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COMPONENTS AND INDUCTIVE DIMENSIONS OF COMPACT SPACES

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Abstract. It is shown that for every pair of natural numbers $m \geqslant n \geqslant 1$, there exists a compact Fréchet space $X_{m,n}$ such that
(a) $\dim X_{m,n} = n$, $\operatorname{ind} X_{m,n} = \operatorname{Ind} X_{m,n} = m$, and
(b) every component of $X_{m,n}$ is homeomorphic to the n-dimensional cube I^n.
This yields new counter-examples to the theorem on dimension-lowering maps in the cases of inductive dimensions.

Keywords: inductive dimension, theorem on dimension-lowering maps, component.

MSC 2010: 54F45

There exist numerous examples of compact spaces with non-coinciding dimensions in literature. The first such examples by A. L. Lunc [15] and O. V. Lokucievskiĭ [14] appeared in 1949, and the first two series of compact spaces with $\dim = n < m = \operatorname{ind}$ and $\dim = n < m = \operatorname{Ind}$ by P. Vopěnka [16] appeared in this journal in 1958.[1] It was not noted, probably anywhere, that Vopěnka's method leads to *compact spaces* X *whose every component* P *has* $\operatorname{Ind} P < \operatorname{ind} X \leqslant \operatorname{Ind} X < \infty$*.* Such spaces, in turn, are domains of counter-examples to the theorem on dimension-lowering maps in the cases of inductive dimensions.

Recently, V. A. Chatyrko [5] has constructed compact spaces $X_{\mathrm{Ch},i}$, where $i = 1, 2$, and a (continuous) map $f_{\mathrm{Ch}}\colon X_{\mathrm{Ch},1} \to A_{\mathfrak{c}}$ onto the compact space $A_{\mathfrak{c}}$ with the only accumulation point μ, $\operatorname{card} A_{\mathfrak{c}} = \mathfrak{c}$, which satisfy the following conditions:

- $\dim X_{\mathrm{Ch},i} = 1 < 2 = \operatorname{ind} X_{\mathrm{Ch},i} = \operatorname{Ind} X_{\mathrm{Ch},i}$ for $i = 1, 2$;

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[1] For Lokucievskiĭ's example see also R. Engelking [9, Examples 2.2.14 and 3.1.31]. For more references see [9], V. A. Chatyrko, K. L. Kozlov, B. A. Pasynkov [6], [7], and V. V. Fedorchuk [10].

- all point-inverses $f_{\mathrm{Ch}}^{-1}\alpha$, $\mu \neq \alpha \in A_{\mathfrak{c}}$, are single points, and $\operatorname{Ind} f_{\mathrm{Ch}}^{-1}\mu = 1$;
- every component of $X_{\mathrm{Ch},2}$ is homeomorphic to the interval $I = [0,1]$; and
- $X_{\mathrm{Ch},i}$ are not hereditarily normal for $i = 1, 2$.

Suppose that d is a dimension function, and $\mathcal{M}$ is a class of maps. One says that the theorem on the dimension-lowering maps holds in $\mathcal{M}$ if $\mathrm{d}X \leqslant \mathrm{d}Y + \mathrm{d}f$ for every map $f\colon X \to Y$ in $\mathcal{M}$ (here, $\mathrm{d}f = \sup\{\mathrm{d}f^{-1}y\colon\ y \in Y\}$; cf. R. Engelking [9, Theorems 1.12.4 and 3.3.10]).

Constructing f_{Ch}, Chatyrko has shown that the theorem on inductive-dimension-lowering maps does not hold[2] even if we consider maps into the "hereditarily nice" space $A_{\mathfrak{c}}$. On the other hand, the present author [13] has proved certain theorems on dimension-lowering maps for Ind, for Charalambous-Filippov-Ivanov inductive dimension Ind_0 (M. G. Charalambous [2], A. V. Ivanov [12]), and *for fully closed maps* from spaces that need not be hereditarily normal (see Section 3 in the present paper).

In this paper we modify Chatyrko's construction, develop a method related to Vopěnka's one [16], and prove

Theorem 1. *For every pair of natural numbers $m \geqslant n \geqslant 1$, there exists a compact Fréchet space $X_{m,n}$ such that*

(a) $\dim X_{m,n} = n$, $\operatorname{ind} X_{m,n} = \operatorname{Ind} X_{m,n} = m$, *and*

(b) *every component of $X_{m,n}$ is homeomorphic to the n-dimensional cube I^n.*

Chatyrko [5] has asked *if there exist compact spaces X, Y and a map $f\colon X \to Y$ such that* $\operatorname{Ind} X > \operatorname{Ind} Y + \operatorname{Ind} f + 1$. The answer to this question is "yes".

