GENERALIZED HERMITEAN ULTRADISTRIBUTIONS

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(Received April 1, 2008)

Abstract. In this paper we define, by duality methods, a space of ultradistributions $\mathcal{G}'_\omega(\mathbb{R}^N)$. This space contains all tempered distributions and is closed under derivatives, complex translations and Fourier transform. Moreover, it contains some multipole series and all entire functions of order less than two. The method used to construct $\mathcal{G}'_\omega(\mathbb{R}^N)$ led us to a detailed study, presented at the beginning of the paper, of the duals of infinite dimensional locally convex spaces that are inductive limits of finite dimensional subspaces.

Keywords: distribution, multipole series, Fourier transform, complex translation, ultradistribution

MSC 2010: 46F05, 32A25, 32A45

1. Introduction

The theory of distributions was first introduced in 1950/51 by the French mathematician Laurent Schwartz in his monograph, republished in 1966 (see [8]), in order to generalize the notion of a function which allows the existence of derivatives for functions that in the usual sense are non differentiable. The key point of this theory is to look at a function as a linear continuous functional over a suitable topological vector space of test functions. The test function space considered by Schwartz, called $\mathcal{D}$, is the vector space of all complex C^∞ functions defined in $\mathbb{R}^N$ and with compact support. The topology of this space is defined in the following manner: a net (φ_j) tends to 0 in $\mathcal{D}$ iff there exists a compact set K in $\mathbb{R}^N$ such that all the φ_j have their supports contained in K and φ_j, as well as each derivative of φ_j, converge uniformly to 0 in K. The space of distributions is the dual space of $\mathcal{D}$, designated by $\mathcal{D}'$.

In order to define the Fourier transform in the context of this theory, Schwartz was led to introduce the space $\mathcal{S}$ of all complex C^∞ functions defined in $\mathbb{R}^N$ and

C. Andrade was partially supported by CIPER, L. Loura was partially supported by FCT/POCTI/FEDER through the Research Unit number 1/89.

such that each derivative decreases to zero at infinity faster than any polynomial. The topology of this space is defined by saying that a net (φ_j) tends to 0 in $\mathcal{S}$ iff φ_j and each derivative of φ_j converge uniformly to 0 in $\mathbb{R}^N$. The space $\mathcal{S}$, called the Schwartz space, contains $\mathcal{D}$ with continuous and dense injection. The dual space $\mathcal{S}'$ of $\mathcal{S}$, called the space of tempered distributions, is continuously and densely embedded in $\mathcal{D}'$. Schwartz constructed $\mathcal{S}$ in such a way that the classical Fourier transform is a vectorial and topological isomorphism in $\mathcal{S}$. By duality methods, Schwartz extended the Fourier transform to the space $\mathcal{S}'$ of tempered distributions. This extension of the classical Fourier transform is a vectorial and topological isomorphism in $\mathcal{S}'$.

A natural way to generalize the Fourier transform of tempered distributions is to construct a vector subspace X of $\mathcal{S}$, with a finer topology than the one induced by $\mathcal{S}$, such that X is dense in $\mathcal{S}$ and the Fourier transform is a vector and topological automorphism in X. The dual space X' of X contains $\mathcal{S}'$, with continuous injection, and the classical Fourier transform defined in X is extended by duality to X'. This extended Fourier transform is a vectorial and topological automorphism in X'. We remark that the construction of this kind of spaces has been done, namely in Andrade [1], Gordon [3], Loura [4], Loura and Viegas [5], and Silvestre [9].

In this paper, we apply the previous technique, constructing the subspace $\mathcal{G}_\omega$ of $\mathcal{S}$ and its dual $\mathcal{G}'_\omega$; this is done in Sections 3 and 4. Since the topology we are interested in introducing in $\mathcal{G}_\omega$ is the finest locally convex topology, in Section 2 we study this topology in a general framework. We prove that the classical Fourier transform is a vectorial and topological automorphism in $\mathcal{G}_\omega$, which is extended to $\mathcal{G}'_\omega$. This extension is a vectorial and topological automorphism in $\mathcal{G}'_\omega$ and maintains all the classical properties of the Fourier transform. Moreover, complex translations can be defined in the space $\mathcal{G}'_\omega$ and the classical properties of the Fourier transform in relation to real translations are also extended to complex translations.

In Section 5, we observe that $\mathcal{G}'_\omega$ contains all locally integrable functions of exponential growth at infinity and also some locally integrable functions whose growth at infinity is faster than exponential which are precisely defined in that section.

In Section 6, we study the behavior of multipole series in $\mathcal{G}'_\omega$ and we obtain a necessary and sufficient condition for the convergence of such series. Multipole series are employed to solve some linear differential equations in $\mathcal{G}'_\omega$.

2. Finest locally convex topology

Let $E \neq \{0\}$ be a vector space over the scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and let fix a base $B = \{b_\alpha\}_{\alpha \in A}$ in E. With each element b_α of B we associate its span $E_\alpha = \operatorname{Span}\{b_\alpha\}$. We introduce in each one-dimensional E_α the only topology such that E_α is isomorphic with $\mathbb{K}$. In the context of such conditions, in E we define the inductive limit topology related to the inductive system $\{E_\alpha\}_{\alpha\in A}$ by way of the inclusions $I_\alpha\colon E_\alpha \to E$, that is, the finest locally convex topology over E such that all I_α are continuous. When E is of infinite dimension, this is a non trivial topology, since it differs from the final topology associated with the I_α, which is not compatible with the vector structure. We designate the inductive limit topology by τ_{ind}.

Given the base B, we can define a family of seminorms over E. Let f be a non-negative real function defined on the index set A. If $x \in E$, then we have

$$x = \sum_{\alpha\in A} x_\alpha b_\alpha,$$

and we define $\mu_f\colon E \to \mathbb{K}$ by

$$\mu_f(x) = \sum_{\alpha\in A} f(\alpha)|x_\alpha|.$$

This is a well defined mapping on E, and furthermore it is a seminorm. The family M of all these seminorms is filtering, and we can consider the topology in E generated by M, which we denote by τ_M. This topology possesses the following property:

Theorem 1. *Let F be a locally convex space and $T\colon (E,\tau_M) \to F$ a linear function. Then T is continuous.*

Proof. Let N be a system of seminorms generating the topology of F. Then, for all ν in N, we have, for each $x \in E$,

$$\text{(1)} \qquad \begin{aligned} \nu(T(x)) = \nu\Bigl(T\Bigl(\sum_{\alpha\in A} x_\alpha b_\alpha\Bigr)\Bigr) &= \nu\Bigl(\sum_{\alpha\in A} x_\alpha T(b_\alpha)\Bigr) \\ &\leqslant \sum_{\alpha\in A} |x_\alpha|\nu(T(b_\alpha)). \end{aligned}$$

The second equality and the inequality in (1) are justified by the fact that, for each x in E, the sum in α has a finite number of terms, being indexed in a finite subset of A (depending on x).

For each seminorm ν of N, knowing the values that T assumes at each element of the base B of E, we can define a non-negative function in A by

$$f_\nu(\alpha) = \nu(T(b_\alpha)),$$

thus obtaining

$$\nu(T(x)) \leqslant \sum_{\alpha\in A} f_\nu(\alpha)|x_\alpha| = \mu_{f_\nu}(x).$$

□

Corollary.

(i) τ_M *is the finest locally convex topology on* E.

(ii) $\tau_M = \tau_{\mathrm{ind}}$.

(iii) τ_M *is independent of the base used for its construction.*

We are, consequently, studying the finest locally convex topology on E which may, as we will see, be defined in different ways. We assume that E is endowed with this topology, which will be denoted by τ.

Theorem 2. *E is a Hausdorff space.*

P r o o f. Let $x \in E$ with $x \neq 0$. Then

$$x = \sum_{\alpha \in A} x_\alpha b_\alpha,$$

and there exists β in A such that $x_\beta \neq 0$. Define $f_\beta\colon A \to [0, +\infty[$ by $f_\beta(\alpha) = 0$ for $\alpha \neq \beta$ and $f_\beta(\beta) = 1$. We have

$$\mu_{f_\beta}(x) = \sum_{\alpha \in A} f_\beta(\alpha)|x_\alpha| = |x_\beta| > 0.$$

□

Theorem 3. *E is a barrelled, bornological and Montel space.*

P r o o f. These properties are invariant under taking inductive limits and are verified by finite-dimensional topological vector spaces. □

The family $\{E_\alpha\}_{\alpha \in A}$ is not an inductive spectrum, since we do not have transition mappings. There is, however, an inductive spectrum for the topology of E. To construct this inductive spectrum, we consider the class $\mathcal{J}$ of all finite subsets of the index set A, and define, for all $J \in \mathcal{J}$,

$$E_J = \operatorname{span} \bigcup_{\alpha \in J} \{b_\alpha\},$$

introducing in each E_J the natural topology associated with this finite-dimensional space. We also define, supposing $J, J' \in \mathcal{J}$ and $J \subset J'$, transition applications $u_{J'J}\colon E_J \to E_{J'}$ as the injections of E_J into $E_{J'}$. The family $\{E_J\}_{J \in \mathcal{J}}$ organizes itself as an inductive spectrum equivalent to the inductive system $\{E_\alpha\}_{\alpha \in A}$. We can see in this way that τ is the inductive limit topology of all finite-dimensional subspaces of E.

Theorem 4. *A subset L of E is bounded iff there exists $J \in \mathcal{J}$ such that L is a bounded subset in E_J.*

P r o o f. If L is a bounded subset of E_J, then by continuity of the inclusion of E_J in E, L is a bounded subset of E.

To prove the converse statement, we will start by proving the following: if L is a bounded subset of E, then there is $J \in \mathcal{J}$ such that $L \subset E_J$. When E is finite dimensional, this is trivial; we assume that E is infinite dimensional. Suppose L is not contained in any E_J, that is,

$$\forall\, J \in \mathcal{J} \qquad \exists\, x_J \in L \qquad x_J \notin E_J.$$

In particular, for all $J \in \mathcal{J}$ we have $x_J \neq 0$.

