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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF HIGHER ORDER DIFFERENCE EQUATIONS

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Abstract. Asymptotic properties of solutions of the difference equation of the form

$$\Delta^m x_n = a_n \varphi(x_{\tau_1(n)}, \ldots, x_{\tau_k(n)}) + b_n$$

are studied. Conditions under which every (every bounded) solution of the equation $\Delta^m y_n = b_n$ is asymptotically equivalent to some solution of the above equation are obtained.

Keywords: difference equation, asymptotic behavior

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1. Introduction

We denote by $\mathbb{Z}, \mathbb{N}, \mathbb{R}$ the set of integers, positive integers and real numbers, respectively. For $p \in \mathbb{Z}$ let $N(p) = \{p, p+1, \ldots\}$.

Let $m, k \in \mathbb{N}$. We consider the difference equation of the form

$$\text{(E)} \qquad \Delta^m x_n = a_n \varphi(x_{\tau_1(n)}, \ldots, x_{\tau_k(n)}) + b_n,$$

$$n \in \mathbb{N}, \quad a_n, b_n \in \mathbb{R}, \quad \varphi\colon \mathbb{R}^k \to \mathbb{R}, \quad \tau_1, \ldots \tau_k\colon \mathbb{N} \to \mathbb{Z},$$

$$\lim_{n\to\infty} \tau_i(n) = \infty \quad \text{for} \quad i = 1, \ldots, k.$$

Let $n_0 = \min\{n \in \mathbb{N}\colon\ \tau_i(j) \geqslant 1 \ \text{ for every } \ j \geqslant n \ \text{ and every } \ i = 1, \ldots, k\}$. By a solution of equation (E) we mean a sequence $x\colon \mathbb{N} \to \mathbb{R}$ if there exists $q \in N(n_0)$ such that the equation (E) is satisfied for all $n \geqslant q$. We say that sequences x, y are asymptotically equivalent if $x_n - y_n = o(1)$. For a given sequence x of real numbers, by $\sum x_n$ we denote the series whose partial sums are $x_1, x_1 + x_2, x_1 + x_2 + x_3$ and so on.

Recently, there has been a great interest in the study of asymptotic and oscillatory behavior of solutions of higher order difference equations, see for example [3], [4], [6], [10]–[17], Chapter 6 of [1], Chapter 7 of [5] and the references cited therein.

The purpose of this paper is to study the asymptotic behavior of solutions of equation (E). Using the Schauder fixed point theorem and some technical results obtained in Section 2 we show that if the sequence (a_n) is sufficiently "small" and the function φ is continuous (uniformly continuous and bounded), then every bounded solution (every solution) of the equation $\Delta^m y_n = b_n$ is asymptotically equivalent to some solution of (E). Moreover, if the sequence (b_n) is also "small", then we may replace the solutions of $\Delta^m y_n = b_n$ by the solutions of $\Delta^m y_n = 0$, i.e. every polynomial sequence of degree $< m$ is asymptotically equivalent to some solution of (E). A similar problem for the first order nonautonomous difference equation was considered in [10].

The results obtained here extend the results of the paper [3] and some of those contained in [4], [11]. For the general theory of difference equations, one can refer to [1], [5], [7]. Many references to some applications of the difference equations can be found in [2], [8], [9].

The space of all sequences $x\colon \mathbb{N} \to \mathbb{R}$ we denote by SQ. The Banach space of all bounded sequences $x \in SQ$ with the norm

$$\|x\| = \sup\{|x_n|\colon\ n \in \mathbb{N}\}$$

we denote by BS. If $B \subseteq \mathbb{R}$ then B^k denotes the set

$$B \times B \times \ldots \times B \subseteq \mathbb{R}^k.$$

Similarly if $c \in \mathbb{R}$, then $c^k = (c, c, \ldots, c) \in \mathbb{R}^k$. The standard (Euclidean) metric on $\mathbb{R}^k$ we denote by d. We choose a constant $\lambda_0 \in \mathbb{R}$ such that

