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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS OF nth-ORDER m-POINT BOUNDARY VALUE PROBLEMS

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Abstract. The paper deals with the existence of multiple positive solutions for the boundary value problem

$$\begin{cases} (\varphi(p(t)u^{(n-1)})(t))' + a(t)f(t,u(t),u'(t),\dots,u^{(n-2)}(t)) = 0, \quad 0<t<1, \\ u^{(i)}(0)=0, \quad i=0,1,\dots,n-3, \\ u^{(n-2)}(0) = \sum\limits_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i), \quad u^{(n-1)}(1)=0, \end{cases}$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and a positive homomorphism with $\varphi(0)=0$. Using a fixed-point theorem for operators on a cone, we provide sufficient conditions for the existence of multiple positive solutions to the above boundary value problem.

Keywords: boundary-value problems, positive solutions, fixed-point theorem, cone

MSC 2010: 34B18

1. Introduction

In this paper we introduce a new operator which improves and generates a p-Laplace operator for some $p>1$, and we study the existence of multiple positive solutions for the nth-order m-point nonlinear boundary value problem of the form

$$(\varphi(p(t)u^{(n-1)})(t))' + a(t)f(t,u(t),u'(t),\dots,u^{(n-2)}(t)) = 0, \quad 0<t<1, \tag{1.1}$$

with the boundary value condition

$$\begin{cases} u^{(i)}(0)=0, \quad i=0,1,\dots,n-3, \\ u^{(n-2)}(0) = \sum\limits_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i), \quad u^{(n-1)}(1)=0, \end{cases}$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with a positive homomorphism with $\varphi(0) = 0$. Here $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$ and α_i satisfy $\alpha_i \in [0,+\infty)$, $0 < \sum\limits_{i=1}^{m-2} \alpha_i < 1$, $p \in C([0,1],(0,+\infty))$, $f \in C([0,1] \times [0,+\infty)^{n-1},\ [0,+\infty))$.

A projection $\varphi\colon \mathbb{R} \to \mathbb{R}$ is called an increasing homeomorphism and a positive homomorphism if the following conditions are satisfied:

(1) if $x \leqslant y$, then $\varphi(x) \leqslant \varphi(y)$ for all $x, y \in \mathbb{R}$;

(2) φ is a continuous bijection and its inverse is also continuous;

(3) $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathbb{R}_+$.

In the above definition, condition (3) can be replaced by the following stronger condition:

(4) $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathbb{R}$, where $\mathbb{R} = (-\infty,+\infty)$.

Remark 1.1. If conditions (1), (2) and (4) hold, then φ is homogeneous and generates a p-Laplace operator, i.e., $\varphi(x) = |x|^{p-2}x$ for some $p > 1$.

Remark 1.2. It is well known that a p-Laplacian operator is odd. However, the operator which we defined above is not necessarily odd, see Example 5.1.

The multi-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [6]. Since then, nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [1], [3], [4], [5], [16] and references therein. Recently, the existence and multiplicity of positive solutions for the p-Laplacian operator, i.e., $p(t) \equiv 1$ and $\varphi(x) = |x|^{p-2}x$ for some $p > 1$, have received wide attention, see [2], [13], [14], [15], [17] and references therein. We know that the oddness of a p-Laplacian operator is key to the proof. However, in this paper we define a new operator which improves and generalizes a p-Laplacian operator for some $p > 1$ and φ is not necessarily odd. Moreover for increasing homeomorphism and positive homomorphism operator research has proceeded very slowly, see [10], [11]. Especially the existence of multiple positive solutions for nth-order m-point boundary value problems still remains unknown.

In [11], Liu and Zhang studied the existence of positive solutions of the quasi-linear differential equation

$$\begin{cases} (\varphi(x'))' + a(t)f(x(t)) = 0, & t \in (0,1),\\ x(0) - \beta x'(0) = 0, & x(1) + \delta x'(1) = 0, \end{cases}$$

subject to linear mixed boundary conditions by a simple application of a fixed-point index theorem in cones, where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and a positive homomorphism with $\varphi(0) = 0$.

