract. The main objective of this paper is to study the boundedness character, the periodicity character, the convergence and the global stability of positive solutions of the difference equation

$$x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k}}, \quad n = 0, 1, 2, \ldots$$

where the coefficients $\alpha_i, \beta_i \in (0, \infty)$ for $i = 0, 1, 2$, and l, k are positive integers. The initial conditions $x_{-k}, \ldots, x_{-l}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers such that $l < k$. Some numerical experiments are presented.

Keywords: difference equation, boundedness, period two solution, convergence, global stability

MSC 2010: 39A10, 39A99, 34C99

1. Introduction

Our goal in this paper is to investigate the boundedness character, the periodicity character, the convergence and the global stability of positive solutions of the difference equation

$$x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k}}, \quad n = 0, 1, 2, \ldots \tag{1}$$

where the coefficients $\alpha_i, \beta_i \in (0, \infty)$ for $i = 0, 1, 2$ and l, k are positive integers. The initial conditions $x_{-k}, \ldots, x_{-l}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers such that $l < k$. We consider numerical examples which represent different types of solutions to the equation (1). The case when any of α_i, β_i for $i = 0, 1, 2$ are allowed to be zero gives different special cases of the equation (1) which have been studied by many authors (see for example [1]–[16]). For the related work see [17]–[38]. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore, the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations. Note that the difference equation (1) has been discussed in [8] when $\alpha_0 = \beta_0 = 0$.

Definition 1. A difference equation of order $(k+1)$ is of the form

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-l}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots$$

with $l < k$ where F is a continuous function which maps some set J^{k+1} into J and J is a set of real numbers. An equilibrium point $\widetilde{x}$ of this equation is a point that satisfies the condition $\widetilde{x} = F(\widetilde{x}, \widetilde{x}, \ldots, \widetilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \widetilde{x}$ for all $n \geqslant -k$ is a solution of that equation.

Definition 2. Let $\widetilde{x} \in (0, \infty)$ be an equilibrium point of the difference equation (1). Then we have the following:

(i) An equilibrium point $\widetilde{x}$ of the difference equation (1) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + \ldots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$ for all $n \geqslant -k$.

(ii) An equilibrium point $\widetilde{x}$ of the difference equation (1) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + \ldots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \gamma$, then

$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\widetilde{x}$ of the difference equation (1) is called a global attractor if for every $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ we have

$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iv) An equilibrium point $\widetilde{x}$ of the equation (1) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{x}$ of the difference equation (1) is called unstable if it is not locally stable.

Definition 3. We say that a sequence $\{x_n\}_{n=-k}^{\infty}$ is bounded and persists if there exist positive constants m and M such that

$$m \leqslant x_n \leqslant M \quad \text{for all } n \geqslant -k.$$

Definition 4. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period r if $x_{n+r} = x_n$ for all $n \geqslant -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period r if r is the smallest positive integer having this property.

2. Local stability of the equilibrium point

In this section we study the local stability character of the solutions of equation (1). Assume that $\widetilde{a} = \alpha_0 + \alpha_1 + \alpha_2$ and $\widetilde{b} = \beta_0 + \beta_1 + \beta_2$. Then the positive equilibrium point $\widetilde{x}$ of the difference equation (1) is given by

$$\widetilde{x} = \frac{\widetilde{a}}{\widetilde{b}}. \tag{2}$$

Let $F\colon (0,+\infty)^3 \longrightarrow (0,+\infty)$ be a continuous function defined by

$$F(u_0, u_1, u_2) = \frac{\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2}{\beta_0 u_0 + \beta_1 u_1 + \beta_2 u_2}. \tag{3}$$

Then the linearized equation associated with the difference equation (1) about the positive equilibrium point $\widetilde{x}$ takes the form

$$y_{n+1} + a_2 y_n + a_1 y_{n-l} + a_0 y_{n-k} = 0, \tag{4}$$

where

$$\begin{cases} \dfrac{(\alpha_0\beta_1 - \alpha_1\beta_0) + (\alpha_0\beta_2 - \alpha_2\beta_0)}{\widetilde{a}\,\widetilde{b}} = -a_2, \\ \dfrac{(\alpha_1\beta_0 - \alpha_0\beta_1) + (\alpha_1\beta_2 - \alpha_2\beta_1)}{\widetilde{a}\,\widetilde{b}} = -a_1, \\ \dfrac{(\alpha_2\beta_0 - \alpha_0\beta_2) + (\alpha_2\beta_1 - \alpha_1\beta_2)}{\widetilde{a}\,\widetilde{b}} = -a_0. \end{cases} \tag{5}$$

The characteristic equation of the linearized equation (4) is

$$\lambda^{n+1} + a_2\lambda^n + a_1\lambda^{n-l} + a_0\lambda^{n-k} = 0. \tag{6}$$

Theorem 1. ([18], [19] The linearized stability theorem) *Suppose F is a continuously differentiable function defined on an open neighbourhood of the equilibrium point $\widetilde{x}$. Then the following statements are true.*

(i) *If all roots of the characteristic equation* (6) *of the linearized equation* (4) *have absolute value less than one, then the equilibrium point $\widetilde{x}$ is locally asymptotically stable.*

(ii) *If at least one root of equation* (6) *has absolute value greater than one, then the equilibrium point $\widetilde{x}$ is unstable.*

Theorem 2. ([18]) *Assume that $p_i \in \mathbb{R}$, $i = 1, 2, \ldots k$. Then*

$$\sum_{i=1}^{k} |p_i| < 1 \tag{7}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \quad n = 0, 1, 2, \ldots \tag{8}$$

Theorem 3. *Assume that*

$$\begin{aligned} &|(\alpha_0\beta_1 - \alpha_1\beta_0) + (\alpha_0\beta_2 - \alpha_2\beta_0)| + |(\alpha_1\beta_0 - \alpha_0\beta_1) + (\alpha_1\beta_2 - \alpha_2\beta_1)| \\ &+ |(\alpha_2\beta_0 - \alpha_0\beta_2) + (\alpha_2\beta_1 - \alpha_1\beta_2)| < \widetilde{a}\,\widetilde{b}. \end{aligned}$$

Then the positive equilibrium point $\widetilde{x}$ of equation (1) *is locally asymptotically stable.*

P r o o f. It is obvious from (5) and the assumption of Theorem 3 that

$$|a_2| + |a_1| + |a_0| < 1.$$

It follows by Theorem 2 that equation (1) is asymptotically stable. □

3. Boundedness of the solutions

In this section we study the bounded and persisting character of the positive solutions of equation (1).

