Czechoslovak Mathematical Journal, 60 (135) (2010), 801–816

ON ZEROS OF CHARACTERS OF FINITE GROUPS

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(Received March 3, 2009)

Abstract. For a finite group G and a non-linear irreducible complex character χ of G write $\upsilon(\chi) = \{g \in G \mid \chi(g) = 0\}$. In this paper, we study the finite non-solvable groups G such that $\upsilon(\chi)$ consists of at most two conjugacy classes for all but one of the non-linear irreducible characters χ of G. In particular, we characterize a class of finite solvable groups which are closely related to the above-mentioned question and are called solvable φ-groups. As a corollary, we answer Research Problem 2 in [Y. Berkovich and L. Kazarin: Finite groups in which the zeros of every non-linear irreducible character are conjugate modulo its kernel. Houston J. Math. 24 (1998), 619–630.] posed by Y. Berkovich and L. Kazarin.

Keywords: finite groups, characters, zeros

MSC 2010: 20C15

1. Introduction

Let G be a finite group and $\upsilon(\chi) := \{g \in G \mid \chi(g) = 0\}$, where χ is an irreducible complex character of G. A classical theorem of Burnside asserts that $\upsilon(\chi)$ is non-empty for all $\chi \in \mathrm{Irr}_1(G)$, where $\mathrm{Irr}_1(G)$ denotes the set of non-linear irreducible characters of G. It makes sense to consider the structure of a finite group whose character table contains a small number of zeros (see [1], [2] and [18] for examples).

Y. Berkovich and L. Kazarin [1] posed the following question: classify the finite groups G with the following property:

$(*)$: $\upsilon(\chi)$ is a conjugacy class for all but one of the non-linear irreducible characters χ of G.

For the question, we define

Project supported by the NNSF of China (Grant No. 10871032) and the NSF of Sichuan University of Science and Engineering (Grant No. 2009XJKRL011).

Definition. A non-abelian group G is said to be a φ-group if G has exactly one non-linear irreducible character φ such that $\varphi_{G'}$ is not irreducible.

We first characterize the solvable φ-groups.

Theorem A. *Let G be a solvable group. Then G has exactly one non-linear irreducible character φ such that $\varphi_{G'}$ is not irreducible if and only if one of the following holds:*

(1) *G is a 2-transitive Frobenius group with kernel G' or an extra-special 2-group.*

(2) $G \cong \mathrm{SL}(2,3)$.

(3) $G \cong \mathrm{S}_4$.

(4) *G is a semidirect product of $\mathrm{SL}(2,3)$ and the natural $\mathrm{SL}(2,3)$-module M. Furthermore, G' is a 2-transitive Frobenius group with kernel M and complement isomorphic to $\mathbb{Q}_8$ (the quaternion group of order 8).*

Indeed, we study the finite groups G with the following property:

$(**)$: $\upsilon(\chi)$ consists of at most two conjugacy classes for all but one of the non-linear irreducible characters χ of G.

Theorem B. *Let G be a finite non-solvable group. Then G satisfies property $(**)$ if and only if G is isomorphic to A_5, S_5, $\mathrm{L}_2(7)$, or A_6.*

By Theorem A and Theorem B, we get the following Corollary, which is the main Theorem of [27].

Corollary. *Let G be a finite non-abelian group. Then G satisfies property $(*)$ if and only if G is one of the following groups:*

(1) *G is a 2-transitive Frobenius group with kernel G' or an extra-special 2-group;*

(2) *G is a Frobenius group with kernel G' of order greater than 3 and complement of order 2;*

(3) $G \cong \mathrm{SL}(2,3)$;

(4) $G \cong \mathrm{S}_4$;

(5) $G \cong \mathrm{A}_5$.

In this paper, G always denotes a finite group. Notation is standard and taken from [9]. In particular, $\mathrm{cd}(G)$ denotes the set of irreducible character degrees of G, and $k_G(N)$ the number of conjugacy classes of G contained in N, where N is a normal subset of G. For $N \lhd G$, set $\mathrm{Irr}(G|N) = \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$.

We shall freely use the following facts: Let $N \lhd G$ and write $\overline{G} = G/N$.

(1) For any $x \in G$, $\bar{x}^{\overline{G}}$ (when viewed as a subset of G, that is, the set $\bigcup_{g\in G} x^g N$) is a union of conjugacy classes of G; furthermore, $k_G(\bar{x}^{\overline{G}}) = 1$ if and only if $\chi(x) = 0$ for all $\chi \in \mathrm{Irr}(G|N)$.

(2) If G has property $(*)$, then so has G/N.
(3) If G has property $(**)$, then so has G/N.

2. On solvable φ-groups

First, we give some lemmas for proving Theorem A.

Lemma 2.1 ([15, Theorem 19.3]). *Suppose that H acts non-trivially on N and fixes every non-linear irreducible character of N. Assume that $(|N|,|H|)=1$. Set $M=[N,H]$. Assume that H is solvable. Then $N'=M'$ and one of the following occurs:*
(1) *N is abelian;*
(2) *M is a p-group of class 2 and $N' \leqslant Z(NH)$; or*
(3) *M is a Frobenius group with kernel M'.*

Lemma 2.2 ([15, Lemma 19.1]). *Let P be a p-group of class $\leqslant 2$ and suppose that P acts non-trivially on some p'-group Qsuch that $C_P(x) \subseteq P'$ for all $x \in Q-\{1\}$. Then the action is Frobenius and P is either cyclic or isomorphic to $\mathbb{Q}_8$.*

Lemma 2.3 ([17, Lemma 1.10]). *Let V, N be non-trivial normal subgroups of G such that G/V is a Frobenius group with cyclic kernel N/V of order b and with a cyclic complement of order a. If N is also a Frobenius group with kernel V, an elementary abelian group, then $ib \in cd(G)$ for any non-trivial divisor i of a.*

Lemma 2.4 ([14, Theorem]). *Let $Z \lhd G$, G/Z be p-solvable and $\lambda \in \mathrm{Irr}(Z)$. Suppose that $p \nmid \chi(1)/\lambda(1)$ for all $\chi \in \mathrm{Irr}(G|\lambda)$. Then the Sylow p-subgroups of G/Z are abelian.*

For a finite group G, if $G' < G$ and $|C_G(g)| = |C_{G/G'}(gG')|$ holds for any $g \in G - G'$, then (G, G') is called a Camina pair.

