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ON VECTOR FUNCTIONS OF BOUNDED CONVEXITY

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Abstract. Let X be a normed linear space. We investigate properties of vector functions $F\colon [a,b] \to X$ of bounded convexity. In particular, we prove that such functions coincide with the delta-convex mappings admitting a Lipschitz control function, and that convexity $K_a^b F$ is equal to the variation of F'_+ on $[a,b)$. As an application, we give a simple alternative proof of an unpublished result of the first author, containing an estimate of convexity of a composed mapping.

Keywords: bounded convexity, delta-convex mapping, bounded variation, Banach space

MSC 2010: 47H99, 26A99

1. Introduction

If C is a convex subset of a (real) normed linear space X, then $f\colon C \to \mathbb{R}$ is called a *d.c. function* (or a delta-convex function) if it can be represented as a difference of two continuous convex functions on C. In [17], the notion of a d.c. function was extended to the notion of a *d.c. mapping between arbitrary Banach spaces* and a theory of such mappings was developed (see Introduction in [6] for a brief review). A well-known result of Roberts and Varberg ([12], [13]) asserts that a function $f\colon [a,b] \to \mathbb{R}$ is a difference of two Lipschitz convex functions if and only if f has a finite convexity $K_a^b f$. The notion of convexity goes back to de la Vallée Poussin (1908) and Riesz (1911) (see [13, p. 28]), and was already studied and applied also in the case of Banach space-valued functions ([14], [17], [4], [7]).

The first aim of this article is to present basic properties of vector functions of bounded convexity. All these properties either are known or follow by known methods, but the proofs need some effort. In particular, using [17, Theorem 9] and the ideas of its proof, we prove a generalization of the above-mentioned result of Roberts and Varberg:

A Banach space-valued function F on $[a, b]$ has a bounded convexity if and only if it is a d.c. mapping (in the sense of [17]*) which has a Lipschitz control function.*

Note that in [17], only d.c. mappings defined on *open* convex sets were studied. However, the definition of d.c. mappings on arbitrary convex sets has a good sense and even some deeper results can be proved for d.c. mappings on closed convex sets (see [19], where some Hartman's [9] results on compositions of d.c. functions are generalized).

We also prove that the equality $K_a^b F = V(F'_+, [a, b))$ holds for each mapping F with a finite convexity $K_a^b F$, which seems to be a new result for vector functions.

As an application of the main Theorem 3.1, we give in Section 4 a simple alternative proof of an unpublished result of the first author, containing an estimate of convexity of a composed mapping; this is the second aim of the present article.

2. Preliminaries

Let X be a Banach space and F an X-valued mapping defined a.e. on $[a, b]$. By the symbol $\int_a^b F$ we denote the Bochner integral (and so, the Lebesgue integral, if $X = \mathbb{R}$) with respect to the Lebesgue measure. If $a > b$, we set $\int_a^b F := -\int_b^a F$.

The least Lipschitz constant of a Lipschitz mapping F between metric spaces will be denoted by $\operatorname{Lip} F$. If X is a normed linear space, $M \subset \mathbb{R}$, and $F\colon M \to X$, we define the *variation of F on M* as

$$V(F, M) := \sup \left\{ \sum_{i=1}^{n} \|F(x_{i-1}) - F(x_i)\| \right\},$$

where the supremum is taken over all finite collections of points $x_0 < \ldots < x_n$ in M. We say that $F\colon [a, b] \to X$ is of bounded variation provided $V_a^b F := V(F, [a, b]) < \infty$. We set $V_a^a F := 0$ and $V_a^b F := -V_b^a F$ if $a > b$.

If F is absolutely continuous and a.e. differentiable on $[a, b]$, then

$$V_a^b F = \int_a^b \|F'(x)\| \, \mathrm{d}x. \tag{1}$$

Indeed, the absolute continuity of F implies (see [5, Theorem 3.3 and Remark 3.4]) that $V_a^b F = \int_a^b \operatorname{md}(F, x) \, \mathrm{d}x$, where $\operatorname{md}(F, x) := \lim_{t \to 0} \|F(x + t) - F(x)\| / |t|$ is the "metric derivative". Since clearly $\operatorname{md}(F, x) = \|F'(x)\|$ if $F'(x)$ exists, we obtain (1). (For mappings F which are Lipschitz and a.e. differentiable, (1) follows from [10, Theorem 7] and [8, 2.10.13].)

The following notion of convexity goes back to de la Vallée Poussin (1908); cf. [13].