Theorem 4. *Every solution of equation* (1) *is bounded and persisting.*

P r o o f. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of equation (1). It follows from equation (1) that

$$\begin{aligned} x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k}} &= \frac{\alpha_0 x_n}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k}} \\ &+ \frac{\alpha_1 x_{n-l}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k}} + \frac{\alpha_2 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k}}. \end{aligned}$$

Then

$$x_n \leqslant \frac{\alpha_0}{\beta_0} + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} = M, \quad n \geqslant 1. \tag{9}$$

Now we wish to show that there exists $m > 0$ such that

$$x_n \geqslant m, \quad n \geqslant 1.$$

The transformation

$$x_n = \frac{1}{y_n}$$

reduces equation (1) to the equivalent form

$$\begin{aligned} y_{n+1} &= \frac{\beta_0 y_{n-l} y_{n-k} + \beta_1 y_n y_{n-k} + \beta_2 y_n y_{n-l}}{\alpha_0 y_{n-l} y_{n-k} + \alpha_1 y_n y_{n-k} + \alpha_2 y_n y_{n-l}} \\ &= \frac{\beta_0 y_{n-l} y_{n-k}}{\alpha_0 y_{n-l} y_{n-k} + \alpha_1 y_n y_{n-k} + \alpha_2 y_n y_{n-l}} \\ &\quad + \frac{\beta_1 y_n y_{n-k}}{\alpha_0 y_{n-l} y_{n-k} + \alpha_1 y_n y_{n-k} + \alpha_2 y_n y_{n-l}} \\ &\quad + \frac{\beta_2 y_n y_{n-l}}{\alpha_0 y_{n-l} y_{n-k} + \alpha_1 y_n y_{n-k} + \alpha_2 y_n y_{n-l}}. \end{aligned}$$

It follows that

$$y_{n+1} \leqslant \frac{\beta_0}{\alpha_0} + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} = H_1, \quad n \geqslant 1.$$

Thus we obtain

$$x_n = \frac{1}{y_n} \geqslant \frac{1}{H_1} = \frac{\alpha_0 \alpha_1 \alpha_2}{\beta_0 \alpha_1 \alpha_2 + \beta_1 \alpha_0 \alpha_2 + \beta_2 \alpha_0 \alpha_1} = m, \quad n \geqslant 1, \tag{10}$$

where H_1 is a positive constant. From (9) and (10) we see that

$$m \leqslant x_n \leqslant M, \; n \geqslant 1.$$

□

4. Periodicity of the solutions

In this section we study the periodic character of the positive solutions of equation (1).

Theorem 5. *Consider the difference equation* (1). *Then the following statements are true:*

(1) *If l and k are even, then equation* (1) *has no positive solutions of prime period two.*

(2) *If l is odd, k is even and $\alpha_0 + \alpha_2 > \alpha_1$, then equation* (1) *has no positive solutions of prime period two.*

(3) *If l is even, k is odd and $\alpha_0 + \alpha_1 > \alpha_2$, then equation* (1) *has no positive solutions of prime period two.*

(4) *If l and k are odd positive integers, $\alpha_1 + \alpha_2 > \alpha_0$ and $\beta_0 > \beta_1 + \beta_2$, then the necessary and sufficient condition for the difference equation* (1) *to have positive solutions of prime period two is that the inequality*

$$4\alpha_0 (\beta_1 + \beta_2) < [(\alpha_1 + \alpha_2) - \alpha_0] [\beta_0 - (\beta_1 + \beta_2)] \tag{11}$$

is valid.

(5) *If l is odd, k is even, $\alpha_1 > \alpha_0 + \alpha_2$ and $\beta_1 < \beta_0 + \beta_2$, then the necessary and sufficient condition for the difference equation* (1) *to have positive solutions of prime period two is that the inequality*

$$4\beta_1 (\alpha_0 + \alpha_2) < [\alpha_1 - (\alpha_0 + \alpha_2)] [(\beta_0 + \beta_2) - \beta_1] \tag{12}$$

is valid.

(6) *If l is even, k is odd, $\alpha_2 > \alpha_0 + \alpha_1$ and $\beta_2 < \beta_0 + \beta_1$, then the necessary and sufficient condition for the difference equation* (1) *to have positive solutions of prime period two is that the inequality*

$$4\beta_2 (\alpha_0 + \alpha_1) < [\alpha_2 - (\alpha_0 + \alpha_1)] [(\beta_0 + \beta_1) - \beta_2] \tag{13}$$

is valid.

Proof. Suppose that there exist positive distinctive solutions of prime period two

$$\dots,\ P,\ Q,\ P,\ Q,\ \dots$$

of the difference equation (1). Now, we discuss the following cases:

Case 1. l and k are even positive integers. In this case $x_n = x_{n-l} = x_{n-k}$. Then there exist a positive period two solution $\{x_n\}$ such that

$$x_{2q} = P, \quad q = -1, 0, 1, \dots$$
$$x_{2q+1} = Q, \quad q = -1, 0, 1, \dots$$

and $P \neq Q$. From the difference equation (1) we have

$$P = Q = \frac{\widetilde{a}}{\widetilde{\widetilde{b}}}.$$

This is a contradiction. Thus, equation (1) has no prime period two solution.

C a s e 2. l is a positive odd integer and k is a positive even integer. In this case $x_{n+1} = x_{n-l}$ and $x_n = x_{n-k}$. From the difference equation (1) we have

$$P = \frac{\alpha_0 Q + \alpha_1 P + \alpha_2 Q}{\beta_0 Q + \beta_1 P + \beta_2 Q}, \qquad Q = \frac{\alpha_0 P + \alpha_1 Q + \alpha_2 P}{\beta_0 P + \beta_1 Q + \beta_2 P}.$$

Consequently, we obtain

$$\alpha_0 Q + \alpha_1 P + \alpha_2 Q = \beta_0 PQ + \beta_1 P^2 + \beta_2 PQ$$

and

$$\alpha_0 P + \alpha_1 Q + \alpha_2 P = \beta_0 PQ + \beta_1 Q^2 + \beta_2 PQ.$$

By subtracting we have

$$P + Q = -\frac{[(\alpha_0 + \alpha_2) - \alpha_1]}{\beta_1}.$$

Since $\alpha_1 < \alpha_0 + \alpha_2$, we have $P + Q < 0$. Thus, we have a contradiction.

C a s e 3. l is a positive even integer and k is a positive odd integer. Then the proof of this case is similar to case 2.