Lemma 2.5 ([10, Theorem 2.1]). *Let (G, G') be a Camina pair. Suppose that G is not a p-group. Then either G is a Frobenius group with kernel G' or G/G' is a p-group for some prime p; in this case, G has a normal p-complement M, $M < G'$ and $C_G(m) \subseteq G'$ for all $m \in M-\{1\}$.*

The following Lemma is a well-known fact.

Lemma 2.6. *Let P be a non-abelian 2-group. If $|P/P'| = 4$ and $\mathrm{Aut}(P)$ is not a 2-group, then $P \cong \mathrm{Q}_8$.*

Proof of Theorem A. Let $\varphi \in \mathrm{Irr}_1(G)$ be the unique irreducible character such that $\varphi_{G'}$ is not irreducible. Then the hypothesis yields that $\chi_{G'} \in \mathrm{Irr}_1(G')$ for every $\chi \in \mathrm{Irr}_1(G) - \{\varphi\}$, which implies $\mathrm{Irr}_1(G/G'') = \{\varphi\}$. Observe that any non-linear irreducible character of G' is extendible to G.

Suppose, first, that $G'' = 1$. Then either G is a 2-transitive Frobenius group with kernel G' or G is an extra-special 2-group (see [21]), and thus G satisfies (1) of the Theorem.

We, now, suppose that $G'' \neq 1$. Then either G/G'' is a 2-transitive Frobenius group with kernel G'/G'' and cyclic complement or G/G'' is an extra-special 2-group (note that $\mathrm{Irr}_1(G/G'') = \{\varphi\}$).

Case 1. G/G'' is a 2-transitive Frobenius group with kernel G'/G''.

Note the set of all non-identity elements of G'/G'' is a conjugacy class of G/G'', and that G'/G'' is an elementary abelian group of order p^m.

By Theorem 12.4 of [9] and its proof, we obtain that for any $\psi \in \mathrm{Irr}_1(G')$, either ψ vanishes on $G' - G''$ or $(\lambda\psi)^G$ is irreducible for some $\lambda \in \mathrm{Irr}(G'/G'')$. Recall that any non-linear irreducible character of G' is extendible to G; it follows that ψ vanishes on $G' - G''$ for all $\psi \in \mathrm{Irr}_1(G')$, and so (G', G'') is a Camina pair. Hence we have to consider the following three cases.

Subcase 1.1. Assume that G' is a p-group.

Then G' is a normal Sylow p-subgroup of G, and thus $G = G'H$, where H is a p-complement of G and $|H| = |G/G'| = p^m - 1$. Furthermore, we have $G' = [G', H]G'' = [G', H]\Phi(G') = [G', H]$. Since H fixes every non-linear irreducible character of G' (note that any non-linear irreducible character of G' is extendible to G), we have $G'' \leqslant Z(G)$ (see Lemma 2.1), and thus H acts trivially on G''. Observe that $G' - G''$ is a conjugcy class of G, so that H acts irreducibly on G'/G''. Since G' is a p-group of class 2 (because $G'' \leqslant Z(G)$), G' is a special p-group (see [12]). By [6, IX, Theorem 6.5], we conclude that $m = 2$, $p = 2$, $|G'/G''| = p^m = 4$ and $|H| = 3$. It follows that $G' \cong \mathbb{Q}_8$, so that $G = G'H \cong \mathrm{SL}(2, 3)$. Hence G satisfies (2) of the Theorem.

Subcase 1.2. Assume that G' is a Frobenius group with kernel G''.

Since G'/G'' is an elementary abelian of order p^m, we conclude that $|G'/G''| = p$ and $|G/G''| = p(p-1)$ (note that G'/G'' is cyclic).

We, first, claim that G'' is a 2-group. Assume otherwise. To reach a contradiction, we may assume that G'' is a minimal normal subgroup of G. Then G'' is an elementary abelian q-group with $q \neq 2$, $G'' = F(G)$ and $|G'|$ is odd. Hence G' has no non-principal real irreducible character, and so every character in $\mathrm{Irr}_1(G) - \{\varphi\}$ is not real (because any non-linear irreducible character of G' is extendible to G). Note that φ must be real (since φ is only one non-linear irreducible character of G/G''). Since G/G' is a cyclic group of order $p-1$, we see that G has exactly three real irreducible characters, namely, 1_G, φ and λ ($\lambda^2 = 1_G$). It follows that G has exactly three real classes. Observing that $G' - G''$ and $\{1\}$ are two real classes of G ($G' - G''$ consists of all elements of order p in G), we conclude that the set of all involutions is a real class. Since G/G'' is a Frobenius group of order $p(p-1)$, G/G'' has p involutions (they are contained in $G/G'' - G'/G''$). Let $z \in G - G'$ be an involution. Suppose that there exists an element $y \in G'' - \{1\}$ such that yz is an involution. Then we have $y^z = y^{-1}$, and thus G has at least 4 real classes, a contradiction. Hence, for every involution $z \in G - G'$ and every element $y \in G'' - \{1\}$, yz is not an involution. Thus we conclude that G has exactly p involutions. Since the p involutions form a conjugacy class, for every involution x in G we obtain that $p = |G\colon C_G(x)|$ and $|C_G(x)| = |G|/p = |G''|(p-1)$. It follows that $G'' \subseteq C_G(x)$, and so $x \in C_G(G'') = C_G(F(G)) \leqslant F(G) = G''$. This implies that $|G''|$ is not odd, a contradiction. Hence our claim is true.