Definition 2.1. Let X be a normed linear space and $F\colon [a,b]\to X$ a mapping. For every partition $D=\{a=x_0<x_1<\ldots<x_n=b\}$ of $[a,b]$, we put

$$K(F,D)=\sum_{i=1}^{n-1}\Bigl\|\frac{F(x_{i+1})-F(x_i)}{x_{i+1}-x_i}-\frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}}\Bigr\|.$$

(If $n=1$, we put $K(F,D):=0$.) Then the *convexity of F on* $[a,b]$ is defined as

$$K_a^bF=\sup K(F,D),$$

where the supremum is taken over all partitions D of $[a,b]$. If $K_a^bF<\infty$, we say that F *has a bounded (or finite) convexity.*

Remark 2.2. Clearly $K_a^bF=K_a^b(-F)=K_{-b}^{-a}\widetilde{F}$, where $\widetilde{F}(x):=F(-x)$, $x\in[-b,-a]$. Indeed, $K(F,D)=K(\widetilde{F},\widetilde{D})$ for $\widetilde{D}:=\{-x_n<\ldots<-x_1<-x_0\}$.

We state the following basic definition from [17] for mappings defined on arbitrary (not necessarily open) convex sets.

Definition 2.3. Let X,Y be normed linear spaces, let $C\subset X$ be a convex set, and let $F\colon C\to Y$ be a continuous mapping. We say that F is *d.c.* (or *delta-convex*) if there exists a continuous (necessarily convex) function $f\colon C\to\mathbb{R}$ such that $y^*\circ F+f$ is convex on C whenever $y^*\in Y^*$, $\|y^*\|=1$. In this case we say that f *controls* F, or that f is a *control function* for F.

Remark 2.4. Similarly to [17], it is easy to check the following fact: *if $Y=\mathbb{R}^n$ (equipped with an arbitrary norm) and $F=(F_1,\ldots,F_n)$, then the mapping F is d.c. if and only if all its components F_i are d.c.*

An elegant alternative equivalent definition of d.c. mappings is given by the property (ii) of the following Proposition 2.5. For the proof, it is sufficient to observe that the proof of the first part of [17, Proposition 1.13] does not use the assumption that A is open.

Proposition 2.5. *Let X, Y be normed linear spaces, let $A\subset X$ be a convex set, and let $F\colon A\to Y$, $f\colon A\to\mathbb{R}$ be continuous. Then the following conditions are equivalent.*

(i) *F is d.c. with a control function f.*

(ii) $\|\lambda F(x)+\mu F(y)-F(\lambda x+\mu y)\|\leqslant\lambda f(x)+\mu f(y)-f(\lambda x+\mu y)$ *whenever* $x\in A$, $y\in A$, $\lambda\geqslant 0$, $\mu\geqslant 0$, $\lambda+\mu=1$.

(iii) $$\Bigl\|\frac{F(z+kv)-F(z)}{k}-\frac{F(z)-F(z-hv)}{h}\Bigr\|\leqslant\frac{f(z+kv)-f(z)}{k}-\frac{f(z)-f(z-hv)}{h}$$

whenever $z\in A$, $v\in X$, $z+kv\in A$, $z-hv\in A$, $k>0$, $h>0$.

The following easy lemma is well-known.

Lemma 2.6. *Let X be a Banach space and $G\colon [a,b] \to X$ with $V(G,[a,b)) < \infty$. Then the limit $\lim_{x\to b-} G(x)$ exists in X. Moreover,*

$$V(G,[a,b]) = V(G,[a,b)) + \|G(b) - \lim_{x\to b-} G(x)\|.$$

Proof. Observe that G is bounded. Further, if $a \leqslant x_0 < x_1 < \ldots < x_n = b$, then $\sum_{i=1}^{n} \|G(x_{i-1}) - G(x_i)\| \leqslant V(G,[a,b)) + 2 \sup_{x\in[a,b]} \|G(x)\|$. Consequently, $V(G,[a,b]) < \infty$. Now the statement of the lemma follows from [2, Lemma 5.2]. □

Remark 2.7. Quite similarly we obtain the symmetric version of Lemma 2.6: $V(G,[a,b]) = V(G,(a,b]) + \|G(a) - \lim_{x\to a+} G(x)\|$ whenever $V(G,(a,b]) < \infty$.

Let us recall the definition of the one-sided strict derivative.

Definition 2.8. Let X be a Banach space. We say that $F\colon [x, x+\delta] \to X$ has at x a *strict right derivative* $A \in X$ if

$$\lim_{\substack{(y,z)\to(x,x)\\ y\neq z,\ y\geqslant x,\ z\geqslant x}} \frac{F(z)-F(y)}{z-y} = A.$$

(The strict left derivative is defined analogously.)

We will need also the following version of the mean value theorem (see Proposition 3 in [1, I.2]).

Lemma 2.9. *Let X be a Banach space, $F\colon [c,d] \to X$ a continuous mapping, and $A \in X$. Suppose that $F'_+(t)$ exists for each $t \in (c,d)$. Then*

$$\left\|\frac{F(d)-F(c)}{d-c} - A\right\| \leqslant \sup_{x\in(c,d)} \|F'_+(x) - A\|. \tag{2}$$

As an easy consequence we obtain the following lemma.