C a s e 4. l and k are odd positive integers. In this case $x_{n+1} = x_{n-l} = x_{n-k}$. From the difference equation (1) we have

$$P = \frac{\alpha_0 Q + \alpha_1 P + \alpha_2 P}{\beta_0 Q + \beta_1 P + \beta_2 P}, \qquad Q = \frac{\alpha_0 P + \alpha_1 Q + \alpha_2 Q}{\beta_0 P + \beta_1 Q + \beta_2 Q}.$$

Consequently, we obtain

$$\alpha_0 Q + \alpha_1 P + \alpha_2 P = \beta_0 PQ + \beta_1 P^2 + \beta_2 P^2$$

and

$$\alpha_0 P + \alpha_1 Q + \alpha_2 Q = \beta_0 PQ + \beta_1 Q^2 + \beta_2 Q^2.$$

By subtracting we have

$$P + Q = \frac{[(\alpha_1 + \alpha_2) - \alpha_0]}{\beta_1 + \beta_2}, \tag{14}$$

while by adding we obtain

$$PQ = \frac{\alpha_0 [(\alpha_1 + \alpha_2) - \alpha_0]}{(\beta_1 + \beta_2) [\beta_0 - (\beta_1 + \beta_2)]} \tag{15}$$

provided that $(\alpha_1+\alpha_2)>\alpha_0$ and $\beta_0>(\beta_1+\beta_2)$. Assume that P and Q are two positive distinct real roots of the quadratic equation

$$t^2-(P+Q)\,t+PQ=0. \tag{16}$$

Thus, we deduce that

$$\left(\frac{[(\alpha_1+\alpha_2)-\alpha_0]}{\beta_1+\beta_2}\right)^2>4\,\frac{\alpha_0\,[(\alpha_1+\alpha_2)-\alpha_0]}{(\beta_1+\beta_2)\,[\beta_0-(\beta_1+\beta_2)]}. \tag{17}$$

From (17) we obtain

$$4\alpha_0\,(\beta_1+\beta_2)<[(\alpha_1+\alpha_2)-\alpha_0]\,[\beta_0-(\beta_1+\beta_2)]\,,$$

and hence the condition (11) is valid. Conversely, suppose that the condition (11) is valid provided that $(\alpha_1+\alpha_2)>\alpha_0$ and $\beta_0>(\beta_1+\beta_2)$. Then we deduce immediately from (11) that the inequality (17) holds. Consequently, there exist two positive distinct real numbers P and Q representing two positive roots of (16) such that

$$P=\frac{[(\alpha_1+\alpha_2)-\alpha_0]+\gamma}{2\,(\beta_1+\beta_2)} \tag{18}$$

and

$$Q=\frac{[(\alpha_1+\alpha_2)-\alpha_0]-\gamma}{2\,(\beta_1+\beta_2)}, \tag{19}$$

where $\gamma=\sqrt{[(\alpha_1+\alpha_2)-\alpha_0]^2-4\alpha_0\,(\beta_1+\beta_2)\,[(\alpha_1+\alpha_2)-\alpha_0]\,/\,[\beta_0-(\beta_1+\beta_2)]}$. Now, we are going to prove that P and Q are positive solutions of prime period two of the difference equation (1). To this end, we assume that $x_{-k}=Q$, $x_{-k+1}=P,\ldots,$ $x_{-l}=Q$, $x_{-l+1}=P,\ldots,x_{-1}=Q$ and $x_0=P$, where $l<k$. Now, we are going to show that $x_1=Q$ and $x_2=P$. From the difference equation (1) we deduce that

$$x_1=\frac{\alpha_0x_0+\alpha_1x_{-l}+\alpha_2x_{-k}}{\beta_0x_0+\beta_1x_{-l}+\beta_2x_{-k}}=\frac{\alpha_0P+(\alpha_1+\alpha_2)\,Q}{\beta_0P+(\beta_1+\beta_2)\,Q}. \tag{20}$$

Substituting (18) and (19) into (20) we deduce that

$$\begin{aligned} x_1-Q&=\frac{\alpha_0P+(\alpha_1+\alpha_2)\,Q-\beta_0PQ-(\beta_1+\beta_2)\,Q^2}{\beta_0P+(\beta_1+\beta_2)\,Q}\\ &=\frac{\alpha_0\left[\frac{[(\alpha_1+\alpha_2)-\alpha_0]+\gamma}{2(\beta_1+\beta_2)}\right]+(\alpha_1+\alpha_2)\left[\frac{[(\alpha_1+\alpha_2)-\alpha_0]-\gamma}{2(\beta_1+\beta_2)}\right]}{\beta_0\left[\frac{[(\alpha_1+\alpha_2)-\alpha_0]+\gamma}{2(\beta_1+\beta_2)}\right]+(\beta_1+\beta_2)\left[\frac{[(\alpha_1+\alpha_2)-\alpha_0]-\gamma}{2(\beta_1+\beta_2)}\right]}\\ &\quad-\frac{\beta_0\left[\frac{\alpha_0[(\alpha_1+\alpha_2)-\alpha_0]}{(\beta_1+\beta_2)[\beta_0-(\beta_1+\beta_2)]}\right]+(\beta_1+\beta_2)\left[\frac{[(\alpha_1+\alpha_2)-\alpha_0]-\gamma}{2(\beta_1+\beta_2)}\right]^2}{\beta_0\left[\frac{[(\alpha_1+\alpha_2)-\alpha_0]+\gamma}{2(\beta_1+\beta_2)}\right]+(\beta_1+\beta_2)\left[\frac{[(\alpha_1+\alpha_2)-\alpha_0]-\gamma}{2(\beta_1+\beta_2)}\right]}. \end{aligned} \tag{21}$$

Multiplying the denominator and numerator of (21) by $4\left(\beta_1+\beta_2\right)^2$ we get

(22)

$$
\begin{aligned}
& x_1-Q \\
& =\frac{2\left(\beta_1+\beta_2\right)\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]\left(\widetilde{a}-\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]\right)}{2 \beta_0\left(\beta_1+\beta_2\right)\left(\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]+\gamma\right)+2\left(\beta_1+\beta_2\right)^2\left(\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]-\gamma\right)} \\
& \quad+\frac{\left(4 \alpha_0\left(\beta_1+\beta_2\right)\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]\left[\left(\beta_1+\beta_2\right)-\beta_0\right]\right) /\left[\beta_0-\left(\beta_1+\beta_2\right)\right]}{2 \beta_0\left(\beta_1+\beta_2\right)\left(\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]+\gamma\right)+2\left(\beta_1+\beta_2\right)^2\left(\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]-\gamma\right)} \\
& \quad+\frac{\left[2\left(\beta_1+\beta_2\right)\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]-2\left(\beta_1+\beta_2\right)\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]\right] \gamma}{2 \beta_0\left(\beta_1+\beta_2\right)\left(\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]+\gamma\right)+2\left(\beta_1+\beta_2\right)^2\left(\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]-\gamma\right)} \\
& =\frac{4 \alpha_0\left(\beta_1+\beta_2\right)\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]}{2 \beta_0\left(\beta_1+\beta_2\right)\left(\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]+\gamma\right)+2\left(\beta_1+\beta_2\right)^2\left(\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]-\gamma\right)} \\
& \quad-\frac{\left(4 \alpha_0\left(\beta_1+\beta_2\right)\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]\left[\beta_0-\left(\beta_1+\beta_2\right)\right]\right) /\left(\beta_0-\beta_1-\beta_2\right)}{2 \beta_0\left(\beta_1+\beta_2\right)\left(\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]+\gamma\right)+2\left(\beta_1+\beta_2\right)^2\left(\left[\left(\alpha_1+\alpha_2\right)-\alpha_0\right]-\gamma\right)}=0 .
\end{aligned}
$$

Hence $x_1=Q$. Similarly, we can show that

$$
x_2=\frac{\alpha_0 x_1+\alpha_1 x_{-l+1}+\alpha_2 x_{-k+1}}{\beta_0 x_1+\beta_1 x_{-l+1}+\beta_2 x_{-k+1}}=\frac{\alpha_0 Q+\left(\alpha_1+\alpha_2\right) P}{\beta_0 Q+\left(\beta_1+\beta_2\right) P}=P .
$$

By using the mathematical induction, we conclude

$$
x_n=Q, \quad x_{n+1}=P, \quad n \geqslant-k .
$$

Finally, we note that the proofs of cases 5 and 6 are similar to that of case 4 and therefore are omitted here. Thus, the proof of Theorem 5 is now complete. □

5. Global stability

In this section we study the global asymptotic stability of the positive solutions of equation (1).