By induction we can build sequences $(x_n)_{n\in\mathbb{N}}$ in L and $(J_n)_{n\in\mathbb{N}}$ in $\mathcal{J}$ such that

$$\forall\, n \in \mathbb{N} \qquad x_n \in \Big(E_{J_{n+1}} \setminus E_{\bigcup\limits_{k=1}^{n} J_k}\Big).$$

Write

$$x_n = \sum_{\alpha\in A} x_{n,\alpha} b_\alpha.$$

Since x_n belongs to $E_{J_{n+1}}$ but not to $E_{\bigcup\limits_{k=1}^{n} J_k}$, we have:

$$\forall\, n \in \mathbb{N} \qquad \exists\, \alpha_n \in A \qquad \alpha_n \in J_{n+1} \qquad \wedge \qquad \alpha_n \notin \Big(\bigcup_{k=1}^{n} J_k\Big) \qquad \wedge \qquad x_{n,\alpha_n} \neq 0.$$

In this way we have constructed a sequnce (α_n), with all α_n different from each other, which is essential for choosing a non-negative function g of a variable in A verifying

$$\forall\, n \in \mathbb{N} \qquad g(\alpha_n) = \frac{n}{|x_{n,\alpha_n}|}.$$

For any $f\colon\ A \to [0,+\infty[$ we have

$$\mu_f(x_n) = \sum_{\alpha\in A} f(\alpha)\,|(x_n)_\alpha| = \sum_{\alpha\in A} f(\alpha)|x_{n,\alpha}| \geqslant f(\alpha_n)|x_{n,\alpha_n}|.$$

Consequently,

$$\mu_g(x_n) \geqslant g(\alpha_n)|x_{n,\alpha_n}| = n,$$

and thus we conclude that L is not a bounded subset of E.

We have proved that if L is a bounded subset of E, then there is $J \in \mathcal{J}$ such that $L \subset E_J$. We now prove that L is a bounded subset of E_J. If L is a subset of E_J, then all its elements can be written in the form

$$x = \sum_{\alpha \in J} x_\alpha b_\alpha.$$

The fact that L is bounded in E implies that, for all $f\colon A \to [0, +\infty[$, there is a positive real number r_f such that

$$\forall x \in L \quad \mu_f(x) \leqslant r_f.$$

In particular, for the characteristic function $f_J = \chi_J$ we have

$$\forall x \in L \qquad \mu_{f_J}(x) = \sum_{\alpha \in A} f_J(\alpha)|x_\alpha| = \sum_{\alpha \in J} |x_\alpha| \leqslant r_{f_J}.$$

Since $\sum\limits_{\alpha \in J} |x_\alpha|$ is one of the equivalent norms defining the usual topology in E_J, L is a bounded subset of E_J. $\square$

By definition we have

$$E = \operatorname{span} \bigcup_{\alpha \in A} E_\alpha.$$

This shows that E can be seen, algebraically, as the direct sum of the E_α's, that is, the set of those elements of the product space

$$\prod_{\alpha \in A} E_\alpha$$

with only a finite number of non-zero coordinates. Moreover, the topology of E, being the inductive limit topology by the injection mappings I_α, can be identified with the direct sum topology of the direct sum E (see [7]). The convex space E is thus the topological direct sum of the E_α's. This implies the following result.

Theorem 5. *The convex space E is complete.*

P r o o f. This result follows from the fact that E is the direct sum of the complete spaces E_α. $\square$

3. The space $\mathcal{G}_\omega$

Let $v \in \mathbb{C}^N$, $v = (v_1, \dots, v_N)$. We will use the notation

$$v^2 = \sum_{k=1}^{N} v_k^2.$$

For each $\omega \in \mathbb{R} \setminus \{0\}$ we define $\mathcal{G}_\omega(\mathbb{R}^N)$ as the space

$$\operatorname{span}\{\varphi \in C(\mathbb{R}^N) \colon\ \varphi(x) = p(x)\mathrm{e}^{-\frac{1}{2}(x-z)^2|\omega|},\ p \in \mathcal{P}(\mathbb{R}^N),\ z \in \mathbb{C}^N\},$$

where $\mathcal{P}(\mathbb{R}^N)$ is the vector space of all complex polynomials defined in $\mathbb{R}^N$. To simplify notation, we study the case $N = 1$, bearing in mind that the results are similar for higher dimensions. We eventually make an exception in the definition of the tensorial product. When there is no confusion possible, we will use the abbreviation

$$\mathcal{G}_\omega = \mathcal{G}_\omega(\mathbb{R}).$$

According to this definition, we have $\mathcal{G}_{-\omega} = \mathcal{G}_\omega$. We define B_ω by

$$B_\omega = \{\varphi \in \mathcal{G}_\omega(\mathbb{R}) \colon\ \varphi(x) = x^n \mathrm{e}^{-\frac{1}{2}(x-z)^2|\omega|},\ n \in \mathbb{N}_0,\ z \in \mathbb{C}\},$$

where $\mathbb{N}_0$ is the set of non-negative integers. It can be shown that B_ω is a base for $\mathcal{G}_\omega$ (see [2]). We can thus explicitly define a nontrivial family of seminorms on $\mathcal{G}_\omega$ which will generate the finest locally convex topology on this space. The exhibition of a base for $\mathcal{G}_\omega$ is, of course, not necessary to define this topology, but the seminorms are quite useful.

Given $\varphi \in \mathcal{G}_\omega$, we have

$$\varphi(x) = \sum_{l=0}^{n} \sum_{j=0}^{k} a_{l,j} x^l \mathrm{e}^{-\frac{1}{2}(x-z_j)^2|\omega|},$$

where $n, k \in \mathbb{N}_0$ and $a_{l,j} \in \mathbb{C}$. For each $f \colon \mathbb{N}_0 \times \mathbb{C} \to [0, +\infty[$, the function $\mu_f \colon \mathcal{G}_\omega \longrightarrow \mathbb{R}$ given by

$$\mu_f(\varphi) = \sum_{l=0}^{n} \sum_{j=0}^{k} f(l, z_j)|a_{l,j}|$$

is clearly a seminorm, and the family M of seminorms associated with B_ω is the collection of such functions.

Proposition 1. *The locally convex space $\mathcal{G}_\omega$ is Hausdorff, barrelled, bornological, Montel and complete.*

P r o o f. This is a consequence of the results in the previous section. $\square$

For $J \subset \mathbb{N}_0 \times \mathbb{C}$, let

$$\mathcal{G}_{\omega,J} = \operatorname{span}\{x^n \mathrm{e}^{-\frac{1}{2}(x-z)^2|\omega|} \colon (n,z) \in J\}.$$

Proposition 2. *A subset L of $\mathcal{G}_\omega$ is bounded in $\mathcal{G}_\omega$ iff there exists a finite subset J of $\mathbb{N}_0 \times \mathbb{C}$ such that $L \subset \mathcal{G}_{\omega,J}$ and L is bounded in $\mathcal{G}_{\omega,J}$.*

P r o o f. This results from Theorem 4. $\square$

This characterization gives us an idea of what are convergent sequences in $\mathcal{G}_\omega$. The sequence $(\varphi_m)_{m\in\mathbb{N}}$ given by

$$\forall m \in \mathbb{N} \qquad \varphi_m(x) = \frac{1}{m}\mathrm{e}^{-\frac{1}{2}(x-m)^2|\omega|}$$

converges uniformly to zero in $\mathbb{R}$, but does not converge to zero in $\mathcal{G}_\omega$. The sequence $(\psi_m)_{m\in\mathbb{N}}$,

$$\forall m \in \mathbb{N} \qquad \psi_m(x) = m\mathrm{e}^{-\frac{1}{2}x^2|\omega|},$$

is not bounded and, thus, not convergent in $\mathcal{G}_\omega$. Finally, the sequence $(\lambda_m)_{m\in\mathbb{N}}$ defined by

$$\forall\ m \in \mathbb{N} \qquad \lambda_m(x) = \sum_{l=0}^{n}\sum_{j=0}^{k} \frac{l+j}{m}\, x^l \mathrm{e}^{-\frac{1}{2}(x-j)^2|\omega|}$$

is convergent in $\mathcal{G}_\omega$ for all non-negative integers n and k.

Proposition 3. *If F is a locally convex space, then any linear mapping*

$$T\colon\ \mathcal{G}_\omega \to F$$

is continuous. In particular, the algebric dual $\mathcal{G}_\omega^*$ coincides with the topological dual $\mathcal{G}'_\omega$.

P r o o f. This is a direct consequence of Theorem 1. $\square$

The derivative operator is denoted by D (or D_x if it is necessary to specify the independent variable x); the nth order derivative is denoted by D^n. Sometimes, if necessary or convenient, instead of Df and $D^2 f$ we will write, respectively, f' and f''; instead of $D^n f$ we will use $f^{(n)}$.

The space $\mathcal{G}_\omega$ is closed with respect to D. In fact, for all polynomials p and all complex numbers z we have

$$D(p(x)\mathrm{e}^{-\frac{1}{2}(x-z)^2|\omega|}) = [Dp(x) - (x-z)|\omega|p(x)]\mathrm{e}^{-\frac{1}{2}(x-z)^2|\omega|}.$$

Indeed by induction, for all $n \in \mathbb{N}$, $\mathcal{G}_\omega$ is closed with respect to D^n. This is a linear operator, and thus, according to Proposition 3, it is also continuous.

If $p \in \mathcal{P}(\mathbb{R})$ is given by

$$\forall\, x \in \mathbb{R} \qquad p(x) = \sum_{k=0}^{N} a_k x^k,$$

we define $p(D)\colon \mathcal{G}_\omega \to \mathcal{G}_\omega$ by

$$\forall\, \varphi \in \mathcal{G}_\omega \qquad p(D)\varphi = \sum_{k=0}^{N} a_k D^k \varphi.$$

This, indeed, is also a continuous linear operator in $\mathcal{G}_\omega$.

Let $\mathcal{X}$ be the space

$$\operatorname{span}\{\psi \in C(\mathbb{R})\colon\ \exists q \in \mathcal{P}(\mathbb{R})\ \exists K \in \mathbb{C}\ \forall x \in \mathbb{R}\ \psi(x) = q(x)\mathrm{e}^{Kx|\omega|}\}.$$

Then $\mathcal{X}$ is a set of multipliers for $\mathcal{G}_\omega$, that is, $\mathcal{X}\mathcal{G}_\omega \subset \mathcal{G}_\omega$.