Now we claim that $p = 3$. We may assume that G'' is a minimal normal subgroup of G. Recall that $\chi_{G'}$ is irreducible for all $\chi \in \mathrm{Irr}_1(G) - \{\varphi\}$, since G' is a Frobenius group with kernel G'' and complement of order p, we easily conclude that $\mathrm{cd}(G) = \{1, p-1, p\}$. It follows by Lemma 2.3 that $p-1$ is a prime and thus $p = 3$.

Next we show that $G \cong \mathrm{S}_4$. Notice that $|G/G'| = 2$. Suppose $G''' \neq \{1\}$. Since G/G''' satisfies the hypothesis $|G/G'''/(G/G''')'| = |G/G'| = 2$, we obtain that $G/G''' \cong \mathrm{S}_4$ by induction, and thus $|G''/G'''| = 4$. We easily see that $G'' \cong \mathbb{Q}_8$ and so $3|(8-1)$, a contradiction. So $G''' = \{1\}$. Observe that $\mathrm{cd}(G) = \mathrm{cd}(G') \cup \varphi(1) = \{1, 2, 3\}$. Hence $G \cong \mathrm{S}_4$ (see [1, Corollary]), and thus G satisfies (3) of the Theorem.

Subcase 1.3. Assume that G' has a normal p-complement M, $M < G''$ and $C_{G'}(m) \subseteq G''$ for all $m \in M - \{1\}$.

Note that G/M is a solvable φ-group and $(G'/M, G''/M)$ is a Camina pair. Since G'/M is a p-group, arguing as in Subcase 1.1, we have that $p = 2$, and that $G/M \cong \mathrm{SL}(2,3) = \mathbb{Q}_8 \rtimes C(3)$. It follows by Lemma 2.2 that $G' = M \rtimes \mathbb{Q}_8$ is a Frobenius group with kernel M and complement isomorphic to $\mathbb{Q}_8$.

Now we show that M is a minimal normal subgroup of G. Otherwise, let $E \lhd G$ be such that M/E is a minimal normal subgroup of G/E. Then as shown in the above two paragraphs, M/E is an elementary abelian group of order 9 and $G/E/M/E \cong \mathrm{SL}(2,3)$. For any non-principal $\lambda \in \mathrm{Irr}(E)$ and $\chi \in \mathrm{Irr}(G|\lambda)$, we have 3 does not divide $\chi(1)$ (in fact, $\chi(1) = 8$). Then it follows by Lemma 2.4 that

G/E has an abelian Sylow 3-subgroup, which is impossible (because, as shown in the above paragraph, M/E is a faithful $G/E/M/E$-module). Hence M is a minimal normal subgroup of G.

Hence M is an elementary abelian q-group and $G/M \cong \mathrm{SL}(2,3)$ acts irreducibly on M. Observe that $\mathrm{cd}(G) = \mathrm{cd}(G') \cup \varphi(1) = \{1, 2, 2^3, 3\}$. It follows from [13] that $q = 3$. Hence M is an irreducible $GF(3)[\mathrm{SL}(2,3)]$-module in which every element of order 3 has a quadratic minimal polynomial, and so M is the standard module for $\mathrm{SL}(2,3)$ (see [4, Corollary 5.2]). This implies that M is an elementary abelian group of order 9, and thus G satisfies (4) of the Theorem.

Case 2. Suppose that G/G'' is an extra-special 2-group.

Now we show that the case does not occur. To reach a contradiction, we may assume that G'' is a minimal normal subgroup of G with order q^s. Suppose that $q = 2$, thus G is a 2-group. We easily conclude that this is impossible. So $q \neq 2$. Let $\lambda \in \mathrm{Irr}_1(G')$. Since λ is extendible to G, we obtain that $\ker(\lambda) = \ker(\chi) \cap \mathrm{G}' \lhd \mathrm{G}$. Note that both G'/G'' and G'' are chief factors of G, so we conclude that λ is faithful for all $\lambda \in \mathrm{Irr}_1(G')$. Since each normal subgroup of G' is an intersection of the kernels of some irreducible characters of G', we see that G'' is the unique minimal normal subgroup of G'. Note that $q \neq 2$ and $|G'/G''| = 2$, so it follows from [9, Corollary 12.3] that G' is a Frobenius group with kernel G'' and complement of order 2. Since $G'/G'' = Z(G/G'')$, all elements of $\mathrm{Irr}(G'/G'')$ are G-invariant. For $\psi \in \mathrm{Irr}_1(G')$, ψ is G-invariant. Therefore, all elements of $\mathrm{Irr}(G')$ are invariant under G, and thus all the conjugacy classes of G' are G-invariant. For any element $x \in G'' - \{1\}$, we have $|x^{G'}| = 2$, and so $|x^G| = 2$, thus $|C_G(x)| = |G|/2$. Suppose that P_1 is a Sylow p-subgroup of $C_G(x)$, we easily conclude that $|P_1| = |P|/2$, where P is a Sylow p-subgroup of G such that $P_1 \subset P$. Note that $P'G'' = G'$ is a Frobenius group, so we have that $P_1 \cap P = \{1\}$, which is impossible since $P' = Z(P)$. □

Remark. Ren and Zhang [20] have studied the solvable φ-groups. Here, we give the complete classification of solvable φ-groups.

3. Non-solvable group with property (∗∗)

In what follows, we shall freely use the following facts:

Suppose that G is a simple group of Lie type. Then by [24, Corollary], for each prime factor p of $|G|$ there exists some $\chi \in \mathrm{Irr}_1(G)$ such that χ is of p-defect zero. For such χ, we have $\{x \in G \mid p \mid o(x)\} \subseteq \upsilon(\chi)$ (see [9, Theorem 8.17]), and thus $k_G(\{x \in G \mid p \mid o(x)\}) \leqslant k_G(\upsilon(\chi))$.