Lemma 2.10. *Let X be a Banach space, let $a < b$ be real numbers, and let $F\colon [a,b] \to X$ be a continuous mapping. Suppose that $F'_+(t)$ exists for each $t \in (a,b)$,* *and the limits* $\lim_{t\to a+} F'_+(t)$, $\lim_{t\to b-} F'_+(t)$ *exist. Then* $F'_+(a)$ *and* $F'_-(b)$ *exist as strict one-sided derivatives, and*

$$F'_+(a) = \lim_{t\to a+} F'_+(t) \quad \text{and} \quad F'_-(b) = \lim_{t\to b-} F'_+(t). \tag{3}$$

P r o o f. If $a \leqslant y < z < b$, we apply Lemma 2.9 with $[c,d] := [y,z]$ and $A := \lim_{t\to a+} F'_+(t)$, and obtain

$$\left\| \frac{F(z)-F(y)}{z-y} - \lim_{t\to a+} F'_+(t) \right\| \leqslant \sup_{x\in(y,z)} \|F'_+(x) - \lim_{t\to a+} F'_+(t)\|,$$

which implies that $F'_+(a) = \lim_{t\to a+} F'_+(t)$ is the strict right derivative of F at a. The latter part of the statement can be proved quite similarly. □

3. Properties of vector functions of bounded convexity

For the following easy fact see [14, Proposition 2.3]; the proof therein does not use the completeness of X.

Proposition A. *Let* X *be a normed linear space and* $F\colon [a,b]\to X$ *a mapping with* $K_a^b F < \infty$. *Then* F *is Lipschitz.*

The following Theorem B is also known; its first part coincides with [17, Theorem 2.3], and the second with [18, Lemma 3.10].

Theorem B. *Let* X *be a Banach space and* $F\colon (a,b)\to X$ *a continuous mapping. Then the following conditions are equivalent.*

(i) F *is d.c. on* (a,b).

(ii) $F'_+(x)$ *exists for each* $x\in(a,b)$, *and* $V_c^d F'_+ < \infty$ *for each* $[c,d]\subset(a,b)$.

(iii) $K_c^d F < \infty$ *for each interval* $[c,d]\subset(a,b)$.

Moreover, if the above equivalent conditions are satisfied and $z\in(a,b)$, *then*

$$F(x) = F(z) + \int_z^x F'_+, \quad x\in(a,b). \tag{4}$$

Using Theorem B and ideas of its proof, we prove the following main result of this article.

Theorem 3.1. *Let X be a Banach space and let $F\colon [a,b] \to X$ be continuous. Then the following conditions are equivalent.*

(i) *F is a d.c. mapping with a Lipschitz control function.*

(ii) $K_a^b F < \infty$.

(iii) *$F'_+(x)$ exists for each $x \in [a,b)$, and $V(F'_+, [a,b)) < \infty$.*

(iv) *There exists a mapping of bounded variation $G\colon [a,b] \to X$ such that $F(x) = F(a) + \int_a^x G$ for each $x \in [a,b]$.*

Moreover, if the above equivalent conditions hold, then

(a) $F(x) = F(a) + \int_a^x F'_+$ *for each* $x \in [a,b]$;

(b) $K_a^b F \leqslant f'_-(b) - f'_+(a)$ *whenever f is a control function of F;*

(c) *the function $f_a(x) := \int_a^x (V_a^t F'_+)\,\mathrm{d}t$, $x \in [a,b]$, is a Lipschitz control function of F;*

(d) $K_a^b F = V(F'_+, [a,b)) = (f_a)'_-(b) - (f_a)'_+(a)$;

(e) $K_a^b F = 2 \cdot \min\{\operatorname{Lip} f\colon\ f \text{ controls } F\}$.

Proof. Let f be a control function for the mapping F. Consider a partition $D = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a,b]$. By Proposition 2.5,

$$K_a^b(F,D) \leqslant \sum_{i=1}^{n-1} \Big(\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \Big)$$
$$= \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \leqslant f'_-(b) - f'_+(a).$$

It follows that (i) implies (ii) and (b).

To prove (ii) $\Rightarrow$ (iii), suppose $K_a^b F < \infty$. Then the right derivative $F'_+(x)$ exists for every $x \in [a,b)$ by [14, Proposition 2.4]. Consider a partition $D = \{a = x_0 < x_1 < \ldots < x_n = d\}$ of an interval $[a,d] \subset [a,b)$ and set

$$V(F'_+, D) := \sum_{i=0}^{n-1} \|F'_+(x_{i+1}) - F'_+(x_i)\|.$$

For an arbitrary $\varepsilon > 0$ find $\delta > 0$ such that $x_i + \delta < x_{i+1}$ $(0 \leqslant i \leqslant n-1)$, $d + \delta < b$, and $\|F'_+(x_i) - \frac{1}{\delta}(F(x_i+\delta) - F(x_i))\| < \varepsilon$ $(0 \leqslant i \leqslant n)$. Then

$$V(F'_+, D) \leqslant 2n\varepsilon + \sum_{i=0}^{n-1} \Big\| \frac{F(x_{i+1}+\delta) - F(x_{i+1})}{\delta} - \frac{F(x_i+\delta) - F(x_i)}{\delta} \Big\|$$
$$\leqslant 2n\varepsilon + \sum_{i=0}^{n-1} \Big(\Big\| \frac{F(x_{i+1}+\delta) - F(x_{i+1})}{\delta} - \frac{F(x_{i+1}) - F(x_i+\delta)}{x_{i+1} - (x_i+\delta)} \Big\|$$
$$+ \Big\| \frac{F(x_{i+1}) - F(x_i+\delta)}{x_{i+1} - (x_i+\delta)} - \frac{F(x_i+\delta) - F(x_i)}{\delta} \Big\| \Big) \leqslant 2n\varepsilon + K_a^{d+\delta} F.$$