$$d(t, s) \leqslant \lambda_0 \max\{|t_i - s_i|\colon\ i = 1, 2, \ldots, k\}$$

for every $t = (t_1, \ldots, t_k)$, $s = (s_1, \ldots, s_k) \in \mathbb{R}^k$ (λ_0 may be any real number $\geqslant \sqrt{k}$). If $X \subseteq \mathbb{R}^k$ then $\varphi|X$ denotes the restriction of the function φ to the set X, i.e., $\varphi|X\colon\ X \to \mathbb{R}$, $(\varphi|X)(t) = \varphi(t)$ for any $t \in X$.

2. Preliminary lemmas

In this section we present some lemmas which are interesting in their own right and will be used in the proofs of the main theorems in Section 3.

For $n \in N(1)$ we define numbers

$$\sigma_n^0 = 1, \qquad \sigma_n^1 = \sigma_1^0 + \sigma_2^0 + \ldots + \sigma_n^0 = n.$$

If $k, n \in N(1)$ then by induction on k we define numbers

$$\sigma_n^{k+1} = \sigma_1^k + \sigma_2^k + \ldots + \sigma_n^k.$$

Moreover, we define $\sigma_n^k = 0$ for $n \in -N(0)$, $k \in N(0)$. By virtue of the equality

$$\sum_{i=1}^{n} \binom{k+i-1}{k} = \binom{k+n}{k+1}$$

it is easy to see that if $k \in N(0)$ and $n \in N(1)$, then

$$\sigma_n^k = \binom{n+k-1}{k} = \frac{n(n+1)\ldots(n+k-1)}{k!}.$$

Obviously, $\sigma_n^k \leqslant n^k$.

Lemma 1. *If $x = o(1)$ then $x_n = -\sum\limits_{i=n}^{\infty} \Delta x_i$ for every $n \in N(1)$.*

P r o o f. If $j \geqslant n$ then $\Delta x_n + \ldots + \Delta x_j = x_{j+1} - x_n \longrightarrow -x_n$. □

Lemma 2. *If $x = o(1)$ then*

$$x_n = (-1)^k \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \ldots \sum_{i_k=i_{k-1}}^{\infty} \Delta^k x_{i_k} \tag{1}$$

for every $k, n \in N(1)$.

P r o o f. If $x = o(1)$ then $\Delta^m x = o(1)$ for every $m \in N(1)$. By Lemma 1, $x_n = -\sum\limits_{i_1=n}^{\infty} \Delta x_{i_1}$, $\Delta x_{i_1} = -\sum\limits_{i_2=i_1}^{\infty} \Delta^2 x_{i_2}$. Hence,

$$x_n = (-1)^2 \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \Delta^2 x_{i_2}.$$

By Lemma 1, $\Delta^2 x_{i_2} = -\sum\limits_{i_3=i_2}^{\infty} \Delta^3 x_{i_3}$. Therefore

$$x_n = (-1)^3 \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \sum_{i_3=i_2}^{\infty} \Delta^3 x_{i_3}$$

and so on. After k steps we obtain (1). □

Lemma 3. *Assume $m \in N(1)$, the series $\sum n^{m-1} x_n$ is absolutely convergent and the sequence z is defined by*

$$z_n = \sigma_1^{m-1} x_n + \sigma_2^{m-1} x_{n+1} + \dots \ .$$

Then $z = o(1)$ and $\Delta^m z = (-1)^m x$.