Lemma 1. *For any values of the quotients α_0/β_0, α_1/β_1 and α_2/β_2, the function $F(u_0,u_1,u_2)$ defined by equation* (3) *is monotonic in its arguments.*

Proof. By differentiating the function $F(u_0,u_1,u_2)$ given by the formula (3) with respect to u_i $(i=0,1,2)$ we obtain

$$
F_{u_0}=\frac{(\alpha_0\beta_1-\alpha_1\beta_0)u_1+(\alpha_0\beta_2-\alpha_2\beta_0)u_2}{(\beta_0u_0+\beta_1u_1+\beta_2u_2)^2}, \tag{23}
$$

$$
F_{u_1}=\frac{-(\alpha_0\beta_1-\alpha_1\beta_0)u_0+(\alpha_1\beta_2-\alpha_2\beta_1)u_2}{(\beta_0u_0+\beta_1u_1+\beta_2u_2)^2}, \tag{24}
$$

and

$$F_{u_2} = \frac{-(\alpha_0\beta_2 - \alpha_2\beta_0)u_0 - (\alpha_1\beta_2 - \alpha_2\beta_1)u_1}{(\beta_0 u_0 + \beta_1 u_1 + \beta_2 u_2)^2}. \tag{25}$$

We consider the following cases:

Case 1. $\alpha_0\beta_1 \geqslant \alpha_1\beta_0$, $\alpha_0\beta_2 \geqslant \alpha_2\beta_0$ and $\alpha_1\beta_2 \geqslant \alpha_2\beta_1$. From the equations (23)–(25) it is easy to see that the function $F(u_0, u_1, u_2)$ is non-decreasing in u_0, non-increasing in u_2 and it is not clear what is going on with u_1.

Case 2. $\alpha_0\beta_1 \geqslant \alpha_1\beta_0$, $\alpha_0\beta_2 \geqslant \alpha_2\beta_0$ and $\alpha_1\beta_2 \leqslant \alpha_2\beta_1$. From the equations (23)–(25) it is easy to see that the function $F(u_0, u_1, u_2)$ is non-decreasing in u_0, non-increasing in u_1 and it is not clear what is going on with u_2.

Case 3. $\alpha_0\beta_1 \leqslant \alpha_1\beta_0$, $\alpha_0\beta_2 \geqslant \alpha_2\beta_0$ and $\alpha_1\beta_2 \geqslant \alpha_2\beta_1$. From the equations (23)–(25) it is easy to see that the function $F(u_0, u_1, u_2)$ is non-decreasing in u_1, non-increasing in u_2 and it is not clear what is going on with u_0.

Case 4. $\alpha_0\beta_1 \geqslant \alpha_1\beta_0$, $\alpha_0\beta_2 \leqslant \alpha_2\beta_0$ and $\alpha_1\beta_2 \leqslant \alpha_2\beta_1$. From the equations (23)–(25) it is easy to see that the function $F(u_0, u_1, u_2)$ is non-decreasing in u_2, non-increasing in u_1 and it is not clear what is going on with u_0.

Case 5. $\alpha_0\beta_1 \leqslant \alpha_1\beta_0$, $\alpha_0\beta_2 \leqslant \alpha_2\beta_0$ and $\alpha_1\beta_2 \geqslant \alpha_2\beta_1$. From the equations (23)–(25) it is easy to see that the function $F(u_0, u_1, u_2)$ is non-decreasing in u_1, non-increasing in u_0 and it is not clear what is going on with u_2.

Case 6. $\alpha_0\beta_1 \leqslant \alpha_1\beta_0$, $\alpha_0\beta_2 \leqslant \alpha_2\beta_0$ and $\alpha_1\beta_2 \leqslant \alpha_2\beta_1$. From the equations (23)–(25) it is easy to see that the function $F(u_0, u_1, u_2)$ is non-decreasing in u_2, non-increasing in u_0 and it is not clear what is going on with u_1. □

Theorem 6. *The positive equilibrium point $\widetilde{x}$ of the difference equation* (1) *is a global attractor if one of the following conditions holds:*

$$\alpha_0\beta_1 \geqslant \alpha_1\beta_0,\ \alpha_0\beta_2 \geqslant \alpha_2\beta_0,\ \alpha_1\beta_2 \geqslant \alpha_2\beta_1 \text{ and } \alpha_2 \geqslant (\alpha_0 + \alpha_1); \tag{26}$$

$$\alpha_0\beta_1 \geqslant \alpha_1\beta_0,\ \alpha_0\beta_2 \geqslant \alpha_2\beta_0,\ \alpha_1\beta_2 \leqslant \alpha_2\beta_1 \text{ and } \alpha_1 \geqslant (\alpha_0 + \alpha_2); \tag{27}$$

$$\alpha_0\beta_1 \leqslant \alpha_1\beta_0,\ \alpha_0\beta_2 \geqslant \alpha_2\beta_0,\ \alpha_1\beta_2 \geqslant \alpha_2\beta_1 \text{ and } \alpha_2 \geqslant (\alpha_0 + \alpha_1); \tag{28}$$

$$\alpha_0\beta_1 \geqslant \alpha_1\beta_0,\ \alpha_0\beta_2 \leqslant \alpha_2\beta_0,\ \alpha_1\beta_2 \leqslant \alpha_2\beta_1 \text{ and } \alpha_1 \geqslant (\alpha_0 + \alpha_2); \tag{29}$$

$$\alpha_0\beta_1 \leqslant \alpha_1\beta_0,\ \alpha_0\beta_2 \leqslant \alpha_2\beta_0,\ \alpha_1\beta_2 \geqslant \alpha_2\beta_1 \text{ and } \alpha_0 \geqslant (\alpha_1 + \alpha_2); \tag{30}$$

$$\alpha_0\beta_1 \leqslant \alpha_1\beta_0,\ \alpha_0\beta_2 \leqslant \alpha_2\beta_0,\ \alpha_1\beta_2 \leqslant \alpha_2\beta_1 \text{ and } \alpha_0 \geqslant (\alpha_1 + \alpha_2). \tag{31}$$

We shall prove this theorem for the first condition; in a similar way we can prove it for the others.