Therefore we easily obtain that

$$V(F'_+,[a,b)) \leqslant K_a^b F \quad \text{whenever} \quad K_a^b F < \infty, \tag{5}$$

and so (iii) follows.

Now, suppose (iii). By Lemma 2.6, the limit $L := \lim_{x\to b-} F'_+(x)$ exists in X. Consider the following extension $G\colon (a-1,b+1) \to X$ of F'_+:

$$G(x) = \begin{cases} F'_+(x) & \text{for } x \in [a,b); \\ F'_+(a) & \text{for } x \in (a-1,a); \\ L & \text{for } x \in [b,b+1). \end{cases}$$

It is easy to see that $V(G,(a-1,b+1)) < \infty$ and G is the right derivative of the following extension of F to $(a-1,b+1)$:

$$\widetilde{F}(x) = \begin{cases} F(x) & \text{for } x \in [a,b]; \\ F(a) + F'_+(a)(x-a) & \text{for } x \in (a-1,a); \\ F(b) + L(x-b) & \text{for } x \in [b,b+1). \end{cases}$$

Now, applying Theorem B to $\widetilde{F}$, we obtain (a). Thus we have proved that (iii) implies (iv) and (a).

Let us prove (iv) $\Rightarrow$ (i). Let G be as in (iv). The function $v(x) := V_a^x G$ is bounded and nondecreasing on $[a,b]$. Consequently, the function $h(x) = \int_a^x v$ is Lipschitz and convex on $[a,b]$. We will show that h controls F. For each norm-one functional $y^* \in X^*$ we have

$$y^*(F(x)) + h(x) = y^*(F(a)) + \int_a^x (y^* \circ G + v), \quad x \in [a,b].$$

To obtain that $y^* \circ F + h$ is convex, it suffices to show that the function $y^* \circ G + v$ is nondecreasing on $[a,b]$ (see [13, I.12, Theorem A, and B on p. 13]); and this is easy. Indeed, $a \leqslant x < y \leqslant b$ implies $(y^* \circ G + v)(y) - (y^* \circ G + v)(x) = V_x^y G + y^*(G(y) - G(x)) \geqslant V_x^y G - \|G(y) - G(x)\| \geqslant 0$.

It remains to prove (c), (d) and (e). Let the (equivalent) conditions (i)–(iv) be satisfied. By (a), the condition (iv) holds with $G(x) = F'_+(x)$ for $x \in [a,b)$, $G(b) = \lim_{x\to b-} F'_+(x)$. Hence, we can use $h = f_a$ in the above proof of (iv) $\Rightarrow$ (i). This implies (c).

To show (d), observe that (5), (b) and (c) imply that

$$V(F'_+,[a,b)) \leqslant K_a^b F \leqslant (f_a)'_-(b) - (f_a)'_+(a) \leqslant (f_a)'_-(b) \tag{6}$$

since f_a is nondecreasing. Let $\{x_n\} \subset (a,b)$ be an increasing sequence of points of continuity of the nondecreasing function $t \mapsto V_a^t F'_+$ such that $x_n \to b$. Then the well-known properties of convex functions (see [13, (6) on p. 7]) and indefinite integrals give

$$(f_a)'_-(b) = \lim_{x\to b-} (f_a)'_+(x) = \lim_{n\to\infty} (f_a)'(x_n) = \lim_{n\to\infty} V_a^{x_n} F'_+ = V(F'_+, [a,b)).$$

Thus (6) implies (d).

The inequality "$\leqslant$" from (e) holds by (b). Set

$$f(x) := f_a(x) - \frac{(f_a)'_-(b) + (f_a)'_+(a)}{2} \cdot x \quad \text{for } x \in [a,b].$$

Then f clearly controls F, and (d) implies that $f'_-(b) = \frac{1}{2} K_a^b F$ and $f'_+(a) = -\frac{1}{2} K_a^b F$. Since f is convex, we have $\operatorname{Lip} f = \max\{|f'_+(a)|, |f'_-(b)|\} = \frac{1}{2} K_a^b F$; so we have proved (e). □

Remark 3.2. Note that the above proof gives that F is of bounded convexity on $[a,b]$ if and only if F is a restriction of a d.c. mapping $G\colon (a-1, b+1) \to X$ (cf. [4, Lemma 5.5.]).

The following result is an important supplement to Theorem 3.1.