Example 1. Let $m > n$. Suppose that $\mathcal{D}$ is the decomposition of $X_{m,n}$ into its components, and $f\colon X_{m,n} \to X_{m,n}/\mathcal{D}$ is the natural quotient projection. Then every point-inverse of f is homeomorphic to I^n, $X_{m,n}/\mathcal{D}$ is zero-dimensional in any sense, and hence, $\operatorname{Ind} X_{m,n} = m > n = \operatorname{Ind} X_{m,n}/\mathcal{D} + \operatorname{Ind} f$.

Section 1 contains a proof of Theorem 1, and in Section 2 we show that $\mathrm{Ind}_0 X_{m,n} = n2^{m-n}$. In Section 3 we indicate modifications of our construction, and prove that *for every triple $k \geqslant m \geqslant n \geqslant 1$, there is a compact Fréchet space $Y_{k,m,n}$ such that* $\dim Y_{k,m,n} = n$, $\operatorname{ind} Y_{k,m,n} = \operatorname{Ind} Y_{k,m,n} = m$, *and* $\mathrm{Ind}_0 Y_{k,m,n} = k$.

Our terminology follows Engelking's monographs [8], [9].

[2] Earlier counter-examples to the theorem were obtained as a by-product of constructions of compact spaces X, Y such that $\operatorname{Ind}(X \times Y) > \operatorname{Ind} X + \operatorname{Ind} Y$ (V. V. Filippov [11]). See also comments in [5, Section 5] and [9, Sections 2.2, 2.4, and p. 205].

1. Proof of Theorem 1

Let wX denote the weight of a (topological) space X. Let $A_{\mathfrak{m}}$ be the one-point compactification of the discrete space of cardinality $\mathfrak{m}$, and let $\mu \in A_{\mathfrak{m}}$ be the only accumulation point.

Lemma 1. *Suppose that X is a space with $wX < \mathfrak{m} > \aleph_0$, and $\pi_X\colon A_{\mathfrak{m}} \times X \to X$ is the projection. If $H \subset A_{\mathfrak{m}} \times X$ is a G_δ-set, then there is a set $A \subset A_{\mathfrak{m}}$ such that* $\operatorname{card}(A_{\mathfrak{m}} \setminus A) < \mathfrak{m}$ *and* $A \times \pi_X[H \cap (\{\mu\} \times X)] \subset H$.

Proof. Let $\mathcal{B}$, where $\operatorname{card} \mathcal{B} = wX$, be a base of open sets for X. First, suppose that $H \subset A_{\mathfrak{m}} \times X$ is open. Let $\mathcal{B}_0$ be the family of all $U \in \mathcal{B}$ for which there is a set $A_U \subset A_{\mathfrak{m}}$ with $\mu \in A_U$, $\operatorname{card}(A_{\mathfrak{m}} \setminus A_U) < \aleph_0$, and $A_U \times U \subset H$. It suffices to take $A = \bigcap\limits_{U \in \mathcal{B}_0} A_U$.

If $H = \bigcap\limits_{n=1}^{\infty} H_n$, where $H_n \subset A_{\mathfrak{m}} \times X$ are open, then for every n there is a set $A_n \subset A_{\mathfrak{m}}$ such that $\operatorname{card}(A_{\mathfrak{m}} \setminus A_n) < \mathfrak{m}$ and $A_n \times \pi_X[H_n \cap (\{\mu\} \times X)] \subset H_n$. It suffices to take $A = \bigcap\limits_{n=1}^{\infty} A_n$. □

The next lemma is a direct consequence of A. V. Arkhangelskiĭ's Example 5.12 and Theorem 5.16 in [1].

Lemma 2. *If X is a compact Fréchet space, then so is $A_{\mathfrak{m}} \times X$.*

Lemma 3 (see Vopěnka [16, p. 320])**.** *If X, Y are compact spaces and* $\operatorname{Ind} X = 0$, *then* $\operatorname{Ind}(X \times Y) = \operatorname{Ind} Y$.

For any pair of *non-empty compact spaces X and Y* we will construct a certain compact space $Z(X,Y)$, and later we will investigate the properties of $Z(X,Y)$.