Proof. For every $q \in \{1, \dots, m\}$ we define the sequence r^q by

$$r_n^q = \sigma_1^{q-1} x_n + \sigma_2^{q-1} x_{n+1} + \dots = \sum_{k=0}^{\infty} \sigma_{k+1}^{q-1} x_{n+k}.$$

The series is absolutely convergent since

$$\sigma_{k+1}^{q-1} \leqslant \sigma_{k+1}^{m-1} \leqslant \sigma_{n+k}^{m-1} \leqslant (n+k)^{m-1}.$$

Since

$$r_n^m = \sum_{k=0}^{\infty} \sigma_{k+1}^{m-1} x_{n+k} = \sigma_1^{m-1} x_n + \sum_{k=1}^{\infty} \sigma_{k+1}^{m-1} x_{n+k},$$

$$r_{n+1}^m = \sum_{k=0}^{\infty} \sigma_{k+1}^{m-1} x_{n+1+k} = \sum_{k=1}^{\infty} \sigma_k^{m-1} x_{n+k}$$

we obtain

$$\begin{aligned} r_n^m - r_{n+1}^m &= \sigma_1^{m-1} x_n + \sum_{k=1}^{\infty} (\sigma_{k+1}^{m-1} - \sigma_k^{m-1}) x_{n+k} \\ &= \sigma_1^{m-2} x_n + \sum_{k=1}^{\infty} \sigma_{k+1}^{m-2} x_{n+k} = \sum_{k=0}^{\infty} \sigma_{k+1}^{m-2} x_{n+k} = r_n^{m-1}. \end{aligned}$$

Therefore $\Delta r^m = -r^{m-1}$. Analogously, $\Delta r^{m-1} = -r^{m-2}$. Hence,

$$\Delta^2 r^m = (-1)^2 r^{m-1}$$

and so on. After $m-1$ steps we obtain $\Delta^{m-1} r^m = (-1)^{m-1} r^1$. Obviously $\Delta r^1 = -x$. Hence $\Delta^m r^m = (-1)^m x$. Moreover, by absolute convergence of the series $\sum \sigma_n^{m-1} x_n$ we obtain $\lim\limits_{n\to\infty} r_n^m = 0$. Note that $z = r^m$. The proof is complete. □

Lemma 4. *Assume $m \in N(1)$ and let the series $\sum n^{m-1}x_n$ be absolutely convergent. Then there exists exactly one sequence z such that $z = o(1)$ and $\Delta^m z = (-1)^m x$. The sequence z is defined by*

$$
\begin{aligned}
z_n &= \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} x_{i_m} \\
&= \sum_{k=0}^{\infty} \sigma_{k+1}^{m-1} x_{n+k} = \sum_{k=0}^{\infty} \binom{m-1+k}{m-1} x_{n+k} \\
&= \sum_{k=0}^{\infty} \frac{(k+1)(k+2)\dots(k+m-1)}{(m-1)!} x_{n+k}.
\end{aligned}
$$

Proof. By Lemma 3, if $z_n = \sum\limits_{k=0}^{\infty} \sigma_{k+1}^{m-1} x_{n+k}$ then $z = o(1)$ and $\Delta^m z = (-1)^m x$. If y is a sequence such that $y = o(1)$ and $\Delta^m y = (-1)^m x$, then $z - y = o(1)$ and $\Delta^m(z - y) = \Delta^m z - \Delta^m y = 0$. Therefore $z - y$ is a convergent to zero polynomial sequence. Hence $z - y = 0$. Hence $y = z$. By Lemma 2,

$$z_n = (-1)^m \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} \Delta^m z_{i_m}.$$

The equality $\Delta^m z = (-1)^m x$ implies

$$z_n = (-1)^m \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} (-1)^m x_{i_m} = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} x_{i_m}.$$

The proof is complete. □

3. Main results

Theorem 1. *Assume φ is continuous and the series $\sum n^{m-1}a_n$ is absolutely convergent. Then for any bounded solution y of the equation $\Delta^m y = b$ there exists a solution x of* (E) *such that $x = y + o(1)$.*

Proof. Assume y is a bounded solution of the equation $\Delta^m y = b$ and Y is the set of values of the sequence y. Choose a number $\mu > 0$. Let

$$U = \{t \in \mathbb{R}^k\colon\ \text{there exists}\ \ s \in Y^k\ \ \text{such that}\ \ d(s,t) < \lambda_0\mu\}.$$