Proposition 3.4. *Let X be a Banach space and let $F\colon [a,b] \to X$ be continuous. Then the following conditions are equivalent.*

(i) *$F'_+(x)$ exists for each $x \in (a,b)$, and $V(F'_+, (a,b)) < \infty$.*

(ii) *$K_a^b F < \infty$.*

(iii) *$F'_-(x)$ exists for each $x \in (a,b)$, and $V(F'_-, (a,b)) < \infty$.*

Moreover, if the above equivalent conditions hold, then

$$K_a^b F = V(F'_+, (a,b)) = V(F'_-, (a,b)). \tag{7}$$

Proof. Suppose that (i) holds. Choose $c \in (a,b)$. Applying Remark 2.7 (with $G(x) := F'_+(x)$ for $x \in (a,c]$ and $G(a) := 0$) we obtain that $\lim_{t\to a_+} F'_+(t)$ exists. By Lemma 2.10, $F'_+(a) = \lim_{t\to a+} F'_+(t)$. So, using Lemma 2.6 once more (now with $G(a) := F'_+(a)$), we obtain that $V(F'_+, (a,c]) = V(F'_+, [a,c])$. Consequently, $V(F'_+, (a,b)) = V(F'_+, [a,c]) + V(F'_+, [c,b)) = V(F'_+, [a,b)) = K_a^b F$ by Theorem 3.1 (iii), (d).

The implication (ii) $\Rightarrow$ (i) is obvious by Theorem 3.1. It remains to prove that (ii) $\Leftrightarrow$ (iii) and $K_a^b F = V(F'_-, (a,b))$, which can be done using Remark 2.2. □

In the rest of this section, we will deduce some further properties of vector functions of bounded convexity, which are well-known in the scalar case. They are mainly consequences of Theorem 3.1(d).

Proposition 3.4. *Let X be a Banach space, $F\colon [a,b] \to X$ with $K_a^b F < \infty$. Then*

(i) *F has a strict right derivative at each point of $[a,b)$, and a strict left derivative at each point of $(a,b]$;*

(ii) $\lim\limits_{t\to x+} F'_{\pm}(t) = F'_{+}(x)$ *for* $x \in [a,b)$, *and* $\lim\limits_{t\to x-} F'_{\pm}(t) = F'_{-}(x)$ *for* $x \in (a,b]$;

(iii) *the set* $A := \{x \in (a,b)\colon\ F'_{+}(x) \neq F'_{-}(x)\}$ *is at most countable and* $\sum\limits_{x\in A} \|F'_{+}(x) - F'_{-}(x)\| \leqslant K_a^b F$;

(iv) $K_a^b F = V_a^b G$, *where* $G(x) := F'_{+}(x)$ *for* $x \in [a,b)$ *and* $G(b) := F'_{-}(b)$;

(v) *if* $[c,d] \subset [a,b)$, *then* $V(F'_{+}, [c,d]) = K_c^d F + \|F'_{+}(d) - F'_{-}(d)\|$;

(vi) $K_a^b F = V_a^b H$, *where* $H(x) := F'_{-}(x)$ *for* $x \in (a,b]$, *and* $H(a) := F'_{+}(a)$.

Proof. For an arbitrary $x \in [a,b)$, Theorem 3.1 and Lemma 2.6 (with $[x,b]$ instead of $[a,b]$) imply existence of $\lim\limits_{t\to x+} F'_{+}(t)$. So, Lemma 2.10 implies that $F'_{+}(x)$ is the strict right derivative. Similarly, if $x \in (a,b]$, Remark 2.7 and Lemma 2.10 imply that $F'_{-}(x)$ exists and is the strict left derivative. Thus we have proved (i).

Obviously, (i) implies (ii); it is sufficient to use the definition of one-sided limits and definitions of one-sided and strict one-sided derivatives.

Let $\{x_1 < x_2 < \ldots < x_n\}$ be an arbitrary finite subset of A. By (ii), there exist points $y_1, y_2, \ldots, y_n$ such that $a < y_1 < x_1 < y_2 < x_2 < \ldots < y_n < x_n$ and $\|F'_{+}(y_i) - F'_{-}(x_i)\| < \varepsilon/n$ for $1 \leqslant i \leqslant n$. Then, by Theorem 3.1(d), we have

$$\begin{aligned}\sum_{i=1}^{n} \|F'_{+}(x_i) - F'_{-}(x_i)\| &\leqslant \sum_{i=1}^{n} \|F'_{+}(x_i) - F'_{+}(y_i)\| + \varepsilon \\ &\leqslant V(F'_{+}, [a,b)) + \varepsilon = K_a^b F + \varepsilon,\end{aligned}$$

which proves that $\sum\limits_{x\in A_0} \|F'_{+}(x) - F'_{-}(x)\| \leqslant K_a^b F$ for each finite set $A_0 \subset A$. Now, (iii) easily follows.