Wang and Hou [15] studied the boundary value problem

$$\begin{cases} (\varphi_p(u'))'(t) + a(t)f(t,u) = 0, \quad 0 < t < 1, \\ \varphi_p(u'(0)) = \sum\limits_{i=1}^{n-2} a_i \varphi_p(u'(\xi_i)), \quad u(1) = \sum\limits_{i=1}^{n-2} b_i u(\xi_i), \end{cases}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$; the authors proved the existence of multiple positive solutions to the above boundary value problem by using a fixed-point theorem for operators on a cone.

In [17], Zhou and Su studied the quasi-linear equation with a p-Laplacian operator

$$\begin{cases} (\varphi_p(u^{(n-1)}))' + g(t)f(u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0, \quad 0 < t < 1, \\ u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-3, \\ u^{(n-1)}(0) - B_0(u^{(n-1)}(\xi)) = 0, \quad n \geqslant 3, \\ u^{(n-1)}(0) - B_1(u^{(n-1)}(\eta)) = 0, \quad n \geqslant 3, \end{cases}$$

where $\varphi_p(s)$ is a p-Laplacian operator. They used the fixed-point index theory to find conditions for the existence of one solution, and of multiple solutions.

In a recent paper [8], using lower and upper solutions methods, Kong and Kong established results for solutions and positive solutions of the following two non-homogeneous boundary value conditions problems:

$$\begin{cases} x''(t)) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x'(0) - \sum\limits_{i=1}^{m} \alpha_i x'(\xi_i) = \lambda_1, \quad x(1) - \sum\limits_{i=1}^{m} \beta_i x(\xi_i) = \lambda_2, \end{cases}$$

and

$$\begin{cases} x''(t)) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(0) - \sum\limits_{i=1}^{m} \alpha_i x(\xi_i) = \lambda_1, \quad x(1) - \sum\limits_{i=1}^{m} \beta_i x(\xi_i) = \lambda_2, \end{cases}$$

respectively.

In [12], using Mawhin's coincidence degree theory, the author studied the more generalized BVPs for higher order differential equations with p-Laplacian subjected to non-homogeneous BCs, in which the nonlinearity f contains $t, x, \ldots, x^{(n-1)}$.

In [7], the authors proved the existence of positive and/or negative solutions of a class of four-point boundary-value problems for fourth order ordinary differential equations by using the continuum property (connectedness and compactness) of the solutions funnel (Knesser's Theorem), combined with the corresponding vector field.

R e m a r k 1.3. On the one hand, we emphasize that the results of the papers [12], [15], [17] are not replaced by φ which we defined above; on the other hand, the assumptions and approach in the paper are different from the paper [7], [8], [12] and the function φ which we defined above is more comprehensive and general than the p-Laplace operator.

But whether or not we can obtain multiple positive solutions of the nth-order m-point boundary value problem (1.1) and (1.2) still remains unknown. So the goal of the present paper is to improve and generalizes p-Laplacian operator and establish some criteria for the existence of multiple positive solutions by means of the classical fixed-point theorem for compact maps.

We shall assume that the following conditions are satisfied.

(C_1) $f \in C([0,1] \times [0,+\infty)^{n-1}, [0,+\infty))$, and α_i satisfy $0 < \sum\limits_{i=1}^{m-2} \alpha_i < 1$;

(C_2) $p \in C([0,1], (0,+\infty))$ is a nondecreasing function;

(C_3) $a(t)$ is a nonnegative measurable function defined in $(0,1)$ and $a(t)$ does not identically vanish on any subinterval of $(0,1)$ and

$$0 < \int_0^1 a(t)\,\mathrm{d}t < +\infty.$$

2. Some definitions and fixed point theorems

In this section we provide background definitions from the cone theory in Banach spaces.

Definition 2.1. Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following conditions are satisfied:

(a) if $y \in P$ and $\lambda \geqslant 0$, then $\lambda y \in P$;

(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leqslant$, that is, $x \leqslant y$ if and only if $y - x \in P$.

Definition 2.2. A map α is said to be a nonnegative, continuous, concave functional on a cone P of a real Banach space E, if

$$\alpha\colon\ P \to [0,\infty)$$

is continuous, and

$$\alpha(tx + (1-t)y) \geqslant t\alpha(x) + (1-t)\alpha(y)$$

for all x, $y \in P$ and $t \in [0,1]$.

Definition 2.3. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

To obtain positive solutions of (1.1) and (1.2) the following fixed-point theorem in cones is fundamental.