This result allows us to define, for each $\psi \in \mathcal{X}$, the linear and continuous operator

$$\begin{aligned} P_\psi\colon\ \mathcal{G}_\omega &\longrightarrow \mathcal{G}_\omega \\ \varphi &\longmapsto \psi\varphi. \end{aligned}$$

Since $\mathcal{G}_\omega$ is closed with respect to complex translation, we may thus define

$$\begin{aligned} \tau_\alpha\colon\ \mathcal{G}_\omega &\longrightarrow \mathcal{G}_\omega \\ \varphi(x) &\longmapsto \varphi(x-\alpha). \end{aligned}$$

We remark that complex translation of a complex-valued real function is well defined provided this function can be extended to the complex plane as an entire function, which is of course the case with polynomials and $\mathrm{e}^{-\frac{1}{2}x^2|\omega|}$.

The space $\mathcal{G}_\omega$ is clearly a vector subspace of $\mathcal{S}$ and, by Proposition 3, the injection of $\mathcal{G}_\omega$ in $\mathcal{S}$ is continuous. Schwartz proved [8] that the Hermite functions are dense in $\mathcal{S}$; since the Hermite functions are in $\mathcal{G}_\omega$, this space is also dense in $\mathcal{S}$:

$$\mathcal{G}_\omega \underset{d}{\hookrightarrow} \mathcal{S}.$$

For real $\omega \neq 0$ consider in $\mathcal{G}_\omega$ the classical Fourier transform $\mathcal{F}_\omega$ defined by

$$\mathcal{F}_\omega\varphi(\xi) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\omega x\xi}\varphi(x)\,\mathrm{d}x. \tag{2}$$

Proposition 4. *Let $\omega \in \mathbb{R}\setminus\{0\}$, $P \in \mathcal{P}(\mathbb{R})$, and $\varphi \in \mathcal{G}_\omega$. Then*

$$P(D)(\mathcal{F}_\omega\varphi) = \mathcal{F}_\omega[P(-\mathrm{i}\omega x)\varphi]; \tag{3}$$

$$\mathcal{F}_\omega(P(D)\varphi) = P(\mathrm{i}\omega\xi)\mathcal{F}_\omega\varphi. \tag{4}$$

Proof. These results are known to be valid in $\mathcal{S}$, hence also in its subset $\mathcal{G}_\omega$. $\square$

In $\mathcal{S}$ complex translation is not possible. The following property, valid in $\mathcal{S}$ only for real translation, is extended in $\mathcal{G}_\omega$ to complex translations:

Proposition 5. *Let $\omega \in \mathbb{R}\setminus\{0\}$ and $\varphi \in \mathcal{G}_\omega$. Then for all $\alpha \in \mathbb{C}$ and $\xi \in \mathbb{R}$ we have*

$$(\mathcal{F}_\omega\tau_\alpha\varphi)(\xi) = \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega\varphi)(\xi). \tag{5}$$

Proof. We prove (5) under four different conditions. We write $\alpha = \alpha_1 + \alpha_2\mathrm{i}$, with α_1 and α_2 real.

(I) *The equality* (5) *is valid for* $\varphi_0 = \mathrm{e}^{-\frac{1}{2}x^2|\omega|}$.

By definition, we have

$$(\mathcal{F}_\omega\tau_\alpha\varphi_0)(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega x\xi}\mathrm{e}^{-\frac{1}{2}(x-\alpha)^2|\omega|}\,\mathrm{d}x = \lim_{n\to\infty}\int_{-n}^{n} \mathrm{e}^{-\mathrm{i}\omega x\xi}\mathrm{e}^{-\frac{1}{2}(x-\alpha)^2|\omega|}\,\mathrm{d}x. \tag{6}$$

By the Cauchy Theorem and noting that the integrand function can be extended as an entire function to $\mathbb{C}$, we have

$$\int_{-n}^{n} \mathrm{e}^{-\mathrm{i}\omega z\xi}\mathrm{e}^{-\frac{1}{2}(z-\alpha)^2|\omega|}\,\mathrm{d}z = \int_{\Gamma_n} \mathrm{e}^{-\mathrm{i}\omega z\xi}\mathrm{e}^{-\frac{(z-\alpha)^2}{2}|\omega|}\,\mathrm{d}z$$

where, for each $n \in \mathbb{N}$, Γ_n is the rectangle with vertices $\pm n$ and $\pm n + \alpha_2\mathrm{i}$, without the line segment $[-n, n]$, with clockwise orientation. This curve is composed of two vertical segments, on which the integral tends to zero as n approaches infinity and one horizontal segment, C_n, where we have

$$\begin{aligned}\lim_{n\to\infty}\int_{C_n} \mathrm{e}^{-\mathrm{i}\omega z\xi}\varphi_0(z-\alpha)\,\mathrm{d}z &= \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}\lim_{n\to\infty}\int_{-n-\alpha_1}^{n-\alpha_1} \mathrm{e}^{-\mathrm{i}\omega s\xi}\varphi_0(s)\,\mathrm{d}s \\ &= \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega\varphi_0)(\xi).\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty}\int_{-n}^{n} \mathrm{e}^{-\mathrm{i}\omega x\xi}\mathrm{e}^{-\frac{1}{2}(x-\alpha)^2|\omega|}\,\mathrm{d}x = \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega\varphi_0)(\xi).$$

(II) *The equality* (5) *is valid for functions of the form* $\varphi(x) = \mathrm{e}^{-\frac{1}{2}(x-\beta)^2|\omega|}$, *where* $\beta \in \mathbb{C}$.

We use (I) to prove this result:

$$\begin{aligned}(\mathcal{F}_\omega\tau_\alpha \mathrm{e}^{-\frac{1}{2}(x-\beta)^2|\omega|})(\xi) &= (\mathcal{F}_\omega\tau_{\alpha+\beta}\varphi_0)(\xi)\\ &= \mathrm{e}^{-\mathrm{i}\omega(\alpha+\beta)\xi}(\mathcal{F}_\omega\varphi_0)(\xi)\\ &= \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}\mathrm{e}^{-\mathrm{i}\omega\beta\xi}(\mathcal{F}_\omega\varphi_0)(\xi)\\ &= \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega\tau_\beta\varphi_0)(\xi)\\ &= \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega \mathrm{e}^{-\frac{1}{2}(x-\beta)^2|\omega|})(\xi).\end{aligned}$$

(III) *The equality* (5) *is valid for functions of the form* $\varphi(x) = x^n\mathrm{e}^{-\frac{1}{2}(x-\beta)^2|\omega|}$, *with* $\beta \in \mathbb{C}$ *and* $n \in \mathbb{N}_0$.

We start by noting that, if (5) is valid for $\varphi \in \mathcal{G}_\omega$, then it is also valid for $x\varphi$. Using (3) we have

$$\begin{aligned}(\mathcal{F}_\omega\tau_\alpha x\varphi)(\xi) &= (\mathcal{F}_\omega(x-\alpha)\tau_\alpha\varphi)(\xi)\\ &= (\mathcal{F}_\omega x\tau_\alpha\varphi)(\xi) - \alpha(\mathcal{F}_\omega\tau_\alpha\varphi)(\xi)\\ &= \frac{\mathrm{i}}{\omega}D_\xi(\mathcal{F}_\omega\tau_\alpha\varphi)(\xi) - \alpha(\mathcal{F}_\omega\tau_\alpha\varphi)(\xi).\end{aligned}$$

Since (5) is valid for φ, we have

$$\begin{aligned}&\frac{\mathrm{i}}{\omega}D_\xi(\mathcal{F}_\omega\tau_\alpha\varphi)(\xi) - \alpha(\mathcal{F}_\omega\tau_\alpha\varphi)(\xi)\\ &\quad= \frac{\mathrm{i}}{\omega}D_\xi[\mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega\varphi)(\xi)] - \alpha\mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega\varphi)(\xi)\\ &\quad= \frac{\mathrm{i}}{\omega}\left[(-\mathrm{i}\omega\alpha)\mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega\varphi)(\xi) + \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}D_\xi(\mathcal{F}_\omega\varphi)(\xi)\right] - \alpha\mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega\varphi)(\xi)\\ &\quad= \alpha\mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega\varphi)(\xi) + \frac{\mathrm{i}}{\omega}\mathrm{e}^{-\mathrm{i}\omega\alpha\xi}D_\xi(\mathcal{F}_\omega\varphi)(\xi) - \alpha\mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega\varphi)(\xi)\\ &\quad= \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}\frac{\mathrm{i}}{\omega}D_\xi(\mathcal{F}_\omega\varphi)(\xi) = \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}(\mathcal{F}_\omega(x\varphi)(\xi).\end{aligned}$$

From (II) it follows that (5) is verified for all $\beta \in \mathbb{C}$ by functions

$$\varphi(x) = x\mathrm{e}^{-\frac{1}{2}(x-\beta)^2|\omega|}.$$

By induction on n, we get (5) for the functions

$$\varphi(x) = x^n \mathrm{e}^{-\frac{1}{2}(x-\beta)^2|\omega|}$$

for all natural n and complex β.

(IV) *The equality* (5) *is valid for all* $\varphi \in \mathcal{G}_\omega$.

We just have to see that if $\varphi \in \mathcal{G}_\omega$, then there exist $N \in \mathbb{N}_0$, $c_0, \ldots, c_N \in \mathbb{C}$ and $\beta_0, \ldots, \beta_N \in \mathbb{C}$ such that for all $x \in \mathbb{R}$ we have

$$\varphi(x) = \sum_{k=0}^{N} c_k x^k \mathrm{e}^{-\frac{1}{2}(x-\beta_k)^2|\omega|}.$$

Then, by (III) and by linearity of $\mathcal{F}_\omega$ and τ_α, Proposition 5 is proved. $\square$

Proposition 6. $\mathcal{F}_\omega(\mathcal{G}_\omega) \subset \mathcal{G}_\omega$.