The properties (iv) and (v) follow immediately from Lemma 2.6, Theorem 3.1(d), and (ii). The property (vi) easily follows from (iv) via Remark 2.2. □

Remark 3.5. Note that properties (i) and (ii) follow immediately also from Remark 3.2 and [17, Note 3.2]. However, [17, Note 3.2] is stated with a hint of the proof only.

Proposition 3.6. *Let X be a Banach space, $a < c < b$, and $F\colon [a,b] \to X$. Then $K_a^b F < \infty$ if and only if both $K_a^c F < \infty$ and $K_c^b F < \infty$, and in this case*

$$K_a^b F = K_a^c F + K_c^b F + \|f'_+(c) - f'_-(c)\|.$$

P r o o f. By Proposition A, we can suppose that F is Lipschitz. By Lemma 2.6 and the first part of Theorem 3.1, the following chain of equivalences holds:

$$\begin{aligned} K_a^b F < \infty &\Leftrightarrow F'_+ \text{ exists on } [a,b) \text{ and } V(F'_+, [a,b)) < \infty \\ &\Leftrightarrow F'_+ \text{ exists on } [a,b) \text{ and } V(F'_+, [a,c)) < \infty,\ V(F'_+, [c,b)) < \infty \\ &\Leftrightarrow K_a^c F < \infty,\ K_c^b F < \infty. \end{aligned}$$

Moreover, if these conditions are satisfied, then Proposition 3.4(v) and Theorem 3.1(d) imply

$$\begin{aligned} K_a^b F &= V(F'_+, [a,b)) = V(F'_+, [a,c]) + V(F'_+, [c,b)) \\ &= K_a^c F + \|F'_+(c) - F'_-(c)\| + K_c^b F. \end{aligned}$$

□

Proposition 3.7. *Let X be a Banach space, $F\colon [a,b] \to X$ with $K_a^b F < \infty$. Let $p(x) := K_a^x F$ for $x \in (a,b]$ and $q(x) := K_x^b F$ for $x \in [a,b)$. Then*
(i) *p is left-continuous at each $x \in (a,b]$ and $p(a+) = 0$;*
(ii) *$p(x+) - p(x) = \|F'_+(x) - F'_-(x)\|$ for each $x \in (a,b)$;*
(iii) *q is right-continuous at each $x \in [a,b)$, $q(b-) = 0$;*
(iv) *$q(x-) - q(x) = \|F'_+(x) - F'_-(x)\|$ for each $x \in (a,b)$.*

P r o o f. Fix $x \in (a,b]$. By Theorem 3.1(d) and the definition of variation, we have

$$p(x-) = \lim_{t \to x-} V(F'_+, [a,t)) = \sup_{t \in (a,x)} V(F'_+, [a,t)) = V(F'_+, [a,x)) = p(x).$$

So the first part of (i) is proved. Further, consider $t \in (a,b)$ and use (7) to obtain

$$0 \leqslant p(t) = V(F'_+, (a,t)) \leqslant V(F'_+, (a,t]) = V(F'_+, (a,b]) - V(F'_+, [t,b]).$$

Since clearly $\lim\limits_{t \to a+} V(F'_+, [t,b]) = V(F'_+, (a,b])$, the second part of (i) follows.

Now, fix $x \in (a,b)$ and observe that, by Proposition 3.6,

$$p(t) = p(x) + \|F'_+(x) - F'_-(x)\| + K_x^t F \quad \text{for } t \in (x,b).$$

Using the second part of property (i) (with $[x,b]$ instead of $[a,b]$), we obtain $\lim\limits_{t\to x+} K_x^t F = 0$, and consequently (ii).

The parts (iii), (iv) can be proved similarly. We can also apply (i) and (ii) to the mapping $\widetilde{F}(x) = F(-x),\ x \in [-b,-a]$, using Remark 2.2. □

The following result generalizes [12, Corollary on p. 571].

Theorem 3.8. *Let X be a Banach space, $F\colon [a,b] \to X$ with $K_a^b F < \infty$. Let F be differentiable on $[a,b]$, and let F' be absolutely continuous and a.e. differentiable. Then*

$$K_a^b F = \int_a^b \|F''(x)\| \,\mathrm{d}x. \tag{8}$$

Proof. It is sufficient to apply Proposition 3.4(iv) and (1). □

Of course, if X has the Radon-Nikodým property, then we can ommit the assumption of a.e. differentiability of F' in Theorem 3.8, since it follows from absolute continuity of F (see [3]).

Finally, we show that for continuous mappings F convexity can be defined in a natural alternative way.

If $F\colon [a,b] \to X$ and $a \leqslant u < v \leqslant b$ are given, then we denote $Q_F(u,v) := (F(v) - F(u))(v-u)^{-1}$. For a partition $D = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a,b]$, we define its norm $\nu(D) := \max\{x_i - x_{i-1}\colon\ 1 \leqslant i \leqslant n\}$.