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of the difference equation (1). To prove this theorem, it suffices to prove that $x_n \to \widetilde{x}$ as $n \to \infty$. Let us prove that as follows: Let $F\colon (0,+\infty)^3 \longrightarrow (0,+\infty)$ be a continuous function defined by the formula (3). With reference to Lemma 6, we notice that if

$$\alpha_0\beta_1 \geqslant \alpha_1\beta_0,\ \ \alpha_0\beta_2 \geqslant \alpha_2\beta_0 \ \text{ and } \ \alpha_1\beta_2 \geqslant \alpha_2\beta_1,$$

then the function $F(u_0,u_1,u_2)$ is non-decreasing in u_0 and non-increasing in u_2. Now, from the difference equation (1) we have

$$\begin{aligned} x_{n+1} &\leqslant \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2(0)}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2(0)} \\ &\leqslant \frac{\alpha_0}{\beta_0} + \frac{\alpha_1}{\beta_1}, \quad n \geqslant 0. \end{aligned}$$

Consequently, we obtain

$$x_n \leqslant \frac{\alpha_0}{\beta_0} + \frac{\alpha_1}{\beta_1} = H, \quad n \geqslant 1, \tag{32}$$

where H is a positive constant. On the other hand, we deduce from the difference equation (1)

$$\begin{aligned} x_{n+1} &\geqslant \frac{\alpha_0(0) + \alpha_1(0) + \alpha_2(H)}{\beta_0(H) + \beta_1(H) + \beta_2(H)} \\ &\geqslant \frac{\alpha_2}{\beta_0 + \beta_1 + \beta_2}, \quad n \geqslant 0. \end{aligned}$$

Consequently, we obtain

$$x_n \geqslant \frac{\alpha_2}{\beta_0 + \beta_1 + \beta_2} = h, \quad n \geqslant 1, \tag{33}$$

where h is a positive constant. From the inequalities (32) and (33) we find that

$$h \ \leqslant x_n \leqslant H, \quad n \geqslant 1.$$

Thus the sequence $\{x_n\}$ is bounded. It follows by the method of full limiting sequences ([10], [16]) that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of the equation (1) with

$$I = I_0 = \lim_{n \dashrightarrow \infty} \inf x_n \leqslant \lim_{n \dashrightarrow \infty} \sup x_n = S_0 = S,$$

where

$$I_n, S_n \in [I,S], \quad n = 0,-1,\dots$$

On the other hand, it follows from the difference equation (1) that

$$I = \frac{\alpha_0 I_{-1} + \alpha_1 I_{-l-1} + \alpha_2 I_{-k-1}}{\beta_0 I_{-1} + \beta_1 I_{-l-1} + \beta_2 I_{-k-1}} \geqslant \frac{\alpha_0 I + \alpha_1 I_{-l-1} + \alpha_2 S}{\beta_0 I + \beta_1 I_{-l-1} + \beta_2 S} \geqslant \frac{(\alpha_0 + \alpha_1)\, I + \alpha_2 S}{\beta_0 I + (\beta_1 + \beta_2)\, S},$$

and consequently, we have

$$(\alpha_0+\alpha_1)I+\alpha_2 S-\beta_0 I^2 \leqslant (\beta_1+\beta_2)SI. \tag{34}$$

Similarly, we deduce from the difference equation (1) that

$$S=\frac{\alpha_0 S_{-1}+\alpha_1 S_{-l-1}+\alpha_2 S_{-k-1}}{\beta_0 S_{-1}+\beta_1 S_{-l-1}+\beta_2 S_{-k-1}} \leqslant \frac{\alpha_0 S+\alpha_1 S_{-l-1}+\alpha_2 I}{\beta_0 S+\beta_1 S_{-l-1}+\beta_2 I} \leqslant \frac{(\alpha_0+\alpha_1)S+\alpha_2 I}{\beta_0 S+(\beta_1+\beta_2)I},$$

and consequently, we have

$$(\alpha_0+\alpha_1)S+\alpha_2 I-\beta_0 S^2 \geqslant (\beta_1+\beta_2)SI. \tag{35}$$

It follows from the inequalities (34) and (35) that

$$(I-S)\left[\beta_0(I+S)+\alpha_2-(\alpha_0+\alpha_1)\right] \geqslant 0, \tag{36}$$

and consequently, we deduce

$$I \geqslant S \quad \text{if} \quad [\beta_0(I+S)+\alpha_2-(\alpha_0+\alpha_1)] \geqslant 0.$$

Now, we know by (26) that

$$\alpha_2 \geqslant \alpha_0+\alpha_1, \tag{37}$$

and so, we find that

$$I \geqslant S. \tag{38}$$

Consequently, we have $I=S$. □

By virtue of Theorems 4 and 7, we arrive at the following result:

Theorem 7. *The positive equilibrium point* $\widetilde{x}$ *of the difference equation* (1) *is globally asymptotically stable.*

6. Numerical experiments on the main results

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical experiments in this section. These experiments represent different types of qualitative behavior of solutions to the nonlinear difference equation (1).

Experiment 1. Figure 1 shows that equation (1) has no prime period two solution if both l, k are even and $l<k$. Choose $l=2$, $k=4$, $x_{-4}=1$, $x_{-3}=2$, $x_{-2}=3$, $x_{-1}=4$, $x_0=5$, $\alpha_0=2$, $\alpha_1=10$, $\alpha_2=20$, $\beta_0=30$, $\beta_1=3$, $\beta_2=4$.

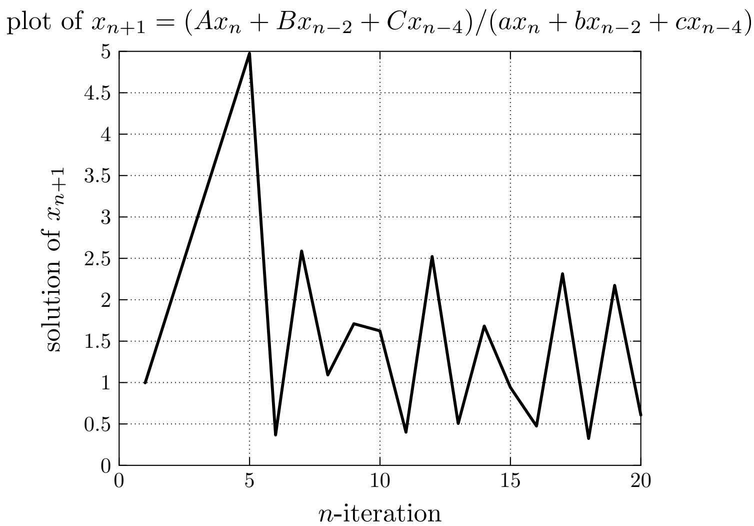

Fig. 1. $x_{n+1} = (2x_n + 10x_{n-2} + 20x_{n-4})/(30x_n + 3x_{n-2} + 4x_{n-4})$

Experiment 2. Figure 2 shows that equation (1) has no prime period two solution if l is odd, k is even and $l < k$. Choose $l = 1$, $k = 2$, $x_{-2} = 1$, $x_{-1} = 2$, $x_0 = 3$, $\alpha_0 = 2$, $\alpha_1 = 10$, $\alpha_2 = 20$, $\beta_0 = 30$, $\beta_1 = 3$, $\beta_2 = 4$.