Since Y^k is a bounded subset of $\mathbb{R}^k$ so U is bounded, too. Hence the closure $\overline{U}$ is compact. Therefore φ is uniformly continuous and bounded on $\overline{U}$. Choose $M > 0$ such that $|\varphi(t)| \leqslant M$ for any $t \in U$. By Lemma 4, there exist numbers

$$\varrho_n = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} |a_{i_m}|$$

and the sequence ϱ is convergent to zero. Hence there exists $q \geqslant n_0$ such that $M\varrho_n < \mu$ for any $n \geqslant q$. Let

$$T = \{x \in BS\colon\ x_n = 0 \ \text{ for } \ n < q \ \text{ and } \ |x_n| \leqslant M\varrho_n \ \text{ for } \ n \geqslant q\},$$
$$S = \{x \in BS\colon\ x_n = y_n \ \text{ for } \ n < q \ \text{ and } \ |x_n - y_n| \leqslant M\varrho_n \ \text{ for } \ n \geqslant q\}.$$

Let us define a mapping $F\colon T \to S$ by $F(x)(n) = x_n + y_n$. Then the formula $\varrho(x,z) = \sup\{|x_n - z_n|\colon\ n \in \mathbb{N}\}$ defines a metric on the set S such that F is an isometry of T onto S. Obviously T is a convex and closed subset of the space BS. Choose an $\varepsilon > 0$. Then there exists $p \in \mathbb{N}$ such that $M\varrho_n < \varepsilon$ for any $n \geqslant p$. For $n = 1, \ldots, p$ let G_n denote a finite ε-net for the interval $[-M\varrho_n, M\varrho_n]$ and let

$$G = \{x \in T\colon\ x_n = 0 \text{ for } n \geqslant p \text{ and } x_n \in G_n \text{ for } n < p\}.$$

Then G is a finite ε-net for T. Hence T is a complete and totally bounded metric space and so T is compact. If $x \in S$, $n \geqslant n_0$, then $(y_{\tau_1(n)}, \ldots, y_{\tau_k(n)}) \in Y^k$ and

$$\begin{aligned} &d((x_{\tau_1(n)}, \ldots, x_{\tau_k(n)}), (y_{\tau_1(n)}, \ldots, y_{\tau_k(n)})) \\ &\qquad \leqslant \lambda_0 \max\{|x_{\sigma_i(n)} - y_{\sigma_i(n)}|\colon\ i = 1, 2, \ldots, k\} < \lambda_0 \mu. \end{aligned}$$

It means that $(x_{\tau_1(n)}, \ldots, x_{\tau_k(n)}) \in U$ for any $x \in S$ and $n \geqslant n_0$. Hence $|\varphi(x_{\tau_1(n)}, \ldots, x_{\tau_k(n)})| \leqslant M$ for every $x \in S$ and $n \geqslant n_0$. Let $x \in S$,

$$x_n^* = \begin{cases} 0 & \text{for } n < n_0 \\ a_n \varphi(x_{\tau_1(n)}, \ldots, x_{\tau_k(n)}) & \text{for } n \geqslant n_0. \end{cases}$$

Let us define a sequence $A(x)$ as follows:

$$A(x)(n) = \begin{cases} y_n & \text{for } n < q \\ y_n + (-1)^m \displaystyle\sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} x_{i_m}^* & \text{for } n \geqslant q. \end{cases}$$

If $n \geqslant q$ then

$$|A(x)(n) - y_n| = \left| \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} x^*_{i_m} \right| \leqslant \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} |x^*_{i_m}|$$
$$\leqslant \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} M|a_{i_m}| = M\varrho_n.$$

It means that $A(x) \in S$. Hence $A(S) \subseteq S$.