Lemma 3.1 ([26, Theorem 3.6]). *Let G be a non-abelian simple group of Lie type except for $\mathrm{L}_2(q)$ where $q \geqslant 4$, $\mathrm{L}_3(4)$, $\mathrm{Sz}(2^{2m+1})$ where $m \geqslant 1$. Then there exist $\xi, \eta \in \mathrm{Irr}_1(G)$ such that ξ is of 2-defect zero and η is of s-defect zero, and $\xi(1) \neq \eta(1)$, where s is an odd prime; furthermore, one of them vanishes on elements of at least four distinct orders, and the other vanishes on elements of at least three distinct orders.*

Lemma 3.2. *Let $G \cong \mathrm{Sz}(2^{2m+1})$ where $m \geqslant 1$, then G does not satisfy property $(**)$.*

Proof. Let $\alpha, \beta \in \mathrm{Irr}(G)$ with $\alpha(1) = (2^{2m+1} - 1)(2^{2m+1} - 2^{m+1} + 1)$ and $\beta(1) = (2^{2m+1} - 1)(2^{2m+1} + 2^{m+1} + 1)$ (see [8, XI, Theorem 5.10]). Note that $\pi_e(G) = \{1, 2, 4, \text{all factors of } (2^{2m+1}-1), (2^{2m+1}-2^{m+1}+1) \text{ and } (2^{2m+1}+2^{m+1}+1)\}$. It follows from the hypothesis and [27, Lemma 2.10] that both $2^{2n+1} - 2^{n+1} + 1$ and $2^{2n+1} + 2^{n+1} + 1$ are prime, so that $G \cong \mathrm{Sz}(8)$, and thus we obtain a contradiction. The contradiction completes the proof. □

The following Lemma is useful in our argument. We shall freely use these results in the rest of the section.

Lemma 3.3. *Let $N \lhd G$. The the following statements hold:*

(1) *For any $\chi \in \mathrm{Irr}(G)$, if G is non-solvable and $k_G(\upsilon(\chi)) \leqslant 2$, then $\chi_{G'}$ is irreducible.*

(2) *For any $\chi \in \mathrm{Irr}(G)$, if $\upsilon(\chi) \subset N$ for some $N \lhd G$, then $\gcd(\chi(1), |G/N|) = 1$.*

(3) *Let G be a non-abelian simple group. Then there exists $\chi \in \mathrm{Irr}_1(G)$ such that $\chi(1)$ is even and χ is of p-defect zero for some prime divisor p of $|G|$.*

(4) *Let $N < M$ be two normal subgroups of G with $k_G(M - N) = 1$. Then M is solvable.*

Proof. See [18].

Remark. Suppose that G is a non-solvable group with property $(**)$. If $k_G(\upsilon(\chi)) \leqslant 2$ for any $\chi \in \mathrm{Irr}_1(G)$, then it follows by [2, Theorem 1.1] that either $G \cong \mathrm{A}_5$ or $G \cong \mathrm{L}_2(7)$. In the following Lemma, we suppose that G has a unique non-linear irreducible character φ such that $\upsilon(\varphi)$ consists of r conjugacy classes of G with $r \geqslant 3$, but $\upsilon(\chi)$ consists of at most two conjugacy classes of G for the other $\chi \in \mathrm{Irr}_1(G)$.

Lemma 3.4. *Let G be non-solvable group with property $(**)$. Suppose that N is a minimal normal subgroup of G, and that N is non-solvable. Set $N = N_1 \times \ldots \times N_s$ a direct product of isomorphic simple groups N_i where $s \geqslant 1$, and set $\theta_i \in \mathrm{Irr}_1(N_i)$ such that $\theta_i(1)$ is even and that θ_i is of p-defect zero for some prime divisor p of $|N_i|$. Then $s = 1$ and G/N is solvable if one of the following conditions holds:*
(1) $N = G'$.
(2) $\varphi(1)$ *is odd.*
(3) $\varphi(1)$ *is even and* $\varphi(1) < 4\theta_1(1)$.

Proof. First we show that if G satisfies one of the conditions above, then $s = 1$. Assume that $s \geqslant 2$. Set $\theta = \theta_1 \times \ldots \times \theta_s$. Let χ be an irreducible constituent of θ^G, let $x_1 \in N_1$ be of a prime order p, $x_2 \in N_2$ be of a prime order q ($q \neq p$), $x_3 \in N_2$ be of a prime order r ($r \neq p$ and $r \neq q$). Clearly θ^g is of p-defect zero for any $g \in G$, thus $\vartheta^g(x_1) = \vartheta^g(x_1x_2) = \vartheta^g(x_1x_3) = 0$. This implies that $\chi(x_1) = \chi(x_1x_2) = \chi(x_1x_3) = 0$. The hypothesis yields that $\chi = \varphi$.

Suppose that $N = G'$. Set $\psi = \theta_1 \times 1_{N_2} \times \ldots \times 1_{N_s}$, where 1_{N_i} is the trivial character of N_i, where $i = 2, \ldots, s$. Let φ be an irreducible constituent of ψ^G. Clearly $\chi \neq \varphi$. It follows from the hypothesis and Lemma 3.3(1) that $\varphi_{G'} = \psi$. Observe that $k_G(\upsilon(\psi)) \geqslant 3$, a contradiction.

Suppose that $\varphi(1)$ is odd. Note that $\chi = \varphi$. Clearly $\chi(1)$ is even, a contradiction.

Suppose that $\varphi(1)$ is even and $\varphi(1) < 4\theta(1)$. Since $\varphi(1) = \chi(1) \geqslant \theta(1) = \theta_1(1) \times \ldots \times \theta_s(1)$ ($s \geqslant 2$), we obtain a contradiction.