Proposition 3.9. *Let X be a normed linear space and $F\colon [a,b] \to X$. Then the following statements hold.*

(i) *If D_1 and D_2 are partitions of $[a,b]$ and D_2 is a refinement of D_1, then $K(F,D_1) \leqslant K(F,D_2)$.*

(ii) *If F is continuous and $(D^n)_1^\infty$ is a sequence of partitions of $[a,b]$ with $\lim\limits_{n\to\infty} \nu(D^n) = 0$, then $\lim\limits_{n\to\infty} K(F,D^n) = K_a^b F$.*

Proof. The statement (i) immediately follows from [14, Lemma 2.2] (considering, e.g., F as the mapping into the completion of X). To prove (ii), consider an arbitrary real number $A < K_a^b F$. Now find a partition $D = \{a = x_0 < x_1 < \ldots < x_k = b\}$ with $K(F,D) > A$. Set $\varepsilon := (K(F,D) - A)/k$. Continuity of F easily gives existence of $\delta > 0$ such that

$$\big|\|Q_F(x_{i-1},x_i) - Q_F(x_i,x_{i+1})\| - \|Q_F(\tilde{x}_{i-1},\tilde{x}_i) - Q_F(\tilde{x}_i,\tilde{x}_{i+1})\|\big| < \varepsilon \tag{9}$$
$$\text{whenever } a \leqslant \tilde{x}_0 \leqslant \tilde{x}_1 \leqslant \ldots \leqslant \tilde{x}_k \leqslant b \text{ and } |x_i - \tilde{x}_i| < \delta \text{ for } i = 0,\ldots,k.$$

Choose $n_0 \in \mathbb{N}$ such that

$$\nu(D^n) < \tilde{\delta} := \min\bigl\{\delta, \tfrac{1}{2}\min\{x_i - x_{i-1}\colon\ 1 \leqslant i \leqslant k\}\bigr\}$$

for each $n > n_0$. Fix an $n > n_0$. Then we can easily find $\tilde{x}_i \in D^n$ such that $|x_i - \tilde{x}_i| < \tilde{\delta}$ for $i = 1, \ldots, k-1$. Then clearly $\widetilde{D} := \{\tilde{x}_0 := a < \tilde{x}_1 < \ldots < \tilde{x}_k := b\}$ is a partition of $[a,b]$. Using (9) and the definition of ε, we obtain $K(F, \widetilde{D}) > A$. Since D^n is a refinement of $\widetilde{D}$, we have $K(F, D^n) > A$ by (i), which completes the proof. □

4. Convexity of a composed mapping

Now we will give a short alternative proof of an unpublished result (Theorem 4.1 below) of the first author [16] on convexity of a composed mapping. We show that this result is an easy consequence of the following Proposition C (originally essentially proved in [15]) and Theorem 3.1.

Proposition C. *Let X, Y, Z be normed linear spaces, $A \subset X$ and $B \subset Y$ convex sets. Let $F\colon\ A \to B$ and $G\colon\ B \to Z$ be d.c. mappings with control functions $f\colon\ A \to \mathbb{R}$ and $g\colon\ B \to \mathbb{R}$, respectively. If G and g are Lipschitz on B with Lipschitz constants L_G and L_g, then $G \circ F$ is d.c. on A with a control function $h = g \circ F + (L_G + L_g)f$.*

P r o o f. This was proved in [17] (Proposition 4.1) assuming that the sets A, B are also open. However, it is easy to see that the proof does not need this additional assumption, since it is based on Proposition 2.5 which holds for an arbitrary convex set A. □

Theorem 4.1. *Let X, Y be Banach spaces, $A \subset X$ a convex set, and let $F\colon\ A \to Y$ be a nonconstant Lipschitz mapping which admits a Lipschitz control function f. Let $\varphi\colon\ [a,b] \to A$ be Lipschitz. Then*

$$K_a^b\,(F \circ \varphi) \leqslant (\operatorname{Lip} F + \operatorname{Lip} f) K_a^b\,\varphi + 2 \operatorname{Lip} f \operatorname{Lip} \varphi. \tag{10}$$

P r o o f. Since F is nonconstant, $\operatorname{Lip} F > 0$, and thus we can suppose $K_a^b \varphi < \infty$. By Theorem 3.1(c),(d) we can choose a Lipchitz control function c of φ such that $K_a^b \varphi = c'_-(b) - c'_+(a)$. By Proposition C we obtain that $h := f \circ \varphi + (\operatorname{Lip} F + \operatorname{Lip} f) \cdot c$ controls $F \circ \varphi$. So, by Theorem 3.1(b), we obtain

$$\begin{aligned} K_a^b(F \circ \varphi) &\leqslant h'_-(b) - h'_+(a) \\ &= ((f \circ \varphi)'_-(b) - (f \circ \varphi)'_+(a)) + (\operatorname{Lip} F + \operatorname{Lip} f)(c'_-(b) - c'_+(a)) \\ &\leqslant (\operatorname{Lip} F + \operatorname{Lip} f) K_a^b \varphi + 2 \operatorname{Lip} f \operatorname{Lip} \varphi. \end{aligned}$$

□

Note that Theorem 4.1 is related to the following open problem ([17, Problem 7]): *Let X, Y be Banach spaces, $A \subset X$ an open convex set, and $F\colon A \to Y$ a Lipschitz mapping. Suppose that there are $\alpha \geqslant 0$, $\beta \geqslant 0$ such that*

$$K_0^1(F \circ \varphi) \leqslant \alpha K_0^1 \varphi + \beta \operatorname{Lip} \varphi \tag{11}$$

whenever $\varphi\colon [0,1] \to A$ is Lipschitz. Is then F d.c. on A?