Let $\varepsilon > 0$. Since the function φ is uniformly continuous on U there exists such $\delta > 0$ that if $s, t \in U$ and $d(s,t) < \lambda_0\delta$, then $|\varphi(t) - \varphi(s)| < \varepsilon$. Assume $x, z \in S$ and $\|x - z\| < \delta$. If $n \geqslant n_0$ then

$$d((x_{\tau_1(n)}, \ldots, x_{\tau_k(n)}), (z_{\tau_1(n)}, \ldots, z_{\tau_k(n)})) < \lambda_0\delta.$$

Let

$$z^*_n = \begin{cases} 0 & \text{for } n < n_0, \\ a_n\varphi(z_{\tau_1(n)}, \ldots, z_{\tau_k(n)}) & \text{for } n \geqslant n_0. \end{cases}$$

Then

$$\|A(x) - A(z)\| = \sup_{n \geqslant q} \left| \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} x^*_{i_m} - \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} z^*_{i_m} \right|$$
$$= \sup_{n \geqslant q} \left| \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} (x^*_{i_m} - z^*_{i_m}) \right| \leqslant \sup_{n \geqslant q} \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} |x^*_{i_m} - z^*_{i_m}|$$
$$= \sum_{i_1=q}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} |x^*_{i_m} - z^*_{i_m}| \leqslant \sum_{i_1=q}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} \varepsilon|a_{i_m}| = \varepsilon\varrho_q.$$

This means that $A\colon\ S \to S$ is continuous. Let $B\colon\ T \to T$, $B = F^{-1} \circ A \circ F$. Then B is continuous and T is a convex and compact subset of the Banach space BS. By the Schauder fixed point theorem there exists $u \in T$ such that $B(u) = u$. Let $x = F(u) \in S$. Then

$$x = F(u) = F(B(u)) = F(F^{-1}AF(u)) = AF(u) = A(x).$$

Hence

$$x_n = y_n + (-1)^m \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} x^*_{i_m}$$

for $n \geqslant q$. By Lemma 4 we obtain

$$\Delta^m x_n = b_n + x^*_n = b_n + a_n\varphi(x_{\tau_1(n)}, \ldots, x_{\tau_k(n)})$$

for $n \geqslant q$. Hence x is a solution of (E). Moreover, by Lemma 4 the sequence

$$w_n = (-1)^m \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} x^*_{i_m}$$

is convergent to zero. Therefore $x = y + o(1)$. The proof is complete.

R e m a r k. Theorem 1 extends Theorem 1 of [3] and Theorem 2 of [4] and Theorem 1 of [11].

Corollary 1. *If the series* $\sum n^{m-1} a_n$, $\sum n^{m-1} b_n$ *are absolutely convergent and* φ *is continuous, then for any* $c \in \mathbb{R}$ *there exists a solution of* (E) *convergent to* c.

P r o o f. Let $c \in \mathbb{R}$,

$$z_n = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} b_{i_m}, \qquad y_n = c + (-1)^m z_n.$$

By Lemma 4, $\lim y_n = c$ and $\Delta^m z = (-1)^m b$. Hence

$$\Delta^m y = \Delta^m c + \Delta^m((-1)^m z) = b.$$

Hence y is a bounded solution of the equation $\Delta^m y = b$. By Theorem 1 it follows that there exists a solution x of (E) such that $x = y + o(1)$. Obviously $\lim x_n = c$. $\square$

Theorem 2. *If the function* φ *is uniformly continuous and bounded, and the series* $\sum n^{m-1} a_n$ *is absolutely convergent, then for every solution* y *of the equation* $\Delta^m y = b$ *there exists a solution* x *of* (E) *such that* $x = y + o(1)$.