Next we show that G/N is solvable. By induction, we may assume that $\mathrm{Sol}(G)$, the maximal solvable normal subgroup of G, is trivial. Now suppose that G/N is non-solvable. Note that $\mathrm{out}(N)$ is solvable by the classification of the finite simple groups, so it follows that $C_G(N)$ is non-solvable and hence contains a non-solvable minimal normal subgroup M of G as $\mathrm{Sol}(C_G(N)) = 1$. Set $T = M \times N$. Let $\psi \in \mathrm{Irr}(M)$ be such that $\psi(1)$ is even and that ψ is of q-defect zero, and let $\theta \in \mathrm{Irr}(N)$ be such that $\theta(1)$ is even and that θ is of p-defect zero, where q, p are prime divisors of $|M|$ and $|N|$ respectively. Let $x \in M$, y, $z \in N$ be of order q, p, r respectively, where $r \neq p$ and $r \neq q$. Then for any irreducible constituent χ of $(\psi \times \theta)^G$, we see that $\chi(x) = \chi(y) = \chi(xy) = \chi(xz) = 0$. Observe that $\chi \neq \varphi$, then we obtain a contradiction. The contradiction completes the proof. □

Proposition 3.5. *Suppose that N is the unique minimal normal subgroup of G and that N is a non-abelian simple group. If G satisfies property $(**)$, then $G \cong \mathrm{A}_5$, S_5, $\mathrm{L}_2(7)$ or A_6.*

Proof. First suppose that $N \cong \mathrm{A}_n$ for some $n \geqslant 8$. Let π be the permutation character of G, and δ be the mapping of G into $\mathbb{N}$ such that $\delta(g)$ is the number of

2-cycles in the standard decomposition of g. Set

$$\lambda = \frac{(\pi-1)(\pi-2)}{2} - \delta, \quad \varrho = \frac{\pi(\pi-3)}{2} + \delta.$$

By [7, V, Theorem 20.6], both λ and ϱ are irreducible characters of G.

For odd n, set

$$\begin{aligned} a_1 &= (1,\dots,n-2),\\ a_2 &= (1,\dots,n-4)(n-3,n-2,n-1),\\ a_3 &= (1,\dots,n-5)(n-4,n-3),\\ b_1 &= (1,\dots,n),\\ b_2 &= (1,\dots,n-3)(n-2,n-1),\\ b_3 &= (1,\dots,n-6)(n-5,n-4,n-3). \end{aligned}$$

For even n, set

$$\begin{aligned} a_1 &= (1,\dots,n-1),\\ a_2 &= (1,\dots,n-2)(n-1,n),\\ a_3 &= (1,\dots,n-5)(n-4,n-3,n-2),\\ b_1 &= (1,\dots,n-3),\\ b_2 &= (1,\dots,n-3)(n-2,n-1,n),\\ b_2 &= (1,\dots,n-4)(n-3,n-2). \end{aligned}$$

We see that $\lambda(a_i) = 0 = \varrho(b_i)$ for any $i = 1,2,3$. Observe that a_1, a_2, a_3 (or b_1, b_2, b_3) lie in distinct conjugacy classes of G.

Let χ be an irreducible constituent of λ^G, and let ψ be an irreducible constituent of ϱ^G. Clearly $\chi \neq \psi$. By the hypothesis, we may assume that $k_G(\upsilon(\chi)) \leqslant 2$. Lemma 3.3(1) implies that $\chi_{G'} = \lambda$. Clearly $k_G(\upsilon(\lambda)) \geqslant 3$, a contradiction.

Next suppose that $N \cong \mathrm{A}_n$ for some $n \leqslant 7$ or one of the sporadic simple groups. If $G = N$, then we conclude that $G \cong \mathrm{A}_5$ or A_6. If $N < G$, then $|G/N| = 2$, and so we obtain that $G \cong \mathrm{S}_5$ from [3].

By the classification theorem of the finite simple groups we can now suppose that N is a simple group of Lie type.

Remark and notation. Let $\chi_p \in \mathrm{Irr}_1(N)$ be of p-defect zero where p is a prime of N, and let ψ be an irreducible constituent of χ_p^G. Observe that $\chi_p^g(x) = 0$ for any $g \in G$ and any $x \in N$ of order divisible by p. It follows that $\psi(x) = 0$ whenever $x \in N$ is of order divisible by p.

Arguing as in the above paragraph, then by Lemma 3.1 and Lemma 3.3(1) we conclude that N is isomorphic to one of the following groups: $\mathrm{L}_2(q)$ where $q \geqslant 4$, $\mathrm{L}_3(4)$, or $\mathrm{Sz}(2^{2m+1})$ where $m \geqslant 1$.

Suppose first that $N \cong \mathrm{L}_2(q)$ where $q \geqslant 4$. Suppose that q is even, so that $N \cong \mathrm{L}_2(2^f)$ for some $f \geqslant 2$. Then $|N| = (2^f - 1)2^f(2^f + 1)$ and N has two cyclic subgroups of orders $2^f - 1$ and $2^f + 1$ (see [7, II, Theorem 8.27]). If both $2^f - 1$ and $2^f + 1$ are prime powers, then by Lemma 3.1 we easily conclude that either $f = 2$ or $f = 3$. From [3], we obtain that $G \cong \mathrm{A}_5$ or S_5.

Now suppose that $\pi(2^f - 1) \geqslant 2$ (resp. $\pi(2^f + 1) \geqslant 2$). By [8, XI, Theorem 5.5], N has 2^{f-1} characters γ_i of degree $2^f - 1$ and $2^{f-1} - 1$ characters β_i of degree $2^f + 1$. Let $\theta \in \mathrm{Irr}_1(N)$ with $\theta(1) = 2^f - 1$ (resp. $2^f + 1$). Observe that θ vanishes on at least three elements of distinct order, and so $k_G(\upsilon(\theta)) \geqslant 3$. It follows from the hypothesis and Lemma 3.3(1) that $\theta^G = e\varphi$, which implies that 2^{f-1} characters γ_i are G-conjugate (resp. $2^{f-1} - 1$ characters β_i are G-conjugate). We easily conclude that $[G\colon I_G(\theta)] = 2^{f-1}$ (resp. $[G\colon I_G(\theta)] = 2^{f-1} - 1$), and thus 2^{f-1} divides $[G\colon G']$ (resp. $2^{f-1} - 1$ divides $[G\colon G']$). Now $G/G' \leqslant \mathrm{Out}(G')$ and $|\mathrm{Out}(G')| = f$, where f is the order of the group of field automorphisms of G'. Then we obtain that 2^{f-1} divides f (resp. $2^{f-1} - 1$ divides f). If 2^{f-1} divides f, then $f = 2$ and thus $2^f - 1 = 3$; this contradicts the assumption that $2^f - 1$ is non-prime. If $2^{f-1} - 1$ divides f, then $f = 2$ or 3, and thus $2^f + 1 = 5$ or 9. Thus since $2^f + 1$ is non-prime we have $2^f + 1 = 9$, so that $N \cong \mathrm{L}_2(8)$, and from [3], we obtain a contradiction.