Since every d.c. mapping on (an open convex set) A is locally Lipschitz, Theorem 4.1 immediately implies that, if F is d.c. on A, then the corresponding α, β always exist locally. More precisely, every $x_0 \in A$ is contained in an open convex set $A_0 \subset A$ such that, for some $\alpha, \beta \geqslant 0$, (11) holds for each Lipschitz curve $\varphi\colon [0,1] \to A_0$.

Note that this problem is only one version of the following natural rough question: *Is it possible to characterize d.c. functions (or even mappings) of more variables in the language of "curves" (i.e. mappings of one real variable) only?*

Another version of this question is the following Problem 6 of [17]: *Let X, Y be Banach spaces, $A \subset X$ an open convex set, and $F\colon A \to Y$ a mapping. Suppose that $F \circ \varphi$ is d.c. on $(0,1)$ whenever $\varphi\colon (0,1) \to A$ is d.c. Is then F locally d.c. on A?*

This problem was answered in negative for $X = \ell^3$, $Y = \ell^\infty$ in [6] and also for $X = \ell^3$ and $Y = \mathbb{R}$ (see [11]), but it is open, e.g., for $X = \mathbb{R}^n$, $Y = \mathbb{R}$.

Finally, we note that (10) can be improved in an interesting special case, and establish an easy estimate for convexity of an inverse function.

Proposition 4.2. *Let $\varphi\colon [a,b] \to [c,d]$ be an increasing continuous bijection with $K_a^b \varphi < \infty$. Let Y be a Banach space, $F\colon [c,d] \to Y$ with $K_c^d F < \infty$. Let f be a control function of F. Then:*

(i)

$$K_a^b\ (F \circ \varphi) \leqslant \operatorname{Lip} F \cdot K_a^b\ \varphi + \operatorname{Lip} \varphi \cdot K_c^d F \leqslant \operatorname{Lip} F \cdot K_a^b\ \varphi + 2 \operatorname{Lip} f \cdot \operatorname{Lip} \varphi;$$

(ii) *the function φ^{-1} has bounded convexity if and only if it is Lipschitz; in this case*

$$K_c^d \varphi^{-1} \leqslant \left(\operatorname{Lip}(\varphi^{-1})\right)^2 \cdot K_a^b \varphi. \tag{12}$$

P r o o f. Let $t_0 < t_1 < \ldots < t_n$ be arbitrary points in $[a,b)$. Observe that $(F\circ\varphi)'_+(t) = F'_+(\varphi(t))\cdot\varphi'_+(t)$ for each $t\in[a,b)$, and thus

$$\begin{aligned}\|(F\circ\varphi)'_+(t_{i+1}) - (F\circ\varphi)'_+(t_i)\| &\leqslant \|F'_+(\varphi(t_{i+1}))\varphi'_+(t_{i+1}) - F'_+(\varphi(t_{i+1}))\varphi'_+(t_i)\| \\ &\quad + \|F'_+(\varphi(t_{i+1}))\varphi'_+(t_i) - F'_+(\varphi(t_i))\varphi'_+(t_i)\| \\ &\leqslant \sup_{x\in[c,d)}\|F'_+(x)\|\cdot|\varphi'_+(t_{i+1}) - \varphi'_+(t_i)| \\ &\quad + \sup_{t\in[a,b)}|\varphi'_+(t)|\cdot\|F'_+(\varphi(t_{i+1})) - F'_+(\varphi(t_i))\|.\end{aligned}$$

Summing over i and using Theorem 3.1(d), we easily obtain the first inequality in (i). Now, the second inequality in (i) follows by Theorem 3.1(e).

To prove (ii), first observe that if φ^{-1} has bounded convexity, then it is Lipschitz by Proposition A. So, suppose that φ^{-1} is Lipschitz. Let $x_0 < x_1 < \ldots < x_n$ be arbitrary points in $[c,d)$, and $t_i := \varphi^{-1}(x_i)$. Then

$$\begin{aligned}\big|(\varphi^{-1})'_+(x_{i+1}) - (\varphi^{-1})'_+(x_i)\big| &= \Big|\frac{1}{\varphi'_+(t_{i+1})} - \frac{1}{\varphi'_+(t_i)}\Big| \\ &= |\varphi'_+(t_{i+1}) - \varphi'_+(t_i)|\cdot|(\varphi^{-1})'_+(x_{i+1})|\cdot|(\varphi^{-1})'_+(x_i)|.\end{aligned}$$

Summing over i and using Theorem 3.1(d), we easily obtain (12). □

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