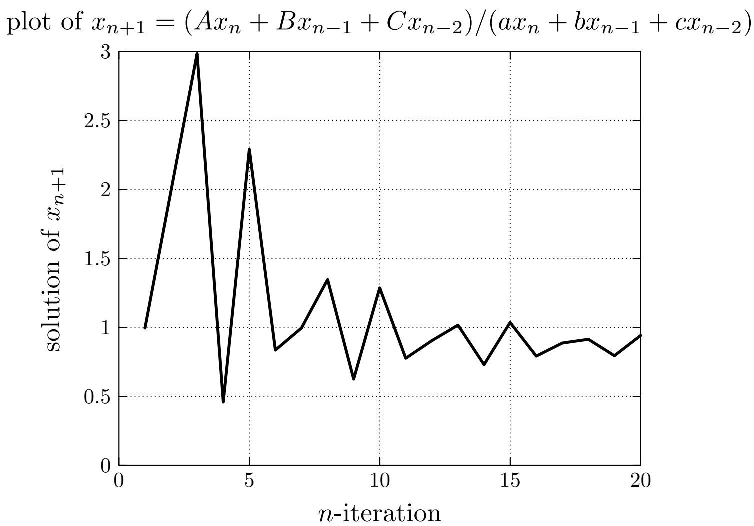

Fig. 2. $x_{n+1} = (2x_n + 10x_{n-1} + 20x_{n-2})/(30x_n + 3x_{n-1} + 4x_{n-2})$

Experiment 3. Figure 3 shows that equation (1) has no prime period two solution if l is even, k is odd and $l < k$. Choose $l = 2$, $k = 3$, $x_{-2} = 1$, $x_{-1} = 2$, $x_0 = 3$, $\alpha_0 = 2$, $\alpha_1 = 20$, $\alpha_2 = 10$, $\beta_0 = 30$, $\beta_1 = 3$, $\beta_2 = 4$.

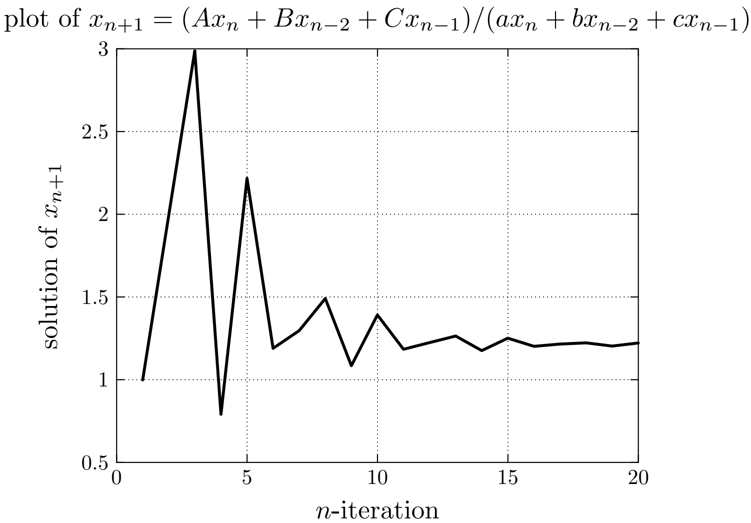

Fig. 3. $x_{n+1} = (2x_n + 20x_{n-2} + 10x_{n-3})/(30x_n + 3x_{n-2} + 4x_{n-3})$

E x p e r i m e n t 4. Figure 4 shows that equation (1) has prime period two solution if both l, k are odd and $l < k$. Choose $l = 1$, $k = 3$, $x_{-3} = 3.6$, $x_{-2} = 0.16$, $x_{-1} = 3.6$, $x_0 = 0.16$, $\alpha_0 = 2$, $\alpha_1 = 10$, $\alpha_2 = 20$, $\beta_0 = 30$, $\beta_1 = 3$, $\beta_2 = 4$. By using equation (1) we get $x_1 = 3.6$.

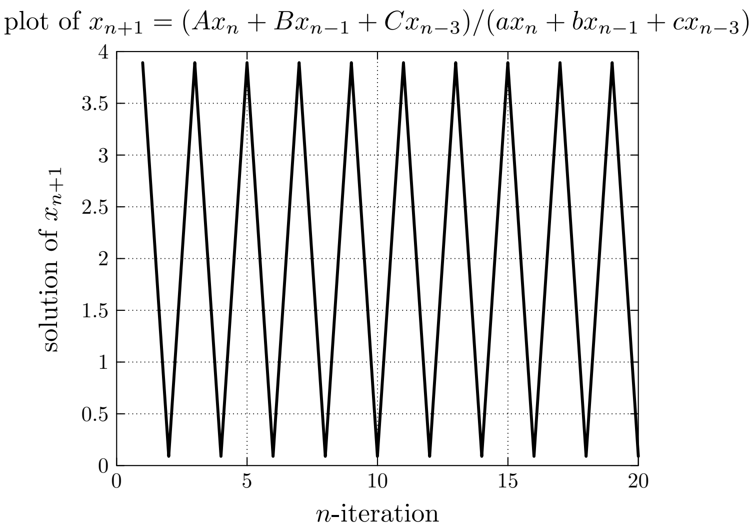

Fig. 4. $x_{n+1} = (2x_n + 10x_{n-1} + 20x_{n-3})/(30x_n + 3x_{n-1} + 4x_{n-3})$

E x p e r i m e n t 5. Figure 5 shows that equation (1) has prime period two solution if l is odd, k is even and $l < k$. Choose $l = 1$, $k = 2$, $x_{-2} = 1.6$, $x_{-1} = 0.7$, $x_0 = 1.6$,

$\alpha_0 = 1$, $\alpha_1 = 10$, $\alpha_2 = 2$, $\beta_0 = 4$, $\beta_1 = 3$, $\beta_2 = 5$. By using equation (1) we get $x_1 = 0.7$.

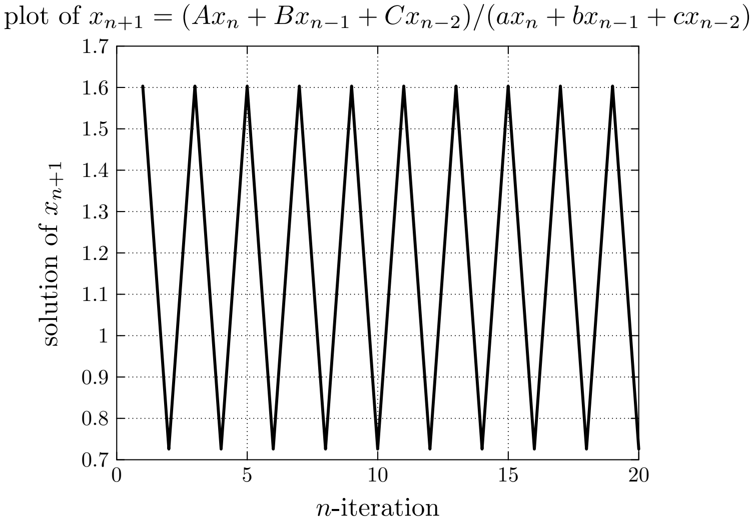

Fig. 5. $x_{n+1} = (x_n + 10x_{n-1} + 2x_{n-2})/(4x_n + 3x_{n-1} + 5x_{n-2})$

Experiment 6. Figure 6 shows that equation (1) has prime period two solution if l is even, k is odd and $l < k$. Choose $l = 2$, $k = 3$, $x_{-3} = 1.6$, $x_{-2} = 0.7$, $x_{-1} = 1.6$, $x_0 = 0.7$, $\alpha_0 = 1$, $\alpha_1 = 2$, $\alpha_2 = 10$, $\beta_0 = 4$, $\beta_1 = 5$, $\beta_2 = 3$. By using equation (1) we get $x_1 = 0.7$.