Theorem 2.1 [9]. *Let K be a cone in a Banach space X. Let D be an open bounded set with $D_k = D \cap K \neq \emptyset$ and $\overline{D}_k \neq K$. Let $T\colon \overline{D}_k \to K$ be a compact map such that $x \neq Tx$ for $x \in \partial D_k$. Then the following results hold:*

(1) *If $\|Tx\| \leqslant \|x\|$ for $x \in \partial D_k$, then $i_k(T, D_k) = 1$.*
(2) *Suppose there is $e \in K$, $e \neq 0$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_k$ and all $\lambda > 0$, then $i_k(T, D_k) = 0$.*
(3) *Let D^1 be open in X such that $\overline{D^1} \subset D_k$. If $i_k(T, D_k) = 1$ and $i_k(T, D^1_k) = 0$, then T has a fixed point in $D_k \setminus \overline{D^1_k}$. The same result holds if $i_k(T, D_k) = 0$ and $i_k(T, D^1_k) = 1$.*

3. The preliminary lemmas

The basic space used in this paper is $E = \{u \in C^{2n-2}[0,1]\colon u^{(i)}(0) = 0,\ i = 0, 1, \ldots, n-3\}$. Thus E is a Banach space when endowed with the norm $\|u\| = \sup\limits_{t\in[0,1]} |u^{(n-2)}(t)|$. By a solution of (1.1), (1.2), we mean a function $u \in C^n[0,1]$ which satisfies (1.1), (1.2).

We can easily get the following lemmas which are useful in the proof of our main results.

Lemma 3.1. *Assume that* (C_1)–(C_3) *hold. Then boundary value problem* (1.1) *and* (1.2) *has a solution $u \in C^n[0,1]$ if and only if $u(t)$ solves the equation*

$$u(t) = \int_0^t \int_0^{\zeta_1} \ldots \int_0^{\zeta_{n-3}} w(\zeta_{n-2})\,\mathrm{d}\zeta_{n-2}\,\mathrm{d}\zeta_{n-3}\ldots\,\mathrm{d}\zeta_1, \tag{3.1}$$

and

$$\inf_{t\in[0,1]} u^{(n-2)}(t) \geqslant \gamma\|u\| \tag{3.2}$$

where

$$w(t) = \int_0^t \frac{1}{p(s)}\varphi^{-1}\Big(\int_s^1 a(\tau)f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \frac{\sum\limits_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} p(s)^{-1}\varphi^{-1}\big(\int_s^1 a(\tau)f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\,\mathrm{d}\tau\big)\,\mathrm{d}s}{1 - \sum\limits_{i=1}^{m-2}\alpha_i},$$

$$\gamma = \frac{\sum\limits_{i=1}^{m-2}\alpha_i\xi_i}{1 - \sum\limits_{i=1}^{m-2}\alpha_i + \sum\limits_{i=1}^{m-2}\alpha_i\xi_i}.$$

P r o o f. Necessity. First suppose that $u \in C^n[0,1]$ is a solution of problem (1.1) and (1.2). From the equation of the boundary condition we have

$$-\varphi(p(1)u^{(n-1)}(1)) + \varphi(p(t)u^{(n-1)}(t)) = \int_t^1 a(s)f(s,u(s),u'(s),\ldots,u^{(n-2)}(s))\,\mathrm{d}s.$$

From the boundary value condition $u^{(n-1)}(1) = 0$ we have

$$u^{(n-1)}(t) = \frac{1}{p(t)}\varphi^{-1}\Big(\int_t^1 a(s)f(s,u(s),u'(s),\ldots,u^{(n-2)}(s))\,\mathrm{d}s\Big). \tag{3.3}$$

Integrating (3.3) from 0 to t, we have

$$u^{(n-2)}(t) - u^{(n-2)}(0) = \int_0^t \frac{1}{p(s)}\varphi^{-1}\Big(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s. \tag{3.4}$$

Substituting $u^{(n-2)}(0) = \sum\limits_{i=1}^{m-2}\alpha_i u^{(n-2)}(\xi_i)$ into (3.4), we obtain

$$\begin{aligned}u^{(n-2)}(t) = &\int_0^t \frac{1}{p(s)}\varphi^{-1}\Big(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\ &+ \sum_{i=1}^{m-2}\alpha_i u^{(n-2)}(\xi_i).\end{aligned}$$

It is easy to see by simple calculation that

$$\begin{aligned}u^{(n-2)}(t) = &\int_0^t \frac{1}{p(s)}\varphi^{-1}\Big(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\ &+ \frac{\sum\limits_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} p(s)^{-1}\varphi^{-1}\big(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,\mathrm{d}\tau\big)\,\mathrm{d}s}{1-\sum\limits_{i=1}^{m-2}\alpha_i}.\end{aligned} \tag{3.5}$$

Then by integrating Eq. (3.5) $n-2$ times on $[0,1]$ we find that for any $t \in [0,1]$, $u(t)$ can be expressed by equation (3.1).