Proof. It is well known that the Fourier transform of $\psi(x) = \mathrm{e}^{-\frac{1}{2}x^2|\omega|}$ is

$$\sqrt{\frac{2\pi}{|\omega|}}\mathrm{e}^{-\frac{1}{2}\xi^2|\omega|}.$$

By Proposition 2, we can compute the Fourier transform of the function φ defined by $\varphi(x) = \mathrm{e}^{-\frac{1}{2}(x-\alpha)^2|\omega|}$:

$$(\mathcal{F}_\omega\varphi)(\xi) = (\mathcal{F}_\omega\tau_\alpha\psi)(\xi) = \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}\sqrt{\frac{2\pi}{|\omega|}}\mathrm{e}^{-\frac{1}{2}\xi^2|\omega|} = \sqrt{\frac{2\pi}{|\omega|}}\mathrm{e}^{-\frac{1}{2}\xi^2|\omega|-\mathrm{i}\omega\alpha\xi}.$$

If sgn denotes the signum function, defined by $\operatorname{sgn}(x) = 1$ for $x > 0$ and $\operatorname{sgn}(x) = -1$ for $x < 0$, we have

$$(\mathcal{F}_\omega\varphi)(\xi) = \sqrt{\frac{2\pi}{|\omega|}}\mathrm{e}^{-\frac{1}{2}\xi^2|\omega|-\mathrm{i}\omega\alpha\xi} = \sqrt{\frac{2\pi}{|\omega|}}\mathrm{e}^{-\frac{1}{2}|\omega|\alpha^2}\mathrm{e}^{-\frac{1}{2}[\xi+\mathrm{i}\alpha\operatorname{sgn}(\omega)]^2|\omega|}.$$

Finally, given any φ in $\mathcal{G}_\omega$, we know that

$$\varphi(x) = \sum_{k=0}^{N} c_k x^k \mathrm{e}^{-\frac{1}{2}(x-\alpha_k)^2|\omega|}$$

and thus

$$
\begin{aligned}
\mathcal{F}_\omega&\Bigg(\sum_{k=0}^{N} c_k x^k \mathrm{e}^{-\frac{1}{2}(x-\alpha_k)^2|\omega|}\Bigg)(\xi)\\
&=\sum_{k=0}^{N} c_k \frac{\mathrm{i}}{\omega} D_\xi^k\bigg(\sqrt{\frac{2\pi}{|\omega|}}\mathrm{e}^{-\frac{1}{2}|\omega|\alpha_k^2}\mathrm{e}^{-\frac{1}{2}[\xi+\mathrm{i}\alpha_k\operatorname{sgn}(\omega)]^2|\omega|}\bigg)\\
&=\sum_{k=0}^{N} p_k(\xi)\mathrm{e}^{-\frac{1}{2}[\xi+\mathrm{i}\alpha_k\operatorname{sgn}(\omega)]^2|\omega|},
\end{aligned}
$$

where $p_0,\dots,p_N$ are complex polynomials. □

Proposition 7. *For all* $\varphi\in\mathcal{G}_\omega$,

$$\varphi=\frac{|\omega|}{2\pi}\mathcal{F}_{-\omega}\mathcal{F}_\omega\varphi=\frac{|\omega|}{2\pi}\mathcal{F}_\omega\mathcal{F}_{-\omega}\varphi.$$

Proof. This result is valid in $\mathcal{S}$, hence also in $\mathcal{G}_\omega$. □

Theorem 6. *The Fourier operator* $\mathcal{F}_\omega\colon\ \mathcal{G}_\omega\to\mathcal{G}_\omega$ *is a topological and vectorial isomorphism.*

Proof. By Propositions 6 and 7 we clearly see that $\mathcal{F}_\omega\colon\ \mathcal{G}_\omega\to\mathcal{G}_\omega$ is a bijection. Besides, both $\mathcal{F}_\omega$ and $\mathcal{F}_\omega^{-1}=|\omega|(2\pi)^{-1}\mathcal{F}_{-\omega}$, being linear mappings in $\mathcal{G}_\omega$, are continuous (Proposition 3). □

Proposition 8. *Let* $\omega\in\mathbb{R}\setminus\{0\}$ *and* $\varphi\in\mathcal{G}_\omega$. *Then we have, for all complex* α,

$$\mathcal{F}_\omega(\mathrm{e}^{-\mathrm{i}\omega\alpha x}\varphi)=\tau_{-\alpha}\mathcal{F}_\omega\varphi.$$

Proof. Let $\varphi\in\mathcal{G}_\omega$. By Theorem 6, we know that $\varphi=\mathcal{F}_{-\omega}\psi$ for a certain $\psi\in\mathcal{G}_\omega$. Then

$$\mathcal{F}_\omega(\mathrm{e}^{-\mathrm{i}\omega\alpha x}\varphi)=\mathcal{F}_\omega(\mathrm{e}^{-\mathrm{i}\omega\alpha x}\mathcal{F}_{-\omega}\psi).$$

From Proposition 2 we have, substituting $-\alpha$ for α and $-\omega$ for ω,

$$\mathcal{F}_{-\omega}\tau_{-\alpha}\psi=\mathrm{e}^{-\mathrm{i}\omega\alpha x}(\mathcal{F}_{-\omega}\psi).$$

Using Proposition 3, we obtain

$$\mathcal{F}_\omega(\mathrm{e}^{-\mathrm{i}\omega\alpha x}\varphi)=\mathcal{F}_\omega(\mathcal{F}_{-\omega}\tau_{-\alpha}\psi)=\frac{2\pi}{|\omega|}\tau_{-\alpha}\psi.$$

But, by Proposition 3 again, we have

$$\varphi = \mathcal{F}_{-\omega}\psi \Leftrightarrow \psi = \frac{|\omega|}{2\pi}\mathcal{F}_\omega\varphi.$$

Consequently

$$\mathcal{F}_\omega(\mathrm{e}^{-\mathrm{i}\omega\alpha x}\varphi) = \tau_{-\alpha}\mathcal{F}_\omega\varphi.$$

This completes the proof. □

4. The space $\mathcal{G}'_\omega$

We define the space $\mathcal{G}'_\omega$ of Generalized Hermitean Ultradistributions as the strong dual of the space $\mathcal{G}_\omega$; we denote by $\langle T,\varphi\rangle_{\mathcal{G}'_\omega,\mathcal{G}_\omega}$ (or by $\langle T,\varphi\rangle$) the duality product between $T\in\mathcal{G}'_\omega$ and $\varphi\in\mathcal{G}_\omega$.

Theorem 7. *The space $\mathcal{G}'_\omega$ is a locally convex complete Hausdorff topological space and a Montel space too. Moreover, $\mathcal{G}_\omega$ is dense in $\mathcal{G}'_\omega$.*

Proof. The only nontrivial statement is the density result. Let $T\in\mathcal{G}''_\omega$ be such that $T=0$ in $\mathcal{G}_\omega$. Since $\mathcal{G}_\omega$ is reflexive, because it is a Montel space, we have $T=\psi$ for some ψ in $\mathcal{G}_\omega$. If $T=0$ in $\mathcal{G}_\omega$, then

$$\forall\ \varphi\in\mathcal{G}_\omega \qquad \langle T,\varphi\rangle = \int_{\mathbb{R}} \psi(x)\varphi(x)\,\mathrm{d}x = 0.$$

In particular, by taking $\varphi=\overline{\psi}$, we get

$$\int_{\mathbb{R}} |\psi(x)|^2\,\mathrm{d}x = 0,$$

whence $\psi=0$. Thus, we have $T=0$, which implies the density of $\mathcal{G}_\omega$ in $\mathcal{G}'_\omega$. □

We can, therefore, write the following inclusions:

$$\mathcal{G}_\omega \underset{d}{\hookrightarrow} \mathcal{S} \underset{d}{\hookrightarrow} \mathcal{S}' \underset{d}{\hookrightarrow} \mathcal{G}'_\omega.$$

For each $n\in\mathbb{N}$, we define the derivative $D^n\colon\ \mathcal{G}'_\omega\to\mathcal{G}'_\omega$ by

$$\forall\ \varphi\in\mathcal{G}_\omega \qquad \langle D^nT,\varphi\rangle = (-1)^n\langle T,D^n\varphi\rangle. \tag{7}$$

This is a linear continuous operator generalizing the derivative operator in $\mathcal{S}'$.

The multiplier set $\mathcal{X}$, defined in Section 3, yields a corresponding set of product operators in $\mathcal{G}'_\omega$ in the following way: for each $\psi \in \mathcal{X}$ we define the operator $P_\psi\colon \mathcal{G}'_\omega \to \mathcal{G}'_\omega$ by

$$\forall\ \varphi \in \mathcal{G}_\omega \qquad \langle P_\psi T, \varphi\rangle = \langle T, P_\psi \varphi\rangle = \langle T, \psi\varphi\rangle,$$

which is a linear continuous operator in $\mathcal{G}'_\omega$ generalizing the corresponding operator in $\mathcal{G}_\omega$.

For each $\alpha \in \mathbb{C}$ we define the translation operator $\tau_\alpha\colon \mathcal{G}'_\omega \to \mathcal{G}'_\omega$ by transposition of the corresponding operator in $\mathcal{G}_\omega$:

$$\forall\ \varphi \in \mathcal{G}_\omega \qquad \langle \tau_\alpha T, \varphi\rangle = \langle T, \tau_{-\alpha}\varphi\rangle,$$

which is also a linear continuous operator in $\mathcal{G}'_\omega$ generalizing the corresponding operator in $\mathcal{G}_\omega$. For real α it generalizes the translation operator in $\mathcal{S}'$.

In what follows, it will be useful to note that the aforementioned multiplier set admits a generalization to $\mathbb{R}^N$ denoted by $\mathcal{X}(\mathbb{R}^N)$ and equal to the span of the set

$$\{\psi \in C(\mathbb{R}^N)\colon\ \exists q \in \mathcal{P}(\mathbb{R}^N)\ \exists K \in \mathbb{C}^N\ \forall x \in \mathbb{R}^N\ \psi(x) = q(x)\mathrm{e}^{K\cdot x|\omega|}\}.$$

Note also that, if $\psi_1 \in \mathcal{X}(\mathbb{R}^N)$ and $\psi_2 \in \mathcal{X}(\mathbb{R}^M)$, then the classical tensor product $\psi_1 \otimes \psi_2$ belongs to $\mathcal{X}(\mathbb{R}^{N+M})$.