Similarly, if q is odd, then arguing as the above paragraph, we obtain a contradiction.

Next we suppose that $N \cong \mathrm{Sz}(2^{2m+1})$ where $m \geqslant 1$. Let χ_0 be the Steinberg character of N, and let ψ be an irreducible constituent of χ_0^G. Let $P \in \mathrm{Syl}_2(N)$.

Assume that $k_G(\upsilon(\psi)) \geqslant 3$. Note that χ_0 is the Steinberg character of N; thus χ_0 is G-invariant. It follows by Lemma 3.3(1) that any non-linear irreducible character of N is extendible to G (note that the outer automorphism group of $\mathrm{Sz}(q)$ is cyclic), so all the elements of $\mathrm{Irr}(N)$ are invariant under G, and thus all the conjugacy classes of N are G-invariant. The hypothesis yields that N satisfies the property $(**)$. But by Lemma 3.2 we obtain a contradiction. Hence $k_G(\upsilon(\psi)) \leqslant 2$.

Since $k_G(\upsilon(\psi)) \leqslant 2$, we see that $\upsilon(\psi) \subseteq N$ and $k_G(\upsilon(\psi)) = 2$. By Lemma 3.3(2), $|G/N|$ is odd and $\psi_N = \chi_0$. Therefore ψ is of 2-defect zero, and $\psi(x) = 0$ for any $x \in G$ of even order. This implies that $P \in \mathrm{Syl}_2(G)$, and $C_G(t)$ is a 2-group for an involution t. By [22] and since P is non-abelian, G is one of the following groups: $\mathrm{Sz}(2^{2m+1})$ where $m \geqslant 1$, $\mathrm{L}_2(q)$ where q is a Fermat prime or Mersenne prime, $\mathrm{L}_3(4)$, $\mathrm{L}_2(9)$. Then we obtain a contradiction.

Finally suppose that N is isomorphic to $\mathrm{L}_3(4)$. Then by [3], we obtain a contradiction. The contradiction completes the proof. □

P r o o f of Theorem B. We need only prove the necessity. Assume first that G satisfies the property $(**)$. By [2, Theorem 1.1], we may suppose that G has a unique non-linear irreducible character φ such that $\upsilon(\varphi)$ consists of r conjugacy classes of G with $r \geqslant 3$, but $\upsilon(\chi)$ consists of at most two conjugacy classes of G for the other $\chi \in \mathrm{Irr}_1(G)$. □

Step 1. G has the unique minimal normal subgroup N such that G/N is solvable.

Assume this is not the case, then G has a minimal normal subgroup N such that G/N is non-solvable. By induction, $G/N \cong \mathrm{A}_5$, S_5, $\mathrm{L}_2(7)$, or $\mathrm{L}_2(9)$.

Case 1. Assume that $G/N \cong \mathrm{A}_5$.

Since $G/N \cong \mathrm{A}_5$, G/N has exactly one conjugacy class of elements of order 3. Choose a 3-element a of G such that $(aN)^{G/N}$ is the conjugacy class of elements of order 3 in G/N. Set $A = (aN)^{G/N}$, and set $P \in \mathrm{Syl}_2(G)$.

We work for a contradiction via several steps.

Step 1.1. $k_G(A) = 2$.

Notice that G/N has two non-linear irreducible characters of degree 3, and that they vanish on A. It follows from the hypothesis that $k_G(A) \leqslant 2$. Suppose that $k_G(A) = 1$. Then each $\chi \in \mathrm{Irr}(G|N)$ vanishes on A. By the second orthogonality relation we have $|C_G(a)| = |C_{G/N}(aN)| = 3$. Hence G has an element a with $C_G(a)$ of order 3. Applying [16, Theorem], we obtain that $G = NA$, where A is isomorphic to $\mathrm{A}_5 \cong \mathrm{SL}(2,4)$ and N is a normal elementary abelian 2-subgroup of order 16; furthermore, N is isomorphic to the natural $\mathrm{SL}(2,4)$-module of dimension 2 over a field of order 4. We easily see that G does not satisfy the hypothesis (see [23, p. 310]). Therefore, our claim is true.

Step 1.2. $\varphi \in \mathrm{Irr}_1(G/N)$.

Assume otherwise. Then $\varphi \in \mathrm{Irr}(G|N)$. Take $\chi_3 \in \mathrm{Irr}(G/N)$ with $\chi_3(1) = 5$. Set $B = \upsilon(\chi_3)$. Note that $k_{\overline{G}}(\upsilon(\chi_3)) = 2$. Then the hypothesis implies that $k_{G/N}(B) = 2 = k_G(B)$, and hence each $\chi \in \mathrm{Irr}(G|N)$ vanishes on B. By the second orthogonality relation, we easily see that there exists a 5-element $b \in G$ such that $|C_G(b)| = 5$. Thus b has order 5 and so $|G|_5 = 5$, and $(5, |N|) = 1$. As $b \notin N$, b acts without fixed points on N and consequently N is nilpotent, and so N is an elementary abelian group.