Sufficiency. If $u(t)$ is a solution of (3.1) direct differentiation of (3.1) we have

$$\begin{aligned}u^{(n-2)}(t) = &\int_0^t \frac{1}{p(s)}\varphi^{-1}\Big(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\ &+ \frac{\sum\limits_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} p(s)^{-1}\varphi^{-1}\big(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,\mathrm{d}\tau\big)\,\mathrm{d}s}{1-\sum\limits_{i=1}^{m-2}\alpha_i}.\end{aligned}$$

Obviously, for $t \in [0,1]$ we have

$$(\varphi(p(t)u^{(n-2)}(t)))' + a(t)f(t, u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0,$$
$$u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n-3,$$
$$u^{(n-2)}(0) = \sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i), \qquad u^{(n-1)}(1) = 0.$$

Finally, we show that (3.2) holds. It is clear that $u^{(n-1)}(t) \geqslant 0$, which implies that

$$\|u\| = u^{(n-2)}(1), \quad \min_{t\in[0,1]} u^{(n-2)}(t) = u^{(n-2)}(0).$$

On the other hand, for given $s_1, s_2 \in [0,1]$ with $s_1 \leqslant s_2$, one can prove that $u^{(n-1)}(s_2) \leqslant u^{(n-1)}(s_1)$. Hence, $u^{(n-1)}(t)$ is nonincreasing on $[0,1]$.

So, for every $i \in \{1, 2, \ldots, m-2\}$ we have

$$\frac{u^{(n-2)}(1) - u^{(n-2)}(0)}{1} \leqslant \frac{u^{(n-2)}(\xi_i) - u^{(n-2)}(0)}{\xi_i},$$

i.e., $u^{(n-2)}(\xi_i) - u^{(n-2)}(0) \geqslant \xi_i u^{(n-2)}(1) - \xi_i u^{(n-2)}(0)$.

Therefore,

$$\sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i) - \sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(0) \geqslant \sum_{i=1}^{m-2} \alpha_i \xi_i u^{(n-2)}(1) - \sum_{i=1}^{m-2} \alpha_i \xi_i u^{(n-2)}(0).$$

This together with the boundary value $u^{(n-2)}(0) = \sum\limits_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i)$ implies that

$$u^{(n-2)}(0) \geqslant \frac{\sum\limits_{i=1}^{m-2} \alpha_i \xi_i}{1 - \sum\limits_{i=1}^{m-2} \alpha_i + \sum\limits_{i=1}^{m-2} \alpha_i \xi_i} u^{(n-2)}(1).$$

The proof is complete. □

Lemma 3.2. *Suppose that conditions* (C_1)–(C_3) *hold. Then the solution $u(t)$ of problem* (1.1), (1.2) *satisfies*

$$u(t) \leqslant u'(t) \leqslant \ldots \leqslant u^{(n-3)}(t) \leqslant u^{(n-2)}(t), \quad 0 \leqslant t \leqslant 1.$$

P r o o f. If $u(t)$ is the solution of problem (1.1), (1.2), then $u^{(n-2)}(t)$ is a concave function and $u^{(i)}(t) \geqslant 0$, $i = 0, 1, 2, \ldots, n-2$, $t \in [0,1]$. Thus we have

$$u^{(i)}(t) = \int_0^t u^{(i+1)}(s)\,\mathrm{d}s \leqslant t u^{(i+1)}(t) \leqslant u^{(i+1)}(t), \quad i = 0, 1, \ldots, n-3,$$

i.e., $u(t) \leqslant u'(t) \leqslant \ldots \leqslant u^{(n-2)}(t)$, $t \in [0,1]$. □

To establish the existence of multiple positive solutions of problem (1.1) and (1.2), we construct a cone K in E by

$$K = \Big\{u \in E\colon\ u^{(n-2)}(t) \geqslant 0,\ \min_{0\leqslant t\leqslant 1} u^{(n-2)}(t) \geqslant \gamma\|u\|\Big\},$$

where γ is defined by Lemma 3.1. Obviously, K is a cone of E.