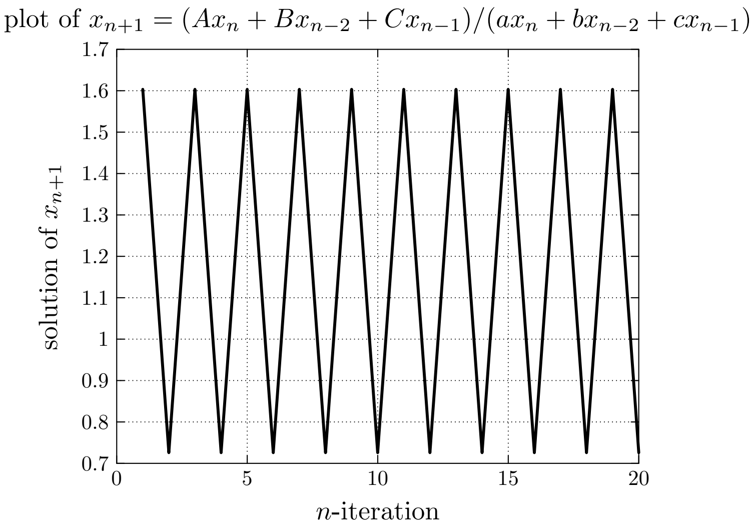

Fig. 6. $x_{n+1} = (x_n + 2x_{n-2} + 10x_{n-3})/(4x_n + 5x_{n-2} + 3x_{n-3})$

E x p e r i m e n t 7. Figure 7 shows that the solution of equation (1) has global stability if $l = 2$, $k = 4$, $x_{-4} = 1$, $x_{-3} = 2$, $x_{-2} = 3$, $x_{-1} = 4$, $x_0 = 5$, $\alpha_0 = 0.5$, $\alpha_1 = 0.25$, $\alpha_2 = 1$, $\beta_0 = 3$, $\beta_1 = 2$, $\beta_2 = 10$.

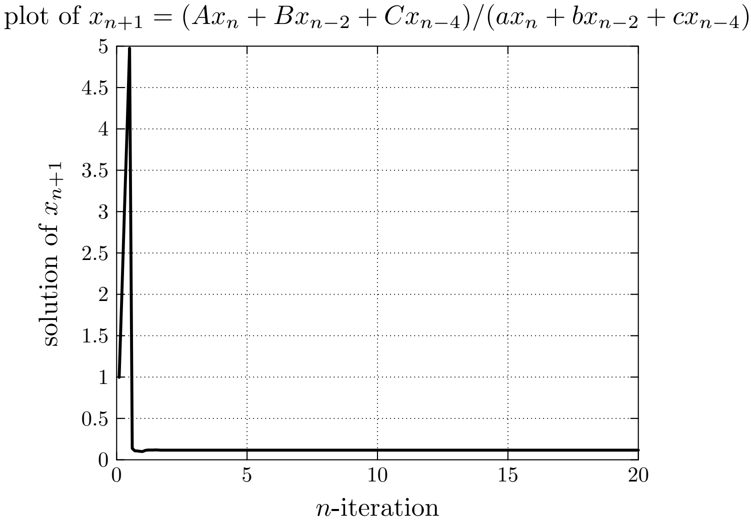

Fig. 7. $x_{n+1} = (0.5x_n + 0.25x_{n-2} + x_{n-4})/(3x_n + 2x_{n-2} + 10x_{n-4})$

Note that the experiments 1, 2, 3 verify Theorem 5 (1, 2, 3), which shows that equation (1) has no prime period two solution, while experiments 4, 5, 6 verify Theorem 5 (4, 5, 6), which shows that equation (1) has prime period two solution. However, the experiment 7 verifies Theorems 7, 8 which shows that if $\alpha_0\beta_1 \geqslant \alpha_1\beta_0$, $\alpha_0\beta_2 \geqslant \alpha_2\beta_0$, $\alpha_1\beta_2 \geqslant \alpha_2\beta_1$ and $\alpha_2 \geqslant (\alpha_0 + \alpha_1)$, then the solution of equation (1) is globally asymptotically stable.

A c k n o w l e d g e m e n t. The authors wish to thank the referee for his suggestions and comments.

References

[1] *M. T. Aboutaleb, M. A. El-Sayed, A. E. Hamza*: Stability of the recursive sequence $x_{n+1} = (\alpha - \beta x_n)/(\gamma + x_{n-1})$. J. Math. Anal. Appl. *261* (2001), 126–133.

[2] *R. Agarwal*: Difference Equations and Inequalities. Theory, Methods and Applications. Marcel Dekker, New York, 1992.

[3] *A. M. Amleh, E. A. Grove, G. Ladas, D. A. Georgiou*: On the recursive sequence $x_{n+1} = \alpha + (x_{n-1}/x_n)$. J. Math. Anal. Appl. *233* (1999), 790–798.

[4] *C. W. Clark*: A delayed recruitment model of population dynamics with an application to baleen whale populations. J. Math. Biol. *3* (1976), 381–391.

[5] *R. Devault, W. Kosmala, G. Ladas, S. W. Schultz*: Global behavior of $y_{n+1} = (p + y_{n-k})/(qy_n + y_{n-k})$. Nonlinear Analysis *47* (2001), 4743–4751.