We now define the tensor product of an element of $\mathcal{G}'_\omega(\mathbb{R}^N)$ by an element of $\mathcal{G}'_\omega(\mathbb{R}^M)$. If $\varphi \in \mathcal{G}_\omega(\mathbb{R}^{N+M})$, then there is $L \in \mathbb{N}$ such that for each $j \in \{1, \ldots, L\}$ there are $c_j \in \mathbb{C} \setminus \{0\}$, $\alpha_j \in \mathbb{C}^N$, $\beta_j \in \mathbb{C}^M$, $p_j \in \mathbb{N}_0^N$ and $q_j \in \mathbb{N}_0^M$, satisfying

$$\forall\ x \in \mathbb{R}^N\ \forall\ y \in \mathbb{R}^M \qquad \varphi(x,y) = \sum_{j=1}^{L} c_j u_j(x) v_j(y),$$

with

$$u_j(x) = x^{p_j}\mathrm{e}^{-\frac{1}{2}(x-\alpha_j)^2|\omega|}$$

and

$$v_j(y) = y^{q_j}\mathrm{e}^{-\frac{1}{2}(y-\beta_j)^2|\omega|}.$$

As the constants c_j and the functions u_j and v_j are uniquely determined by φ, we can define the tensor product of $T \in \mathcal{G}'_\omega(\mathbb{R}^N)$ and $S \in \mathcal{G}'_\omega(\mathbb{R}^M)$ as the element $T \otimes S$ of $\mathcal{G}'_\omega(\mathbb{R}^{N+M})$ given by

$$\langle T \otimes S, \varphi\rangle_{N+M} = \sum_{j=1}^{L} c_j \langle T, u_j\rangle_N \langle S, v_j\rangle_M.$$

The mapping $(T,S) \to T \otimes S$ is bilinear, separately continuous from $\mathcal{G}'_\omega(\mathbb{R}^N) \times \mathcal{G}'_\omega(\mathbb{R}^M)$ to $\mathcal{G}'_\omega(\mathbb{R}^{N+M})$, and has the following properties:

Proposition 9. *Let $T \in \mathcal{G}'_\omega(\mathbb{R}^N)$ and $S \in \mathcal{G}'_\omega(\mathbb{R}^M)$. Then for all $n \in \mathbb{N}_0^N$, $m \in \mathbb{N}_0^M$, $\psi_1 \in \mathcal{X}(\mathbb{R}^N)$, $\psi_2 \in \mathcal{X}(\mathbb{R}^M)$ and $\varphi \in \mathcal{G}_\omega(\mathbb{R}^N)$ we have*

$$\partial_x^n \partial_y^m (T \otimes S) = (\partial_x^n T) \otimes (\partial_y^m S); \ (\psi_1 \otimes \psi_2)(T \otimes S) = (\psi_1 T) \otimes (\psi_2 S);$$
$$\langle T \otimes S, \varphi(x,y)\rangle_{N+M} = \langle T_x, \langle S_y, \varphi(x,y)\rangle_M\rangle_N = \langle S_y, \langle T_x, \varphi(x,y)\rangle_N\rangle_M.$$

We define the Fourier transform $\mathcal{F}_\omega \colon \mathcal{G}'_\omega \to \mathcal{G}'_\omega$ by

$$\forall\, T \in \mathcal{G}'_\omega \ \forall \varphi \in \mathcal{G}_\omega \qquad \langle \mathcal{F}_\omega T, \varphi\rangle = \langle T, \mathcal{F}_\omega \varphi\rangle,$$

where the operator is a vectorial and topological isomorphism in $\mathcal{G}'_\omega$ and generalizes the usual Fourier transform in $\mathcal{S}'$. For all $T \in \mathcal{G}'_\omega$, $P, Q \in \mathcal{P}(\mathbb{R})$ and all $\alpha \in \mathbb{C}$ we have

$$\mathcal{F}_\omega[P(D)Q(x)T] = P(\mathrm{i}\omega x) Q\Big(\frac{\mathrm{i}}{\omega} D\Big)\mathcal{F}_\omega T;$$
$$\mathcal{F}_\omega(\tau_\alpha T) = \mathrm{e}^{-\mathrm{i}\omega\alpha\xi}\mathcal{F}_\omega T;$$
$$\mathcal{F}_\omega(\mathrm{e}^{\mathrm{i}\omega\alpha x} T) = \tau_\alpha \mathcal{F}_\omega T.$$

5. Characterization of some elements of $\mathcal{G}'_\omega$

Let V^{2-} be the vector space

$$\{f \in L^1_{\mathrm{loc}}(\mathbb{R}) \colon \exists \varrho \in]0,2[\ \exists C > 0 \ \exists \sigma \in \mathbb{R} \ |f(x)| \leqslant C\mathrm{e}^{\sigma|x|^\varrho}\}.$$

Lemma 1. *Let $f \in V^{2-}$. Then for all $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$, the integral*

$$\int_{\mathbb{R}} f(x) x^n \mathrm{e}^{-\frac{1}{2}(x-z)^2|\omega|}\,\mathrm{d}x$$

is absolutely convergent.

Proof. If $z = a + \mathrm{i}b$ with $a, b \in \mathbb{R}$, we have

$$\big|\mathrm{e}^{-\frac{1}{2}(x-z)^2|\omega|}\big| = \mathrm{e}^{-\frac{1}{2}((x-a)^2-b^2)|\omega|}.$$

Since f is an element of V^{2-}, there are $\varrho \in]0,2[$, $C > 0$ and $\sigma \in \mathbb{R}$ such that

$$|f(x)x^n \mathrm{e}^{-\frac{1}{2}(x-z)^2|\omega|}| \leqslant C|x|^n \mathrm{e}^{\sigma|x|^\varrho}\mathrm{e}^{-\frac{1}{2}((x-a)^2-b^2)|\omega|}.$$

Since the integral

$$\int_{\mathbb{R}} |x|^n \mathrm{e}^{\sigma|x|^\varrho - \frac{1}{2}((x-a)^2-b^2)|\omega|}\,\mathrm{d}x$$

is absolutely convergent, the lemma is completely proved. □

This result implies that for each $f \in V^{2-}$ and $\varphi \in \mathcal{G}_\omega$, the integral

$$\int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x$$

is absolutely convergent. We can, thus, define the following functional in $\mathcal{G}_\omega$:

$$\begin{aligned} T_f\colon\ \mathcal{G}_\omega &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \langle T_f, \varphi\rangle \end{aligned}$$

such that

$$\forall \varphi \in \mathcal{G}_\omega \qquad \langle T_f, \varphi\rangle = \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x,$$

which, being linear in $\mathcal{G}_\omega$, is continuous in this space, defining therefore an element of $\mathcal{G}'_\omega$.

Lemma 2. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ and suppose that for each $\xi \in \mathbb{C}$, the integral*

$$\int_{\mathbb{R}} f(x)\mathrm{e}^{-\frac{1}{2}(x-\xi)^2}\,\mathrm{d}x$$

vanishes. Then f is zero almost everywhere, that is, f is the zero vector of $L^1_{\mathrm{loc}}(\mathbb{R})$.

Proof. If the above integral is zero for all $\xi \in \mathbb{C}$, then in particular, for each $b \in \mathbb{R}$,

$$\int_{\mathbb{R}} f(x)\mathrm{e}^{-\frac{1}{2}(x+\mathrm{i}b)^2}\,\mathrm{d}x = 0.$$

We thus have

$$\int_{\mathbb{R}} f(x)\mathrm{e}^{-\frac{1}{2}(x+\mathrm{i}b)^2}\,\mathrm{d}x = \mathrm{e}^{\frac{1}{2}b^2}\int_{\mathbb{R}} f(x)\mathrm{e}^{-\frac{1}{2}x^2}\mathrm{e}^{-\mathrm{i}bx}\,\mathrm{d}x = 0.$$

Defining $F(x) = f(x)\mathrm{e}^{-\frac{1}{2}x^2}$, we obtain a function of $L^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} F(x)\mathrm{e}^{-\mathrm{i}bx}\,\mathrm{d}x = 0,$$

which means that the image of F by the Fourier transform is zero:

$$\forall\ b \in \mathbb{R} \qquad \mathcal{F}_1 F(b) = 0.$$

The injectivity of $\mathcal{F}_1$ shows that $F = 0$, whence $f = 0$. □

Consider now the mapping

$$\begin{aligned} \mathcal{T}\colon\ V^{2-} &\longrightarrow \mathcal{G}'_\omega \\ f &\longmapsto T_f. \end{aligned}$$

By Lemma 2, we conclude that this mapping is injective. Consequently, we have the following result:

Theorem 8. *Each element of* V^{2-} *can be identified in a unique way with an element of* $\mathcal{G}'_\omega$, *that is,* $V^{2-} \subset \mathcal{G}'_\omega$.

In this way we see that each locally integrable function verifying the condition $|f(x)| \leqslant C\mathrm{e}^{\sigma|x|^\varrho}$ with $C > 0$, $\sigma \in \mathbb{R}$ and $0 < \varrho < 2$, defines an element of $\mathcal{G}'_\omega$. There is a similar result for the limit case $\varrho = 2$:

Theorem 9. *Let* $V^{2,\sigma}$ *be the vector space*

$$V^{2,\sigma} = \{f \in L^1_{\mathrm{loc}}(\mathbb{R})\colon \exists C > 0 \qquad |f(x)| \leqslant C\mathrm{e}^{\sigma|x|^2}\}.$$

Then $\sigma < \frac{1}{2}|\omega|$ *implies* $V^{2,\sigma} \subset \mathcal{G}'_\omega$.