Remark 3.1. We note that $u^i(t) \leqslant \|u\|$, $t \in [0,1]$ for all $u \in K$, $i = 1,\ldots,n-3$ by virtue of the definition of E.

Define $T\colon\ K \to E$ by

$$(Tu)(t) = \int_0^t \int_0^{\zeta_1} \ldots \int_0^{\zeta_{n-3}} w(\zeta_{n-2})\,\mathrm{d}\zeta_{n-2}\,\mathrm{d}\zeta_{n-3}\ldots\,\mathrm{d}\zeta_1, \tag{3.6}$$

where

$$\begin{aligned} w(t) = &\int_0^t \frac{1}{p(s)}\varphi^{-1}\Big(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ &+ \frac{\sum\limits_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} p(s)^{-1}\varphi^{-1}\big(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,\mathrm{d}\tau\big)\,\mathrm{d}s}{1-\sum\limits_{i=1}^{m-2}\alpha_i}. \end{aligned}$$

From (3.6) and Lemma 3.1, it is easy to obtain the following result.

Lemma 3.3. *Let conditions* (C_1)–(C_3) *hold. Then* $T\colon\ K \to K$ *is completely continuous.*

We define

$$\begin{aligned} K_\varrho &= \{u \in K\colon\ \|u\| \leqslant \varrho\}, \\ \Omega_\varrho &= \Big\{u \in K\colon\ \min_{t\in[0,1]} u^{(n-2)}(t) < \gamma\varrho\Big\} \\ &= \Big\{u \in E\colon\ \gamma\|u\| \leqslant \min_{t\in[0,1]} u^{(n-2)}(t) < \gamma\varrho\Big\}. \end{aligned}$$

Lemma 3.4 [9]**.** *Ω_ϱ has the following properties:*
(a) *Ω_ϱ is open relative to K,*
(b) $K_{\gamma\varrho} \subset \Omega_\varrho \subset K_\varrho$,
(c) *$u \in \partial\Omega_\varrho$ if and only if* $\min\limits_{t\in[0,1]} u^{(n-2)}(t) = \gamma\varrho$,
(d) *$u \in \partial\Omega_\varrho$, then $\gamma\varrho \leqslant u(t) \leqslant \varrho$ for $t \in [0,1]$.*

Now, we introduce the following notation. Let $i = 1, 2, \ldots, n-3$, and

$$f_{\gamma\varrho}^{\varrho} = \min\Big\{\min_{t\in[0,1]} \frac{f(t, u(t), u'(t), \ldots, u^{(n-2)}(t))}{\varphi(\varrho)}\colon\ u^{(i)}(t) \in [0, \varrho],\ u^{(n-2)} \in [\gamma\varrho, \varrho]\Big\},$$

$$f_{0}^{\varrho} = \max\Big\{\max_{t\in[0,1]} \frac{ft, (u(t), u'(t), \ldots, u^{(n-2)}(t))}{\varphi(\varrho)}\colon\ u^{(i)}(t) \in [0, \varrho],\ u^{(n-2)} \in [0, \varrho]\Big\},$$

$$f^{\alpha} = \lim_{u^{(n-2)}\to\alpha} \sup \max_{t\in[0,1]} \frac{f(t, u(t), u'(t), \ldots, u^{(n-2)}(t))}{\varphi(u^{(n-2)})},$$

$$f_{\alpha} = \lim_{u^{(n-2)}\to\alpha} \min \min_{t\in[0,1]} \frac{f(t, u(t), u'(t), \ldots, u^{(n-2)}(t))}{\varphi(u^{(n-2)})} \quad (\alpha := \infty \text{ or } 0^{+}),$$

$$\frac{1}{l} = \frac{1}{p(0)}\varphi^{-1}\Big(\int_0^1 a(\tau)\,\mathrm{d}\tau\Big)\Big(1 + \sum_{i=1}^{m-2}\alpha_i\xi_{m-2}\Big/\Big(1 - \sum_{i=1}^{m-2}\alpha_i\Big)\Big),$$

$$\frac{1}{L} = \frac{1}{p(1)}\int_0^1 \varphi^{-1}\Big(\int_s^1 a(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s.$$

4. Main results

The main results of this paper are the following.