Since $k_G(A) = 2$, we easily conclude that $|C_G(a)| = 6$. As a is a 3-element, a must have order 3, and so $|G|_3 = 3$ and $(3, |N|) = 1$. Let t be the unique involution in $C_G(a)$. As $|C_{G/N}(aN)| = 3$, $t \in N$ and consequently N is an elementary abelian 2-group.

Recall that a fixes exactly one non-identity element of N. So if we set $|N| = 2^m$, then $2^m \equiv 2 \pmod 3$. As powers of 4 are congruent to 1 modulo 3, $m = 2l+1$ is odd, for some integer l. Recall that G has an element of order 5 acting fixed point freely on N, so $2^m \equiv 1 \pmod 5$. On the other hand $2^m = 2 \cdot 4^l \equiv \pm 2 \pmod 5$, a contradiction. Hence $\varphi \in \mathrm{Irr}_1(G/N)$.

Step 1.3. N is an elementary abelian 2-group.

Since $\varphi \in \mathrm{Irr}_1(G/N)$, $\varphi(1) = 4$, 3, or 5. By Lemma 3.4, we see that N is solvable, and so N is an elementary abelian group. Recall that $k_G(A) = 2$; observe that N is an elementary abelian 2-group.

Step 1.4. $\varphi(1) = 5$.

Take $\chi_3 \in \mathrm{Irr}(G/N)$ with $\chi_3(1) = 5$. Assume that $\varphi(1) = 4$ or 3. The hypothesis implies that $k_G(v(\chi_3)) = 2$. Arguing as in Claim 1.2, we obtain a contradiction. Hence $\varphi(1) = 5$.

Step 1.5. $G = G'$ and there exists $1_N \neq \lambda \in \mathrm{Irr}(N)$ such that $P \leqslant I_G(\lambda) < G$.

Note that $G/G' \cap N \leqslant G/N \times G/G'$. It follows from the hypothesis that $G/G' \cap N \cong \mathrm{A}_5$. Then $N \leqslant G'$, and so $G = G'$.

For $1_N \neq \lambda \in \mathrm{Irr}(N)$, if λ is G-invariant, then $N = Z(G)$, and since $G = G'$ we conclude that N is subgroup of the Schur multiplier of A_5, and so $G \cong \mathrm{SL}(2,5)$. By [3], $\mathrm{SL}(2,5)$ does not satisfy the property $(**)$, a contradiction. Therefore, $I_G(\lambda) < G$ for any non-principal $\lambda \in \mathrm{Irr}(N)$, and in particular we have that $|N| > 2$. Since $N \cap Z(P) \neq 1$, there exists $1_N \neq \lambda \in \mathrm{Irr}(N)$ such that $P \leqslant I_G(\lambda) < G$.

Step 1.6. We obtain a contradiction.

Since $P \leqslant I_G(\lambda) < G$, we see that either $I_G(\lambda)/N$ is a 2-group or $I_G(\lambda)/N \cong \mathrm{A}_4$. Set $T := I_G(\lambda)$. Let ω be a irreducible constituent of λ^T, and let $\chi = \omega^G$. Observe that $\chi \neq \varphi$. It follows from the definition of induced character that χ vanishes on $v(\varphi)$. Recall that $k_G(v(\varphi)) = r \geqslant 3$, we obtain a contradiction.

Case 2. Assume that $G/N \cong \mathrm{S}_5$.

Then $\varphi(1) = 6$. Choose two 2-elements a, b of G such that aN is an involution in G/N with $|C_{G/N}(aN)| = 8$, and that bN is an element of order 4 in G/N. Set $A = (aN)^{G/N}$ and $B = (bN)^{G/N}$. The hypothesis yields that A and B are a conjugacy class of G, respectively, and thus $\chi(a) = 0 = \chi(b)$ for each $\chi \in \mathrm{Irr}(G|N)$.

Choose a 5-element c of G such that cN is an element of order 5 in G/N. Set $C = (cN)^{G/N}$. The hypothesis yields that $k_G(C) \leqslant 2$. Suppose that $k_G(C) = 1$. Then $\chi(c) = 0$ for each $\chi \in \mathrm{Irr}(G|N)$. Note that $\chi(a) = 0 = \chi(b)$ for each $\chi \in \mathrm{Irr}(G|N)$; thus we obtain a contradiction, which shows that $k_G(C) = 2$.

Observe that $|C_G(d)| = 10$. As d is a 5-element, d must have order 5, and so $|G|_5 = 5$ and $(5, |N|) = 1$. Let t be the unique involution in $C_G(d)$. As $|C_{G/N}(dN)| = 5$, $t \in N$ and consequently N is an elementary abelian 2-group.

Recall that $\chi(b) = 0$ for any $\chi \in \mathrm{Irr}(G|N)$. By the second orthogonality relation we have $|C_G(b)| = |C_{G/N}(bN)| = 4$. Hence G has an element b with $T = C_G(b)$ of order 4. Clearly $T \subseteq C_G(T) \subseteq C_G(b) = T$. Recall that $G/N \cong S_5$, and that N is an elementary abelian 2-group; then $O(G) = 1$, where $O(G)$ is the largest normal subgroup of odd order in G. Since G is non-solvable and $G' < G$, we use [25, Theorem 1, 2] (where $O(G)$ is denoted by K), to conclude that G has a normal subgroup M with $M \cong \mathrm{PSL}(2, q)$, $G \subseteq \mathrm{Aut}(M)$ and $|G\colon M| = 2$. It is easy to see that $G \cong \mathrm{S}_5$, we obtain a contradiction.

Case 3. Assume that $G/N \cong \mathrm{L}_2(7)$.