[6] *R. Devault, G. Ladas, S. W. Schultz*: On the recursive sequence $x_{n+1} = \alpha + (x_n/x_{n-1})$. Proc. Amer. Math. Soc. *126(11)* (1998), 3257–3261.
[7] *R. Devault, S. W. Schultz*: On the dynamics of $x_{n+1} = (\beta x_n + \gamma x_{n-1})/(Bx_n + Dx_{n-2})$. Comm. Appl. Nonlinear Anal. *12* (2005), 35–39.
[8] *E. M. Elabbasy, H. El-Metwally, E. M. Elsayed*: On the difference equation $x_{n+1} = (\alpha x_{n-l} + \beta x_{n-k})/(Ax_{n-l} + Bx_{n-k})$. Acta Mathematica Vietnamica *33* (2008), 85–94.
[9] *H. El-Metwally, E. A. Grove, G. Ladas*: A global convergence result with applications to periodic solutions. J. Math. Anal. Appl. *245* (2000), 161–170.
[10] *H. El-Metwally, G. Ladas, E. A. Grove, H. D. Voulov*: On the global attractivity and the periodic character of some difference equations. J. Differ. Equ. Appl. *7* (2001), 837–850.
[11] *H. A. El-Morshedy*: New explicit global asymptotic stability criteria for higher order difference equations. J. Math. Anal. Appl. *336* (2007), 262–276.
[12] *H. M. EL-Owaidy, A. M. Ahmed, M. S. Mousa*: On asymptotic behavior of the difference equation $x_{n+1} = \alpha + (x^p_{n-1}/x^p_n)$. J. Appl. Math. Comput. *12* (2003), 31–37.
[13] *H. M. EL-Owaidy, A. M. Ahmed, Z. Elsady*: Global attractivity of the recursive sequence $x_{n+1} = (\alpha - \beta x_{n-k})/(\gamma + x_n)$. J. Appl. Math. Comput. *16* (2004), 243–249.
[14] *C. H. Gibbons, M. R. S. Kulenovic, G. Ladas*: On the recursive sequence $x_{n+1} = (\alpha + \beta x_{n-1})/(\gamma + x_n)$. Math. Sci. Res. Hot-Line *4(2)* (2000), 1–11.
[15] *E. A. Grove, G. Ladas*: Periodicities in Nonlinear Difference Equations. Vol. 4. Chapman & Hall / CRC, 2005.
[16] *G. Karakostas*: Convergence of a difference equation via the full limiting sequences method. Differ. Equ. Dyn. Syst. *1* (1993), 289–294.
[17] *G. Karakostas, S. Stević*: On the recursive sequences $x_{n+1} = A + f(x_n, \ldots, x_{n-k+1})/x_{n-1}$. Comm. Appl. Nonlinear Anal. *11* (2004), 87–100.
[18] *V. L. Kocic, G. Ladas*: Global Behavior of Nonlinear Difference Equations of Higher Order with Applications. Kluwer Academic Publishers, Dordrecht, 1993.
[19] *M. R. S. Kulenovic, G. Ladas*: Dynamics of Second Order Rational Difference Equations with Open Problems and conjectures. Chapman & Hall / CRC, 2001.
[20] *M. R. S. Kulenovic, G. Ladas, W. S. Sizer*: On the recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1})/(\gamma x_n + \delta x_{n-1})$. Math. Sci. Res. Hot-Line *2* (1998), 1–16.
[21] *S. A. Kuruklis*: The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$. J. Math. Anal. Appl. *188* (1994), 719–731.
[22] *G. Ladas, C. H. Gibbons, M. R. S. Kulenovic, H. D. Voulov*: On the trichotomy character of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + x_n)$. J. Difference Equ. Appl. *8* (2002), 75–92.
[23] *G. Ladas, C. H. Gibbons, M. R. S. Kulenovic*: On the dynamics of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n)$. Proceeding of the Fifth International Conference on Difference Equations and Applications, Temuco, Chile, Jan. 3–7, 2000. Taylor and Francis, London, 2002, pp. 141–158.
[24] *G. Ladas, E. Camouzis, H. D. Voulov*: On the dynamic of $x_{n+1} = (\alpha + \gamma x_{n-1} + \delta x_{n-2})/(A + x_{n-2})$. J. Difference Equ. Appl. *9* (2003), 731–738.
[25] *G. Ladas*: On the rational recursive sequence $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n + Cx_{n-1})$. J. Difference Equ. Appl. *1* (1995), 317–321.
[26] *W. T. Li, H. R. Sun*: Global attractivity in a rational recursive sequence. Dyn. Syst. Appl. *11* (2002), 339–346.
[27] *S. Stević*: On the recursive sequence $x_{n+1} = x_{n-1}/g(x_n)$. Taiwanese J. Math. *6* (2002), 405–414.
[28] *S. Stević*: On the recursive sequence $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$. Appl. Math. Letter *15* (2002), 305–308.
[29] *S. Stević*: On the recursive sequence $x_{n+1} = (\alpha + \beta x_n)/(\gamma - x_{n-k})$. Bull. Inst. Math. Acad. Sin. *32* (2004), 61–70.

[30] *S. Stević*: On the recursive sequences $x_{n+1} = \alpha + (x_{n-1}^p / x_n^p)$. J. Appl. Math. Comput. *18* (2005), 229–234.
[31] *X. Yang, W. Su, B. Chen, G. M. Megson, D. J. Evans*: On the recursive sequence $x_{n+1} = (ax_{n-1} + bx_{n-2})/(c + dx_{n-1}x_{n-2})$. J. Appl. Math. Comput. *162* (2005), 1485–1497.
[32] *E. M. E. Zayed, M. A. El-Moneam*: On the rational recursive sequence $x_{n+1} = (D + \alpha x_n + \beta x_{n-1} + \gamma x_{n-2})/(Ax_n + Bx_{n-1} + Cx_{n-2})$. Comm. Appl. Nonlinear Anal. *12* (2005), 15–28.
[33] *E. M. E. Zayed, M. A. El-Moneam*: On the rational recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1} + \gamma x_{n-2} + \delta x_{n-3})/(Ax_n + Bx_{n-1} + Cx_{n-2} + Dx_{n-3})$. J. Appl. Math. Comput. *22* (2006), 247–262.
[34] *E. M. E. Zayed, M. A. El-Moneam*: On the rational recursive sequence $x_{n+1} = \Big(A + \sum\limits_{i=0}^{k} \alpha_i x_{n-i}\Big) \Big/ \Big(B + \sum\limits_{i=0}^{k} \beta_i x_{n-i}\Big)$. Int. J. Math. Math. Sci. *2007* (2007), 12, Article ID23618.
[35] *E. M. E. Zayed, M. A. El-Moneam*: On the rational recursive sequence $x_{n+1} = ax_n - bx_n / (cx_n - dx_{n-k})$. Comm. Appl. Nonlinear Anal. *15* (2008), 47–57.
[36] *E. M. E. Zayed, M. A. El-Moneam*: On the rational recursive sequence $x_{n+1} = \Big(A + \sum\limits_{i=0}^{k} \alpha_i x_{n-i}\Big) \Big/ \sum\limits_{i=0}^{k} \beta_i x_{n-i}$. Math. Bohem. *133* (2008), 225–239.
[37] *E. M. E. Zayed, M. A. El-Moneam*: On the rational recursive sequence $x_{n+1} = Ax_n + (\beta x_n + \gamma x_{n-k})/(Cx_n + Dx_{n-k})$. Comm. Appl. Nonlinear Anal. *16* (2009), 91–106.
[38] *E. M. E. Zayed, M. A. El-Moneam*: On the rational recursive sequence $x_{n+1} = (\alpha + \beta x_{n-k})/(\gamma - x_n)$. J. Appl. Math. Comput. *31* (2009), 229–237.

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