Theorem 4.1. *Assume that one of the following conditions holds:*

(C_4) *There exist* $\varrho_1, \varrho_2, \varrho_3 \in (0, \infty)$ *with* $\varrho_1 < \gamma\varrho_2$ *and* $\varrho_2 < \varrho_3$ *such that*

$$f_0^{\varrho_1} < \varphi(l), \quad f_{\gamma\varrho_2}^{\varrho_2} > \varphi(L), \quad f_0^{\varrho_3} < \varphi(l).$$

(C_5) *There exist* $\varrho_1, \varrho_2, \varrho_3 \in (0, \infty)$ *with* $\varrho_1 < \varrho_2 < \gamma\varrho_3$ *such that*

$$f_{\gamma\varrho_1}^{\varrho_1} > \varphi(L), \quad f_0^{\varrho_2} < \varphi(l), \quad f_{\gamma\varrho_3}^{\varrho_3} > \varphi(L).$$

Then (1.1) *and* (1.2) *has two positive solutions* u_1, u_2 *with* $u_1 \in \Omega_{\varrho_2} \setminus \overline{K}_{\varrho_1}$, $u_2 \in K_{\varrho_3} \setminus \overline{\Omega}_{\varrho_2}$.

Proof. We only consider the condition (C_4). If (C_5) holds, then the proof is similar to the case when (C_4) holds. Let T be a completely continuous operator that was defined by (3.6).

First, we show that $i_k(T, K_{\varrho_1}) = 1$. In fact, $f_0^{\varrho_1} < \varphi(l)$ by (3.6) and Remark 3.1 and we have for $u \in K_{\varrho_1}$,

$$
\begin{aligned}
\|Tu\| = |(Tu)^{(n-2)}(1)| = {} & \int_0^1 \frac{1}{p(s)} \varphi^{-1}\left(\int_s^1 a(\tau) f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\,\mathrm{d}\tau \right) \mathrm{d}s \\
& + \frac{\sum\limits_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} p(s)^{-1} \varphi^{-1}(\int_s^1 a(\tau) f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s}{1 - \sum\limits_{i=1}^{m-2} \alpha_i} \\
\leqslant {} & \int_0^1 p(0)^{-1} \varphi^{-1}\left(\int_0^1 a(\tau) f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\,\mathrm{d}\tau \right) \mathrm{d}s \\
& + \frac{\sum\limits_{i=1}^{m-2} \alpha_i \int_0^{\xi_{m-2}} p(0)^{-1} \varphi^{-1}(\int_0^1 a(\tau) f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s}{1 - \sum\limits_{i=1}^{m-2} \alpha_i} \\
< {} & \varphi^{-1}(\varphi(l)\varphi(\varrho_1)) \frac{1}{p(0)} \varphi^{-1}\left(\int_0^1 a(\tau)\,\mathrm{d}\tau \right) \left(1 + \sum_{i=1}^{m-2} \alpha_i \xi_{m-2} \Big/ \left(1 - \sum_{i=1}^{m-2} \alpha_i \right) \right) \\
= {} & \frac{l\varrho_1}{p(0)} \varphi^{-1}\left(\int_0^1 a(\tau)\,\mathrm{d}\tau \right) \left(1 + \sum_{i=1}^{m-2} \alpha_i \xi_{m-2} \Big/ \left(1 - \sum_{i=1}^{m-2} \alpha_i \right) \right) = \varrho_1 = \|u\|.
\end{aligned}
$$

This implies that $\|Tu\| < \|u\|$ for $u(t) \in \partial K_{\varrho_1}$. By (1) of Theorem 2.1, we have $i_k(T, K_{\varrho_1}) = 1$.

Secondly, we show that $i_k(T, \Omega_{\varrho_2}) = 0$.