P r o o f. Assume y is a solution of the equation $\Delta^m y = b$. Choose $M > 0$ such that $|\varphi(t)| \leqslant M$ for any $t \in \mathbb{R}^k$. Let $\varrho\colon \mathbb{N} \to \mathbb{R}$,

$$\varrho_n = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} |a_{i_m}|, \qquad T = \{x \in BS\colon |x| \leqslant M\varrho\},$$

$$S = \{x \in SQ\colon |x - y| \leqslant M\varrho\}, \qquad F\colon T \to S, \qquad F(x) = y + x.$$

For $x \in S$ let

$$x^*_n = \begin{cases} 0 & \text{for } n < n_0, \\ a_n \varphi(x_{\tau_1(n)}, \dots, x_{\tau_k(n)}) & \text{for } n \geqslant n_0, \end{cases}$$

$$A(x)(n) = y_n + (-1)^m \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} x^*_{i_m}.$$

The rest of the proof is analogous to the second part of the proof of Theorem 1 and we omit it.

R e m a r k. Theorem 2 extends Theorem 2 of [3].

Corollary 2. *If the series $\sum n^{m-1}a_n$, $\sum n^{m-1}b_n$ are absolutely convergent, and φ is uniformly continuous and bounded, then for every polynomial $\beta(n)$ with $\deg(\beta) < m$ there exists a solution x of* (E) *such that $x_n = \beta(n) + o(1)$.*

P r o o f. Let $\beta(n)$ be a polynomial (with real coefficients) such that $\deg(\beta) < m$ and let

$$z_n = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} b_{i_m}, \qquad y_n = \beta(n) + (-1)^m z_n$$

for $n \in \mathbb{N}$. Since $\Delta^m \beta = 0$, we obtain by Lemma 4

$$\Delta^m y = \Delta^m \beta + (-1)^m \Delta^m z = b.$$

Hence y is a solution of the equation $\Delta^m y = b$. By Theorem 2 there exists a solution x of (E) such that $x = y + o(1)$. Since, by Lemma 4 we have $z = o(1)$, we obtain

$$x = \beta + o(1) + o(1) = \beta + o(1).$$

$\square$

R e m a r k. Corollary 2 extends Theorem 2 of [11].

Theorem 3. *Assume the series $\sum n^{m-1}a_n$ is absolutely convergent and the function $\varphi|[c,\infty)^k$ is uniformly continuous and bounded for some $c \in \mathbb{R}$. Then for every solution y of equation $\Delta^m y = b$ which diverges to infinity there exists a solution x of* (E) *such that $x = y + o(1)$.*

P r o o f. Assume $\Delta^m y = b$ and $\lim y_n = \infty$. Choose $M > 0$ such that $|\varphi(t)| < M$ for any $t \in [c,\infty)^k$. For $n \in \mathbb{N}$ let

$$\varrho_n = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} |a_{i_m}|.$$

Choose $q \geqslant n_0$ such that $y_n \geqslant c + M\varrho_1$ for any $n \geqslant q$. Let

$$S = \{x \in SQ\colon\ x_n = y_n \text{ for } n < q \text{ and } |x_n - y_n| \leqslant M\varrho_n \text{ for } n \geqslant q\}.$$

If $x \in S$ and $n \geqslant q$, then $x_n \geqslant y_n - M\varrho_n \geqslant c + M\varrho_1 - M\varrho_n \geqslant c$. Hence $(x_{\tau_1(n)}, \dots, x_{\tau_k(n)}) \in [c,\infty)^k$ for every $x \in S$ and every $n \geqslant q$. The rest of the proof is analogous to the proof of Theorem 1. $\square$

R e m a r k. Theorem 3 extends Theorem 3 of [3].