We now recall the definition of the order and type of an entire function, and some related properties. An entire function f has finite order if

$$\exists \mu > 0 \qquad \exists R_\mu > 0 \qquad \forall r > R_\mu \qquad M(r) < \mathrm{e}^{r^\mu}, \tag{8}$$

where $M(r) = \max\limits_{|z|=r} |f(z)|$. If f has finite order, we define the order of f by

$$\varrho = \inf\{\mu > 0\colon \forall r > R_\mu \qquad M(r) < \mathrm{e}^{r^\mu}\}.$$

If f has finite order ϱ and there are positive numbers k and R_k such that

$$\forall r > R_k \qquad M(r) < \mathrm{e}^{kr^\varrho}, \tag{9}$$

then f is said to be of finite type. In this case we define the type of f as

$$\sigma = \inf\{k > 0\colon \forall r > R_k \qquad M(r) < \mathrm{e}^{kr^\varrho}\}.$$

It can be shown (see [6]) that

$$\varrho = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r)}{\log r}, \tag{10}$$

$$\sigma = \overline{\lim_{r\to+\infty}} \frac{\log M(r)}{r^\varrho}. \tag{11}$$

The following lemma is also proved in [6]:

Lemma 3. *Let* $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$ *be an entire function with order* ϱ *and type* σ. *Then*

(i) $\varrho = \overline{\lim} \log n / \log |a_n|^{-1/n}$;

(ii) $\sigma = (\mathrm{e}\varrho)^{-1} \overline{\lim}\, n |a_n|^{\varrho/n}$;

(iii) *if* $\varrho < +\infty$, *then*

$$\forall \varepsilon > 0 \qquad \exists C_\varepsilon > 0 \qquad \forall z \in \mathbb{C} \qquad |f(z)| \leqslant C_\varepsilon \mathrm{e}^{|z|^{\varrho+\varepsilon}};$$

(iv) *if* $\varrho < +\infty$ *and* $\sigma < +\infty$, *then*

$$\forall \varepsilon > 0 \qquad \exists C_\varepsilon > 0 \qquad \forall z \in \mathbb{C} \qquad |f(z)| \leqslant C_\varepsilon \mathrm{e}^{(\sigma+\varepsilon)|z|^{\varrho}}.$$

The next theorem is crucial for the study of the convergence of multipole series in $\mathcal{G}'_\omega$.

Theorem 10. *Let* $f\colon \mathbb{R} \to \mathbb{C}$ *and suppose that* f *can be extended to* $\mathbb{C}$ *as an entire function. Let* $(p_n)_{n\in\mathbb{N}_0}$ *be the McLaurin polynomials of* f.

(i) *If the entire function extending* f *has finite order* $\varrho < 2$, *then*

$$\forall\, \omega \in \mathbb{R} \setminus \{0\} \qquad f \in \mathcal{G}'_\omega \ \text{ and } \ p_n \to f \ \text{ in } \ \mathcal{G}'_\omega.$$

(ii) *If the entire function extending* f *has order* $\varrho = 2$ *and finite type* σ, *then*

$$\forall\, \omega \in \mathbb{R} \setminus \{0\} \qquad \sigma < \tfrac{1}{2}|\omega| \Rightarrow f \in \mathcal{G}'_\omega \ \text{ and } \ p_n \to f \ \text{ in } \ \mathcal{G}'_\omega.$$

P r o o f. If f has finite order $\varrho < 2$ then, by Lemma 3 (iii), for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\forall z \in \mathbb{C} \qquad |f(z)| \leqslant C_\varepsilon \mathrm{e}^{|z|^{\varrho+\varepsilon}}.$$

By choosing ε such that $\varrho + \varepsilon < 2$, we have $f \in V^{2-}$. Hence by Theorem 8, $f \in \mathcal{G}'_\omega$.

Similarly, if f verifies $\varrho = 2$ and has finite type σ then, by Lemma 3 (iv), for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\forall\, x \in \mathbb{R} \qquad |f(x)| \leqslant C_\varepsilon \mathrm{e}^{(\sigma+\varepsilon)|x|^2},$$

and f is an element of $V^{2,\sigma+\varepsilon}$. If $\sigma < \frac{1}{2}|\omega|$, we can choose ε in such a way that $\sigma + \varepsilon < \frac{1}{2}|\omega|$, and thus by Theorem 9, $f \in \mathcal{G}'_\omega$.

We will now study the convergence of the McLaurin polynomials of f in $\mathcal{G}'_\omega$ which, being a Montel space, implies that the convergence mentioned is identical to the weak convergence, which offers a way of proving

$$\forall\varphi\in\mathcal{G}_\omega\qquad\int_{\mathbb{R}}[f(x)-p_n(x)]\varphi(x)\,\mathrm{d}x\to 0. \tag{12}$$

Since f is entire, we clearly have

$$\forall x\in\mathbb{R}\qquad\forall\varphi\in\mathcal{G}_\omega\qquad[f(x)-p_n(x)]\varphi(x)\to 0.$$

Besides, if f has finite order, then by Lemma 3 (i) we see that, for each $n\in\mathbb{N}$, the order of $f-p_n$ equals the order of f, since

$$\forall z\in\mathbb{C}\qquad(f-p_n)(z)=\sum_{j=0}^{+\infty}a_jz^j-\sum_{j=0}^{n}a_jz^j=\sum_{j=n+1}^{+\infty}a_jz^j,$$

and the upper limit of a subsequence of the given sequence, obtained by the supression of a finite number of terms, remains unchanged. In the same way, we see that if f is of finite type, then the type of $f-p_n$ equals the type of f.

Let $R>0$ and $n\in\mathbb{N}$. If C_R denotes the circumference of radius R and center at the origin, then we can apply the Cauchy integral formula for the derivatives of f:

$$|a_n|=\left|\frac{1}{2\pi\mathrm{i}}\int_{C_R}\frac{f(\lambda)}{\lambda^{n+1}}\,\mathrm{d}\lambda\right|\leqslant\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(R\mathrm{e}^{\mathrm{i}\theta})|}{R^n}\,\mathrm{d}\theta. \tag{13}$$

Considering Lemma 3 (i) we have, for ν such that $\varrho<\nu<2$,

$$\exists C_\nu=C\qquad\forall z\in\mathbb{C}\qquad|f(z)|\leqslant C\mathrm{e}^{|z|^\nu},$$

whence

$$\forall n\in\mathbb{N}_0\qquad\forall R>0\qquad|a_n|\leqslant C\frac{\mathrm{e}^{R^\nu}}{R^n},$$

and with $R=2|z|$,

$$|f(z)-p_n(z)|\leqslant\sum_{k=0}^{+\infty}|a_k||z|^k\leqslant\sum_{k=0}^{+\infty}C\,\frac{\mathrm{e}^{(2|z|)^\nu}}{2^k}=2C\mathrm{e}^{(2|z|)^\nu}$$

and so, for all x in $\mathbb{R}$, all n in $\mathbb{N}_0$ and all q in $\mathcal{P}(\mathbb{R})$, we have

$$|(f-p_n)(x)||q(x)|\mathrm{e}^{-\frac{1}{2}x^2|\omega|}\leqslant 2C|q(x)|\mathrm{e}^{-x^2(\frac{1}{2}|\omega|-2^\nu|x|^{\nu-2})}$$

and by the dominated convergence we conclude, for all functions $\varphi \in \mathcal{G}_\omega$,

$$\left| \int_{\mathbb{R}} [f(x) - p_n(x)]\varphi(x)\,\mathrm{d}x \right| \leqslant \int_{\mathbb{R}} |f(x) - p_n(x)||\varphi(x)|\,\mathrm{d}x \to 0.$$

Now consider Lemma 3 (ii): f is of order 2 and finite type σ. If $\sigma < \frac{1}{2}|\omega|$, then we can choose ε such that $0 < \varepsilon < \frac{1}{2}|\omega| - \sigma$. We know that

$$\exists C > 0 \qquad \forall z \in \mathbb{C} \qquad |f(z)| \leqslant C\mathrm{e}^{(\sigma+\varepsilon)|z|^2},$$

and applying (13) we get

$$\forall n \in \mathbb{N}_0 \qquad \forall R > 0 \qquad |a_n| \leqslant C\frac{\mathrm{e}^{(\sigma+\varepsilon)R^2}}{R^n}.$$

With $R = |z| + \frac{1}{2}$ we infer

$$\begin{aligned}
\forall n \in \mathbb{N}_0 \qquad \forall z \in \mathbb{C} \qquad |f(z) - p_n(z)| &\leqslant \sum_{k=0}^{+\infty} |a_k||z|^k \\
\leqslant \sum_{k=0}^{+\infty} C\frac{\mathrm{e}^{(\sigma+\varepsilon)(|z|+\frac{1}{2})^2}}{(|z|+\frac{1}{2})^k}|z|^k &\leqslant C_1\mathrm{e}^{(\sigma+\varepsilon)|z|^2}\mathrm{e}^{(\sigma+\varepsilon)|z|}\sum_{k=0}^{+\infty}\left(\frac{2|z|}{2|z|+1}\right)^k \\
= C_1(2|z|+1)\mathrm{e}^{(\sigma+\varepsilon)|z|^2}\mathrm{e}^{(\sigma+\varepsilon)|z|}&
\end{aligned}$$

and finally, if $x \in \mathbb{R}$, $n \in \mathbb{N}_0$ and $q \in \mathcal{P}(\mathbb{R})$,

$$|(f - p_n)(x)||q(x)|\mathrm{e}^{-\frac{1}{2}x^2|\omega|} \leqslant C_1(2|x|+1)|q(x)|\mathrm{e}^{-x^2\left(\frac{|\omega|}{2} - (\sigma+\varepsilon) - \frac{\sigma+\varepsilon}{|x|}\right)}.$$

By repeating the use of the Lebesgue dominated convergence theorem, we prove (12). $\square$

In order to derive the essential convergence conditions, in what follows we present some relevant results.

For each $k \in \mathbb{N}$, let $\mathcal{H}_{\mathrm{e}}^k$ be the space of entire functions which satisfy

$$\exists C > 0\ \exists \alpha > 0\ \forall z \in \mathbb{C} \qquad |f(z)| \leqslant C\mathrm{e}^{\alpha|z|^k}. \tag{14}$$

Lemma 4. *A function f is an element of $\mathcal{H}_{\mathrm{e}}^k$ if and only if it has order smaller than k, or has an order equal to k and finite type.*

Proof. If $f \in \mathcal{H}_{\mathrm{e}}^k$, then

$$M(r) = \max_{|z|=r} |f(z)| \leqslant C\mathrm{e}^{\alpha r^k},$$

which implies that f has finite order ϱ, and further, $\varrho \leqslant k$. Besides,

$$\frac{\log M(r)}{r^\varrho} \leqslant \frac{\log C + \alpha r^k}{r^\varrho} = \frac{\log C}{r^\varrho} + \frac{\alpha}{r^{\varrho-k}},$$

and, in the particular case $\varrho = k$, we have, by (11),

$$\sigma = \overline{\lim_{r\to+\infty}} \frac{\log M(r)}{r^k} \leqslant \lim_{r\to+\infty} \Bigl(\frac{\log C}{r^k} + \alpha\Bigr) = \alpha.$$

The converse is a direct consequence of Lemma 3. □

We now study the cases when $k = 1$ and $k = 2$. The space $\mathcal{H}_{\mathrm{e}}^1$ is the space of entire functions of exponential growth. The following lemma is proved in [6].