Let $e^{(n-2)}(t) \equiv 1$ for $t \in [0,1]$. Then $e \in \partial K_1$, and we claim that

$$
u \neq Tu + \lambda e, \quad u \in \partial\Omega_{\varrho_2}, \quad \lambda > 0.
$$

In fact, if not, there exist $u_0 \in \partial\Omega_{\varrho_2}$ and $\lambda_0 > 0$ such that $u_0 = Tu_0 + \lambda_0 e$. By Lemma 3.3, Remark 3.1 and (3.6) we have

$$
\begin{aligned}
\gamma\varrho_2 = \min_{t\in[0,1]} u_0^{(n-2)}(t) &= (Tu_0)^{(n-2)}(t) + \lambda_0 e^{(n-2)}(t) \geqslant \gamma\|Tu_0\| + \lambda_0 \\
&= \gamma \int_0^1 \frac{1}{p(s)} \varphi^{-1}\left(\int_s^1 a(\tau) f(\tau, u_0(\tau), u_0'(\tau), \ldots, u_0^{(n-2)}(\tau))\,\mathrm{d}\tau \right) \mathrm{d}s + \lambda_0 \\
&> \frac{\gamma}{p(1)} \varphi^{-1}(\varphi(L)\varphi(\varrho_2)) \int_0^1 \varphi^{-1}\left(\int_s^1 a(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s + \lambda_0 \\
&= \frac{\gamma}{p(1)} L\varrho_2 \int_0^1 \varphi^{-1}\left(\int_s^1 a(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s + \lambda_0 \\
&= \gamma\varrho_2 + \lambda_0.
\end{aligned}
$$

This implies that $\gamma\varrho_2 > \gamma\varrho_2 + \lambda_0$, which is a contradiction. Hence by (2) of Theorem 2.1, we have $i_k(T, \Omega_{\varrho_2}) = 0$.

Finally, similarly to the proof of $i_k(T, K_{\varrho_1}) = 1$, we can prove that $i_k(T, K_{\varrho_3}) = 1$. Since $\varrho_1 < \gamma\varrho_2$, we have $\overline{K}_{\varrho_1} \subset K_{\gamma\varrho_2} \subset \Omega_{\varrho_2}$. Therefore, Theorem 2.1 implies that problem (1.1), (1.2) has at least two positive solutions u_1, u_2 with $u_1 \in \Omega_{\varrho_2} \setminus \overline{K}_{\varrho_1}$, $u_2 \in K_{\varrho_3} \setminus \overline{\Omega}_{\varrho_2}$.

If (C_5) holds, the proof is similar to the above. □

R e m a r k 4.1. From the proof of Theorem 4.1 we know that $i_k(T, K_{\varrho_1}) = 1 \neq 0$, and we can obtain that problem (1.1) and (1.2) has a third nonnegative solution u_3 with $u_3 \in K_{\varrho_1}$.

As a special case of Theorem 4.1 we obtain the following result.

Corollary 4.1. *If there exist $\varrho, \varrho' \in (0, \infty)$ with $\varrho' < \gamma\varrho$ such that one of the following conditions holds:*

(C_6) $0 < f_0^{\varrho'} < \varphi(l)$, $f_{\gamma\varrho}^{\varrho} > \varphi(L)$, $0 \leqslant f^{\infty} < \varphi(l)$.

(C_7) $\varphi(L) < f_{\gamma\varrho'}^{\varrho'} \leqslant \infty$, $f_0^{\varrho} < \varphi(l)$, $\varphi(L) < f_{\infty} \leqslant \infty$.

Then (1.1) *and* (1.2) *has two positive solutions in K.*

P r o o f. We show that (C_6) implies (C_4). Let $a \in (f^{\infty}, \varphi(l))$. Then there exists $r > a$ such that $\max\limits_{t\in[0,1]} f(t, u, u', \ldots, u^{(n-2)}) \leqslant a\varphi(u^{(n-2)})$ for $u^{(n-2)} \in [r, \infty)$ since $0 \leqslant f^{\infty} < \varphi(l)$. Let

$$\beta = \max\Big\{ \max_{t\in[0,1]} f(t, u, u', \ldots, u^{(n-2)})\colon\ 0 \leqslant u, u', \ldots, u^{(n-2)} \leqslant r\Big\}$$

and

$$\varrho_3 > \max\Big\{\varphi^{-1}\Big(\frac{\beta}{\varphi(l) - a}\Big),\ \varrho\Big\}.$$

Then we have

$$\begin{aligned}\sup_{t\in[0,1]} f(t, u, u', \ldots, u^{(n-2)}) &\leqslant a\varphi(u^{(n-2)}) + \beta \\ &\leqslant a\varphi(\varrho_3) + \beta < \varphi(l)\varphi(\varrho_3) \text{ for } u^{(n-2)} \in [0, \varrho_3].\end{aligned}$$

This implies that $f_0^{\varrho_3} < \varphi(l)$ and (C_4) holds. Similarly, (C_7) implies (C_5). □

By an argument similar to that of Theorem 4.1 we obtain the following results.