Corollary 3. *Assume the series $\sum n^{m-1}a_n$, $\sum n^{m-1}b_n$ are absolutely convergent and the function $\varphi|[c,\infty)^k$ is uniformly continuous and bounded for some $c \in \mathbb{R}$. Then for every polynomial $\beta(n)$ such that $\deg\beta < m$ and $\lim\beta(n) = \infty$ there exists a solution x of* (E) *such that $x_n = \beta(n) + o(1)$.*

Proof. The proof is analogous to the proof of Corollary 2 and is omitted. □

Theorem 4. *Assume the series $\sum n^{m-1}a_n$ is absolutely convergent and the function $\varphi|(-\infty,c]^k$ is uniformly continuous and bounded for some $c \in \mathbb{R}$. Then for any solution y of the equation $\Delta^m y = b$ which diverges to $-\infty$ there exists a solution x of* (E) *such that $x = y + o(1)$.*

Proof. The proof is analogous to the proof of Theorem 3. □

Corollary 4. *Assume the series $\sum n^{m-1}a_n$, $\sum n^{m-1}b_n$ are absolutely convergent and the function $\varphi|(-\infty,c]^k$ is uniformly continuous and bounded for some $c \in \mathbb{R}$. Then for every polynomial $\beta(n)$ such that $\deg\beta < m$ and $\lim\beta(n) = -\infty$ there exists a solution x of* (E) *such that $x_n = \beta(n) + o(1)$.*

Proof. The proof is analogous to the proof of Corollary 2. □

Example. Let $m \geqslant 2$, $k = 1$, $\tau_1(n) = n$, $a_n = n^{-m-1}$, $b_n = 0$, $\varphi\colon \mathbb{R} \to \mathbb{R}$,

$$\varphi(t) = \begin{cases} t & \text{for } t \geqslant 1, \\ \dfrac{2}{|t|+1} & \text{for } t \leqslant 1. \end{cases}$$

Then the equation (E) takes on the form

$$\Delta^m x_n = \frac{1}{n^{m+1}}\varphi(x_n). \tag{E1}$$

The function φ is uniformly continuous and, by Corollary 4, for every polynomial $\beta(n)$ such that $\deg\beta < m$ and $\lim\beta(n) = -\infty$ there exists a solution x of (E1) such that $x_n = \beta(n) + o(1)$. We will show that in that case the equation (E1) has no solutions of the form $x_n = \beta(n) + o(1)$ where $\beta(n)$ is a polynomial such that $\deg\beta = m-1$ and $\lim\beta(n) = \infty$. Assume $c > 0$, $\alpha(n)$ is a polynomial, $\deg\alpha < m-1$, $x_n = cn^{m-1} + \alpha(n) + o(1)$ and x is a solution of (E1). Then $\Delta^m x_n = a_n\varphi(x_n) = a_n x_n = cn^{-2} + o(n^{-2})$ for large n.

Since $\Delta^m(cn^{m-1} + \alpha(n)) = 0$, there exists a sequence z such that $z_n = o(1)$ and $\Delta^m z_n = \Delta^m x_n = cn^{-2} + o(n^{-2})$. Then

$$\frac{\Delta^m z_n}{\Delta 1/n} = \frac{cn^{-2} + o(n^{-2})}{-1/(n^2+n)} = \frac{c + n^2 o(n^{-2})}{-n^2/(n^2+n)} \longrightarrow -c.$$

By discrete l'Hospital theorem we obtain $\lim_{n\to\infty} n\Delta^{m-1} z_n = -c$. Hence

$$\frac{\Delta^{m-1} z_n}{\Delta \ln n} = \frac{\Delta^{m-1} z_n}{\ln(n+1) - \ln n} = \frac{n\Delta^{m-1} z_n}{n \ln(1+1/n)} = \frac{n\Delta^{m-1} z_n}{\ln(1+1/n)^n} \longrightarrow -c.$$

Hence $\lim_{n\to\infty} \Delta^{m-2} z_n / \ln n = -c \neq 0$. On the other hand,

$$\lim_{n\to\infty} \frac{\Delta^{m-2} z_n}{\ln n} = \lim_{n\to\infty} \frac{\Delta^{m-2} o(1)}{\ln n} = \lim_{n\to\infty} \frac{o(1)}{\ln n} = 0.$$

This contradiction shows that x is not a solution of the equation (E1).

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