Lemma 5. *Let $f \in \mathcal{H}_{\mathrm{e}}^1$ with*

$$\forall z \in \mathbb{C} \qquad f(z) = \sum_{j=0}^{+\infty} a_j z^j. \tag{15}$$

Then the sequence $\bigl(\sqrt[j]{j!|a_j|}\bigr)_{j\in\mathbb{N}}$ is bounded.

Conversely, if $(a_j)_{j\in\mathbb{N}}$ is a sequence of complex numbers such that $\bigl(\sqrt[j]{j!|a_j|}\bigr)_{j\in\mathbb{N}}$ is a bounded sequence, then the series in (15) is absolutely convergent for all $z \in \mathbb{C}$ and the function defined by (15) is in $\mathcal{H}_{\mathrm{e}}^1$.

This result can be extended to $\mathcal{H}_{\mathrm{e}}^k$:

Lemma 6. *Suppose that $f \in \mathcal{H}_{\mathrm{e}}^k$ verifies (15). Then for each $p \in \{0, \ldots, k-1\}$, the sequence $\bigl(\sqrt[n]{n!|a_{kn+p}|}\bigr)_{n\in\mathbb{N}}$ is bounded. Conversely, if $(a_j)_{j\in\mathbb{N}}$ is a sequence of complex numbers such that for all $p \in \{0, \ldots, k-1\}$, the sequence $\bigl(\sqrt[n]{n!|a_{kn+p}|}\bigr)_{n\in\mathbb{N}}$ is bounded, then the series in (15) is absolutely convergent for all $z \in \mathbb{C}$ and the function defined by (15) is in $\mathcal{H}_{\mathrm{e}}^k$.*

Theorem 11. *Let* $f(z)=\sum\limits_{k=0}^{+\infty}a_kz^k$ *be an entire function and let* $(p_n)_{n\in\mathbb{N}}$ *be the sequence of McLaurin polynomials of* f.

(i) *If the sequence* $\left(\sqrt[n]{n!|a_n|}\right)_{n\in\mathbb{N}}$ *is bounded, then*

$$\forall\omega\in\mathbb{R}\setminus\{0\}\quad f\in\mathcal{G}'_\omega\qquad\text{and}\qquad p_n\to f\quad\text{in } \mathcal{G}'_\omega.$$

(ii) *If the sequences* $\left(\sqrt[n]{n!|a_{2n}|}\right)_{n\in\mathbb{N}}$ *and* $\left(\sqrt[n]{n!|a_{2n+1}|}\right)_{n\in\mathbb{N}}$ *are bounded, then*

$$\forall\omega\in\mathbb{R}\setminus\{0\}\quad |\omega|>K\Rightarrow f\in\mathcal{G}'_\omega\qquad\text{and}\qquad p_n\to f\quad\text{in } \mathcal{G}'_\omega,$$

where $K=2\max\Big\{\sup\limits_{n\in\mathbb{N}}\sqrt[n]{n!|a_{2n}|},\ \sup\limits_{n\in\mathbb{N}}\sqrt[n]{n!|a_{2n+1}|}\Big\}$.

P r o o f. (i) If the sequence $\left(\sqrt[n]{n!|a_n|}\right)_{n\in\mathbb{N}}$ is bounded, then by Lemma 5, $f\in\mathcal{H}^1_{\mathrm{e}}$ whence, by Lemma 4, f has order less than or equal to 1, and by Theorem 10, (i) is proved.

(ii) If the sequences

$$\left(\sqrt[n]{n!|a_{2n}|}\right)_{n\in\mathbb{N}}\quad\text{and}\quad\left(\sqrt[n]{n!|a_{2n+1}|}\right)_{n\in\mathbb{N}}$$

are bounded, then by Lemma 6, $f\in\mathcal{H}^2_{\mathrm{e}}$ and, by Lemma 4, we obtain $\varrho<2$, or $\varrho=2$ and $\sigma<+\infty$. If $\varrho<2$, we obtain the result for all $\omega\neq0$.

Now suppose $\varrho=2$ and $\sigma<+\infty$. Denoting

$$L=\sup_{n\in\mathbb{N}}\sqrt[n]{n!|a_{2n}|},\quad M=\sup_{n\in\mathbb{N}}\sqrt[n]{n!|a_{2n+1}|}\quad\text{and}\quad\alpha=\max\{L,M\},$$

we have

$$\forall\varepsilon>0\ \exists R_\varepsilon>0\ \forall z\in\mathbb{C}\qquad|f(z)|\leqslant R_\varepsilon\mathrm{e}^{(\alpha+\varepsilon)|z|^2},$$

which implies $\sigma\leqslant\alpha$. Then

$$|\omega|>2\alpha\Rightarrow|\omega|>2\sigma,$$

from which we conclude, by Theorem 10, that $p_n\to f$ in $\mathcal{G}'_\omega$. The theorem is thus completely proved. $\square$

6. Multipole series

A multipole series is a series of the form $\sum\limits_{k=0}^{+\infty}a_k\delta^{(k)}$, where a_k are complex numbers. This series is not convergent in $\mathcal{S}'$ or $\mathcal{D}'$ unless all but a finite number of the coefficients a_k are null. In our space $\mathcal{G}'_\omega$, we have much more convergent multipole series.

Theorem 12. *The series* $\sum\limits_{k=0}^{+\infty} a_k\delta^{(k)}$ *converges in* $\mathcal{G}'_\omega$ *if and only if for each* $p \in \mathbb{N}$ *and for each* $\alpha \in \mathbb{C}$*, the series*

$$\sum_{k=0}^{+\infty}(-1)^k a_{k+p}(k+p)! \sum_{2l+j=k} \frac{(-1)^l \alpha^j |\omega|^{l+j}}{2^l l! j!} \tag{16}$$

is convergent in $\mathbb{C}$.

Proof. Given $\varphi \in \mathcal{G}_\omega$, we have, formally,

$$\left\langle \sum_{k=0}^{+\infty} a_k\delta^{(k)}, \varphi \right\rangle = \sum_{k=0}^{+\infty}(-1)^k a_k \varphi^{(k)}(0).$$

We just have to compute $\varphi^{(k)}(0)$ for a function φ defined by

$$\forall x \in \mathbb{R} \qquad \varphi(x) = x^p \mathrm{e}^{-\frac{1}{2}(x-\alpha)^2|\omega|}.$$

For each $x \in \mathbb{R}$ we have

$$\begin{aligned} x^p \mathrm{e}^{-\frac{1}{2}(x-\alpha)^2|\omega|} &= x^p \mathrm{e}^{-\frac{1}{2}x^2|\omega|}\mathrm{e}^{\alpha x|\omega|}\mathrm{e}^{-\frac{1}{2}\alpha^2|\omega|} \\ &= \mathrm{e}^{-\frac{1}{2}\alpha^2|\omega|} x^p \sum_{n=0}^{+\infty} \frac{(-1)^n|\omega|^n}{2^n n!} x^{2n} \sum_{m=0}^{+\infty} \frac{\alpha^m|\omega|^m}{m!} x^m. \end{aligned} \tag{17}$$

Since both series in (17) are absolutely convergent in $\mathbb{C}$, we can compute their product, knowing that it will define also an absolutely convergent series. Then

$$\begin{aligned} x^p \mathrm{e}^{-\frac{1}{2}(x-\alpha)^2|\omega|} &= \mathrm{e}^{-\frac{1}{2}\alpha^2|\omega|} x^p \sum_{k=0}^{+\infty} \sum_{2l+j=k} \frac{(-1)^l \alpha^j |\omega|^{l+j}}{2^l l! j!} x^k \\ &= \sum_{k=0}^{+\infty} \mathrm{e}^{-\frac{1}{2}\alpha^2|\omega|} \sum_{2l+j=k} \frac{(-1)^l \alpha^j |\omega|^{l+j}}{2^l l! j!} x^{k+p}, \end{aligned}$$

and so we obtain an expression for $\varphi^{(k)}(0)$:

$$\varphi^{(s)}(0) = \begin{cases} 0 & \text{if } s < p, \\ s! c_{s-p} & \text{if } s \geqslant p \end{cases}$$

where

$$c_{s-p} = \mathrm{e}^{-\frac{1}{2}\alpha^2|\omega|} \sum_{2l+j=s-p} \frac{(-1)^l \alpha^j |\omega|^{l+j}}{2^l l! j!}.$$

The convergence of the given multipole series in $\mathcal{G}'_\omega$ is thus equivalent to the convergence, for all $p \in \mathbb{N}$ and for all $\alpha \in \mathbb{C}$, of the series

$$\begin{aligned}\sum_{s=0}^{+\infty}(-1)^s a_s \varphi^{(s)}(0) &= \sum_{s=p}^{+\infty}(-1)^s a_s s! c_{s-p} = \sum_{k=0}^{+\infty}(-1)^{k+p} a_{k+p}(k+p)! c_k \\ &= (-1)^p \mathrm{e}^{-\frac{1}{2}\alpha^2|\omega|}\sum_{k=0}^{+\infty}(-1)^k a_{k+p}(k+p)! \sum_{2l+j=k}\frac{(-1)^l\alpha^j|\omega|^{l+j}}{2^l l! j!}.\end{aligned}$$

This completely proves the theorem. □

Although the Theorem 12 does give us a necessary and sufficient condition for a multipole series to belong to $\mathcal{G}'_\omega$, yet (16) is far too complex to be useful. Therefore, we will obtain simpler sufficient conditions.

Theorem 13. *Let* $\sum_{k=0}^{+\infty} a_k\delta^{(k)}$ *be a multipole series such that the function*

$$\forall z \in \mathbb{C} \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is entire of order ϱ and type σ.