Theorem 4.2. *Assume that one of the following conditions holds:*

(C_8) *There exist* $\varrho_1, \varrho_2 \in (0,\infty)$ *with* $\varrho_1 < \varrho_2$ *such that* $f_0^{\varrho_1} \leqslant \varphi(l)$ *and* $f_{\gamma\varrho_2}^{\varrho_2} \geqslant \varphi(L)$.

(C_9) *There exist* $\varrho_1, \varrho_2 \in (0,\infty)$ *with* $\varrho_1 < \varrho_2$ *such that* $f_{\gamma\varrho_1}^{\varrho_1} \geqslant \varphi(L)$ *and* $f_0^{\varrho_2} \leqslant \varphi(l)$.

Then (1.1), (1.2) *has a positive solution in* K.

As a special case of Theorem 4.2 we obtain the following result.

Corollary 4.2. *Assume that one of the following conditions holds:*

(H_9) $0 \leqslant f^0 < \varphi(l)$ *and* $\varphi(L) < f_\infty \leqslant \infty$.

(H_{10}) $0 \leqslant f^\infty < \varphi(l)$ *and* $\varphi(L) < f_0 \leqslant \infty$.

Then (1.1) *and* (1.2) *has a positive solution in* K.

Remark 4.2. This result strictly includes the sublinear and superlinear cases.

Theorem 4.1 can be generalized to obtain many solutions.

Theorem 4.3. *Suppose that* (C_1)–(C_3) *hold. Then we have the following assertions.*

(1) *There exist* $\{\varrho_i\}_{i=1}^{2k_0} \subset (0,\infty)$ *with* $\varrho_1 < \gamma\varrho_2 < \varrho_2 < \varrho_3 < \gamma\varrho_4 < \ldots < \varrho_{2k_0}$ *such that*

$$f_0^{\varrho_{2k-1}} < \varphi(l), \quad f_{\gamma\varrho_{2k}}^{\varrho_{2k}} > \varphi(L) \quad (k = 1, 2, \ldots, k_0).$$

Then problem (1.1), (1.2) *has at least* $2k_0$ *solutions in* K.

(2) *There exist* $\{\varrho_i\}_{i=1}^{2k_0} \subset (0,\infty)$ *with* $\varrho_1 < \varrho_2$ *and* $\varrho_2 < \gamma\varrho_3 < \varrho_4 < \gamma\varrho_5 < \ldots < \varrho_{2k_0+2}$ *such that*

$$f_{\gamma\varrho_{2k-1}}^{\varrho_{2k-1}} > \varphi(L), \quad f_0^{\varrho_{2k}} < \varphi(l) \quad (k = 1, 2, \ldots, k_0).$$

Then problem (1.1), (1.2) *has at least* $2k_0 - 1$ *solutions in* K.

Remark 4.3. If $\varphi(u) = u$, the problem is a second boundary value problem. If $\varphi(u) = |u|^{p-2}u$, $p > 1$, the problem is a boundary value problem with a p-Laplacian. Then our results of Theorem 4.1 and 4.2 are also new.

5. Example

Example 5.1. As an example we mention the boundary value problem

$$(5.1)\qquad \begin{cases} (\varphi(p(t)u^{(n-1)})(t))' + a(t)f(t, u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0, \quad 0 < t < 1, \\ u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n-3, \\ u^{(n-2)}(0) = \sum\limits_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i), \quad u^{(n-1)}(1) = 0, \end{cases}$$

where

$$\varphi(u) = \begin{cases} \dfrac{u^6}{1+u^4}, & u \leqslant 0, \\ u^6, & u > 0. \end{cases}$$

Here α_i, $a(t) \in ((0,1),[0,\infty))$ and f satisfies the conditions of Theorem 4.1. It is clear that $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and a positive homomorphism with $\varphi(0) = 0$.

Remark 5.1. Because the p-Laplacian operator is odd but the operator which we define by (5.2) is not odd, so the p-Laplacian operators does not apply to our Example 5.1. Hence we generalize boundary value problems with p-Laplacian operators and the results [2], [13], [15], [17] do not apply to Example 5.1.

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