Observe that $\varphi \in \mathrm{Irr}(G/N)$. Suppose that $\varphi(1) = 6$. Set $\chi_1, \chi_2 \in \mathrm{Irr}(G/N)$ with $\chi_1(1) = 7$ and $\chi_2(1) = 8$. The hypothesis yields that $k_{G/N}(\upsilon(\chi_1)) = 2 = k_G(\upsilon(\chi_1))$, and that $k_{G/N}(\upsilon(\chi_2)) = 2 = k_G(\upsilon(\chi_2))$. Hence $\chi(\upsilon(\chi_1)) = 0 = \chi(\upsilon(\chi_2))$ for each $\chi \in \mathrm{Irr}(G|N)$, and so $k_G(\upsilon(\chi)) \geqslant 4$, a contradiction.

For the case when $\varphi(1) = 7$, 8, or 3, arguing as in the above paragraph, we also obtain a contradiction.

Case 4. Assume that $G/N \cong \mathrm{L}_2(9)$.

In this case, arguing as in the case 3, we also obtain a contradiction.

Hence G has the unique minimal normal subgroup N such that G/N is solvable. This implies that $N \leqslant G' < G$. In particular, $G \leqslant \mathrm{Aut}(N)$ and $G/N \leqslant \mathrm{Out}(N)$.

Step 2. $N = G'$.

Assume the contrary, that $N < G'$. Then G/N is a non-abelian solvable group.

We first show that $\varphi \in \mathrm{Irr}(G/N)$. Suppose that $\varphi \in \mathrm{Irr}(G|N)$. Then it follows from the hypothesis and Lemma 3.3(1) that $\chi_{G'}$ is irreducible for any $\chi \in \mathrm{Irr}_1(G/N)$. On the other hand, since G/N is a non-abelian solvable group, there exists $\chi \in \mathrm{Irr}_1(G/N)$ such that $\chi_{G'/N}$ is not irreducible, and thus $\chi_{G'}$ is not irreducible, a contradiction. Therefore, $\varphi \in \mathrm{Irr}(G/N)$.

Recall that G/N is a solvable group. It follows from the hypothesis and Lemma 3.3(1) that $\chi_{G'}$ is irreducible for any $\chi \in \mathrm{Irr}_1(G/N) - \{\varphi\}$. Observe that $\varphi_{G'}$ is not irreducible. Then G/N satisfies the hypothesis of Theorem A. Hence we have to consider the following four cases.

Case 1. Suppose that G/N is a 2-transitive Frobenius group with kernel G'/N or G/N is an extra-special 2-group.

Then we easily see that G'/N is abelian, and thus $N = G''$.

Subcase 1.1. Assume that G/G'' is a 2-transitive Frobenius group with kernel G'/G''.

Then by the proof of Theorem A, we conclude that (G', G'') is a Camina pair. Note that $N = G''$ is the unique minimal normal subgroup of G, so it follows by Lemma 2.5 that either G' is a p-group or G' is a Frobenius group with kernel G''. But G is solvable, a contradiction.

Subcase 1.2. Assume that G/G'' is an extra-special 2-group.

Since $G'/G'' = Z(G/G'')$, all the elements of $\mathrm{Irr}(G'/G'')$ are G-invariant. Note that any non-linear irreducible character of G' is extendible to G, so all the elements of $\mathrm{Irr}(G')$ are invariant under G, and thus all the conjugacy classes of G' are G-invariant. The hypothesis yields that $\upsilon(\chi)$ consists of at most two conjugacy classes of G' for all $\chi \in \mathrm{Irr}_1(G')$. Note that G' is non-solvable. By [2, Theorem 1.1], we have $G' \cong \mathrm{A}_5$ or $\mathrm{L}_2(7)$. Thus $G' = G'' = N$, a contradiction.

Case 2. Suppose that $G/N \cong \mathrm{SL}(2,3)$.

Recall that $\varphi \in \mathrm{Irr}(G/N)$. The hypothesis implies that $\varphi(1) = 3$. By Lemma 3.4, N is a non-abelian simple group. Applying Proposition 3.5, we obtain a contradiction.

Case 3. Suppose that $G/N \cong \mathrm{S}_4$.

Note that $\varphi \in \mathrm{Irr}(G/N)$. Hence $\varphi(1) = 2$ or 3. Arguing as in Case 2, we also obtain a contradiction.

Case 4. Suppose that G/N is a semidirect product of $\mathrm{SL}(2,3)$ and the natural $\mathrm{SL}(2,3)$-module.

Let M be the inverse image of the natural $\mathrm{SL}(2,3)$-module in G. Note that G'/N is a 2-transitive Frobenius group with kernel M/N and complement isomorphic to $\mathbb{Q}_8$. Set $\theta \in \mathrm{Irr}_1(G'/N)$ with $\theta(1) = 2$, and set $\chi \in \mathrm{Irr}(G/N)$ such that $\chi_{G'/N} = \theta$.

Note that θ vanishes on $G'/N - M/N$, thus χ vanishes on $G'/N - M/N$. Since $M/N < G''/N < G'/N$, $k_{G/N}(G'/N - M/N) \geqslant 2$. On the other hand, $\chi \neq \varphi$, so it follows from the hypothesis that $k_{G/N}(G'/N - M/N) = 2 = k_G(G'/N - M/N)$. Hence $k_G(G'' - M) = 1$, and so G'' is solvable by Lemma 3.3(4). Hence we obtain a contradiction.

The final contradiction show that $N = G'$. Then Lemma 3.4 yields that G' is a non-abelian simple group. Then Proposition 3.5 implies that G is one of the following groups: A_5, S_5, $\mathrm{L}_2(7)$ or A_6. The proof is complete. □

Remark. Assume that G satisfies the property $(*)$. If G is non-solvable, then $G \cong \mathrm{A}_5$ by Theorem B. If G is solvable, then we easily see that G is a φ-group. Observe that if G has the structure described in Theorem A(4), then G does not satisfy the property $(*)$. Hence, we obtain the Corollary.

Acknowledgment. The authors would like to thank the referee for his/her help. In addition, The authors would like to Professors Wujie Shi and Yongcai Ren for their helps. This paper is supported by the NNSF of China (Grant No. 10871032) and the NSF of Sichuan University of Science and Engineering (Grant No. 2009XJKRL011).

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