(i) *If $\varrho < 2$, then for all $\omega \in \mathbb{R}\setminus\{0\}$ the multipole series is convergent in $\mathcal{G}'_\omega$.*

(ii) *If $\varrho = 2$ and σ is a real number, then for all $\omega \in]-(2\sigma)^{-1}, 0[\,\cup\,]0,(2\sigma)^{-1}[$ the multipole series is convergent in $\mathcal{G}'_\omega$.*

P r o o f. We define a function $g\colon \mathbb{C}\to\mathbb{C}$ as follows:

$$\forall z \in \mathbb{C} \qquad g(z) = \frac{|\omega|}{2\pi}\sum_{k=0}^{+\infty}\left(\frac{\omega}{\mathrm{i}}\right)^k a_k z^k = \frac{|\omega|}{2\pi} f\left(\frac{\omega}{\mathrm{i}}z\right).$$

This is an entire function with order $\varrho_g = \varrho$ and type $\sigma_g = |\omega|^\varrho\sigma$. We now apply Theorem 10 to this function.

(i) If $\varrho < 2$, then for all $\omega \in \mathbb{R}\setminus\{0\}$ we have $g \in \mathcal{G}'_\omega$ and the McLaurin series of g converges to g in $\mathcal{G}'_\omega$. This implies that we can compute the Fourier transform of the restriction of g to $\mathbb{R}$:

$$\mathcal{F}_\omega g = \mathcal{F}_\omega\left[\frac{|\omega|}{2\pi}\sum_{k=0}^{+\infty}\left(\frac{\omega}{\mathrm{i}}\right)^k a_k x^k\right].$$

Since the McLaurin series of g converges in $\mathcal{G}'_\omega$ and $\mathcal{F}_\omega$ is linear continuous, we have

$$\mathcal{F}_\omega g = \frac{|\omega|}{2\pi} \sum_{k=0}^{+\infty} \left(\frac{\omega}{\mathrm{i}}\right)^k a_k \mathcal{F}_\omega(x^k),$$

and from the properties of the Fourier transform we get

$$\mathcal{F}_\omega g = \frac{|\omega|}{2\pi} \sum_{k=0}^{+\infty} \left(\frac{\omega}{\mathrm{i}}\right)^k a_k \frac{2\pi}{|\omega|} \left(\frac{\mathrm{i}}{\omega}\right)^k \delta^{(k)} = \sum_{k=0}^{+\infty} a_k \delta^{(k)},$$

which implies the convergence of the multipole series in $\mathcal{G}'_\omega$.

(ii) If $\varrho = 2$, then for $|\omega| > 2\sigma_g$ we have the same result as in (i). However,

$$|\omega| > 2\sigma_g \Leftrightarrow |\omega| > 2|\omega|^2\sigma \Leftrightarrow 1 > 2|\omega|\sigma \Leftrightarrow |\omega| < \frac{1}{2\sigma}.$$

□

Corollary 1. *If the sequence* $\left(\sqrt[k]{k!|a_k|}\right)_{k\in\mathbb{N}}$ *is bounded, then for all* ω *in* $\mathbb{R} \setminus \{0\}$ *the multipole series* $\sum_{k=0}^{+\infty} a_k \delta^{(k)}$ *is convergent in* $\mathcal{G}'_\omega$.

Corollary 2. *Suppose the series* $\sum_{k=0}^{+\infty} a_k \delta^{(k)}$ *is such that the sequences* $\left(\sqrt[n]{n!|a_{2n}|}\right)_{n\in\mathbb{N}}$ *and* $\left(\sqrt[n]{n!|a_{2n+1}|}\right)_{n\in\mathbb{N}}$ *are bounded. Let* $\alpha > 0$ *be given by*

$$\alpha = \max\left\{\sup_{n\in\mathbb{N}} \sqrt[n]{n!|a_{2n}|},\ \sup_{n\in\mathbb{N}} \sqrt[n]{n!|a_{2n+1}|}\right\}.$$

Then for $0 \neq |\omega| < (2\alpha)^{-1}$ *this series is convergent in* $\mathcal{G}'_\omega$.

7. Some examples

The function defined by $f(z) = \mathrm{e}^{kz^2}$ has order 2 and type k. Therefore, by Theorem 13, we conclude that for ω in $]-(2k)^{-1}, 0[\,\cup\,]0, (2k)^{-1}[$ the multipole series

$$\sum_{n=0}^{+\infty} \frac{k^n}{n!} \delta^{(2n)}$$

is convergent in $\mathcal{G}'_\omega$.

Let $\varrho > 0$ and $\sigma > 0$. The function

$$f(z) = \sum_{n=1}^{+\infty} \left(\frac{\mathrm{e}\varrho\sigma}{n}\right)^{n/\varrho} z^n$$

constitutes an example of an entire function with order ϱ and type σ. By Theorem 13, we have, for all $\omega \in \mathbb{R} \setminus \{0\}$ and for each $\varrho \in]0, 2[$ the convergence of the multipole series

$$\sum_{n=1}^{+\infty} \left(\frac{\mathrm{e}\varrho\sigma}{n}\right)^{n/\varrho} \delta^{(n)}$$

in $\mathcal{G}'_\omega$. We also conclude that for $\omega \in]-(2\sigma)^{-1}, 0[\cup]0, (2\sigma)^{-1}[$ the multipole series

$$\sum_{n=1}^{+\infty} \left(\frac{2\mathrm{e}\sigma}{n}\right)^{n/2} \delta^{(n)}$$

converges in $\mathcal{G}'_\omega$.

Let $\alpha \in \mathbb{C}$. The function

$$\mathrm{e}^{\mathrm{i}\omega\alpha z} = \sum_{n=0}^{+\infty} \frac{(\mathrm{i}\omega\alpha)^n}{n!} z^n$$

is an entire function of order 1. Thus, by Theorem 10, it defines an element of $\mathcal{G}'_\omega$. Since its McLaurin series is convergent in this space, we can compute

$$\begin{aligned}
\mathcal{F}_\omega(\mathrm{e}^{\mathrm{i}\omega\alpha x}) &= \sum_{n=0}^{+\infty} \frac{(\mathrm{i}\omega\alpha)^n}{n!} \mathcal{F}_\omega(x^n) \\
&= \sum_{n=0}^{+\infty} \frac{(\mathrm{i}\omega\alpha)^n}{n!} \left(\frac{\mathrm{i}}{\omega}\right)^n \frac{2\pi}{|\omega|} \delta^{(n)} \\
&= \frac{2\pi}{|\omega|} \sum_{n=0}^{+\infty} \frac{(-\alpha)^n}{n!} \delta^{(n)}.
\end{aligned}$$

On the other hand,

$$\mathcal{F}_\omega(\mathrm{e}^{\mathrm{i}\omega\alpha x}) = \tau_\alpha \mathcal{F}_\omega 1 = \tau_\alpha \frac{2\pi}{|\omega|} \delta = \frac{2\pi}{|\omega|} \delta_\alpha,$$

whereby we obtain, in this new context, the well known equality

$$\delta_\alpha = \sum_{n=0}^{+\infty} \frac{(-\alpha)^n}{n!} \delta^{(n)}. \tag{18}$$

Let $\alpha \in \mathbb{C}$ and consider the space $\mathcal{G}_{\alpha,\omega}$ defined by

$$\mathcal{G}_{\alpha,\omega} = \operatorname{span}\{\varphi \in C(\mathbb{R})\colon\ \varphi(x) = p(x)\mathrm{e}^{-\frac{1}{2}(x-\alpha)^2|\omega|},\ p \in \mathcal{P}(\mathbb{R})\}.$$

In [5] the case $\alpha = 0$ was studied in detail, and it was shown that a necessary and sufficient condition for a multipole series

$$\sum_{k=0}^{+\infty} a_k \delta^{(k)} \tag{19}$$

to be convergent in $\mathcal{G}'_{0,\omega}$ corresponds to (16) with $\alpha = 0$. Suppose then that the multipole series above verifies (16) only for $\alpha = 0$. We note that the series is convergent in $\mathcal{G}'_{0,\omega}$ if and only if the series

$$\sum_{k=0}^{+\infty} a_k \delta_\alpha^{(k)}$$

is convergent in $\mathcal{G}'_{\alpha,\omega}$. The series in (19) is not convergent in $\mathcal{G}'_\omega$, but there is an element T of this space such that

$$T_{|\mathcal{G}_{\alpha,\omega}} = \sum_{k=0}^{+\infty} a_k \delta_\alpha^{(k)}.$$

This happens because the space $\mathcal{G}_\omega$ can be written as the topological direct sum of the $\mathcal{G}_{\alpha,\omega}$'s

$$\mathcal{G}_\omega = \bigoplus_{\alpha \in \mathbb{C}} \mathcal{G}_{\alpha,\omega}.$$

Thus, given $\psi \in \mathcal{G}_\omega$, we have

$$\psi = \sum_{\alpha \in A_\psi} \psi_\alpha,$$

where $A_\psi \subset \mathbb{C}$ and $\{\psi_\alpha\}_{\alpha \in A}$ are uniquely determined by ψ, while A_ψ is a finite set depending on ψ. We define the mapping T in the following manner:

$$\begin{aligned} T\colon\ \mathcal{G}_\omega &\longrightarrow \mathbb{R} \\ \psi &\longmapsto T\psi \end{aligned}$$

with

$$\forall \psi \in \mathcal{G}_\omega \qquad T\psi = \sum_{\alpha \in A_\psi} \sum_{n=0}^{+\infty} (-1)^n a_n \psi_\alpha^{(n)}(\alpha).$$

This is a linear map, therefore it is continuous, that is, $T \in \mathcal{G}'_\omega$, and

$$\forall \psi \in \mathcal{G}_{\alpha,\omega} \qquad T_{|\mathcal{G}_{\alpha,\omega}}\psi = \sum_{n=0}^{+\infty}(-1)^n a_n \psi^{(n)}(\alpha),$$

which means

$$T_{|\mathcal{G}_{\alpha,\omega}} = \sum_{k=0}^{+\infty} a_k \delta_\alpha^{(k)},$$

which suggests a different, and possibly new, notion of convergence.

Acknowledgement. Authors are thankful to the referees for linguistic corrections and valuable comments.

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