Write $\mathcal{S}_X$ for the family of all subspaces of X that are either finite or homeomorphic to $A_{\aleph_0}$ (hence, the empty set is a member of $\mathcal{S}_X$), and take any cardinal number $\mathfrak{m} \geqslant \max\{\aleph_0, (wX)^+, (wY)^+, \operatorname{card} \mathcal{S}_X\}$. Consider the set $M = A_{\mathfrak{m}} \times X \times Y$ with the product topology, the union

$$N = (\{\mu\} \times X) \cup [(A_{\mathfrak{m}} \setminus \{\mu\}) \times X \times Y],$$

and the function $\pi_1\colon M \to N$:

$$\pi_1(\alpha, x, y) = \begin{cases} (\alpha, x) & \text{if } \alpha = \mu, \\ (\alpha, x, y) & \text{if } \alpha \neq \mu. \end{cases}$$

The decomposition of M into all point-inverses of π_1 is upper semi-continuous. Hence, if we equip N with the largest topology such that π_1 is continuous (the quotient topology), then N is a Hausdorff compact space. The unique function $\pi_2\colon N \to A_{\mathfrak{m}} \times X$ such that $\pi_2\pi_1(\alpha, x, y) = (\alpha, x)$ is continuous. Note that *if $x \in X$, then all sets $\pi_2^{-1}(A \times U)$, where $A \ni \mu$ and $U \ni x$ are open in A and X, respectively, form a neighborhood base for the point $(\mu, x) \in N$.* Indeed, if $V \ni (\mu, x)$ is open in N, then $(\mu, x) \in (A_{\mathfrak{m}} \times X) \setminus \pi_2(N \setminus V)$ and there are open sets $A \ni \mu$ and $U \ni x$ such that $(\mu, x) \in A \times U \subset (A_{\mathfrak{m}} \times X) \setminus \pi_2(N \setminus V)$. Hence, $(\mu, x) \in \pi_2^{-1}(A \times U) \subset V$.

Consider any function $\varphi\colon A_{\mathfrak{m}} \setminus \{\mu\} \to \mathcal{S}_X$ such that $\operatorname{card} \varphi^{-1} S = \mathfrak{m}$ for every $S \in \mathcal{S}_X$. Put

$$H(\alpha) = \begin{cases} \{\mu\} \times X & \text{for } \alpha = \mu, \\ \{\alpha\} \times \varphi\alpha \times Y & \text{for } \alpha \neq \mu, \text{ and} \end{cases}$$

$$Z(X, Y) = \bigcup_{\alpha \in A_{\mathfrak{m}}} H(\alpha).$$

$Z(X, Y)$ inherits topology from N, and is closed in N as every $\varphi\alpha \subset X$ is closed. Note that *$Z(X, Y)$ depends*[3] *on the choice of $\mathfrak{m}$ and φ.*

Let $\pi_{A_{\mathfrak{m}}}\colon A_{\mathfrak{m}} \times X \to A_{\mathfrak{m}}$ and $\pi_X\colon A_{\mathfrak{m}} \times X \to X$ be projections. If we consider the restriction $h = \pi_{A_{\mathfrak{m}}}\pi_2 | Z(X, Y)\colon Z(X, Y) \to A_{\mathfrak{m}}$, we have $h^{-1}\alpha = H(\alpha)$ for every $\alpha \in A_{\mathfrak{m}}$.

Lemma 4. *Every component of $Z(X, Y)$ is homeomorphic to some component of X or Y, and hence, $\dim Z(X, Y) = \max\{\dim X, \dim Y\}$.*

Proof. The equality is a consequence of the theorem on dimension-lowering maps for dim (see [9, Theorem 3.3.10]). □

Lemma 5. *If X and Y are Fréchet spaces, then so is $Z(X, Y)$.*

Proof. Suppose that $H \subset Z(X, Y)$ and $p \in \operatorname{cl} H$. If $p \in H(\alpha)$, where $\alpha \neq \mu$, then an application of Lemma 2 completes the proof since $H(\alpha)$ is homeomorphic to a subspace of $A_{\aleph_0} \times Y$. Suppose $p = (\mu, x) \in \{\mu\} \times X$. If $p \in \operatorname{cl}[H \cap (\{\mu\} \times X)]$, then the proof is complete as X is Fréchet. So, we can assume that $H \cap (\{\mu\} \times X) = \emptyset$. Then $(\mu, x) = \pi_2(\mu, x) \in \operatorname{cl} \pi_2 H$, and by Lemma 2, a certain sequence of points $(\alpha_n, x_n) \in \pi_2 H$ converges to (μ, x). It is easily seen that also any sequence of points $(\alpha_n, x_n, y_n) \in H$ converges to (μ, x) if we consider the topology in N. □

[3] We could write $Z(X, Y, \mathfrak{m}, \varphi)$, but it is easily shown that the dependence on φ is superficial. If $\varphi, \psi\colon A_{\mathfrak{m}} \setminus \{\mu\} \to \mathcal{S}_X$ and $\operatorname{card} \varphi^{-1} S = \operatorname{card} \psi^{-1} S = \mathfrak{m}$ for every $S \in \mathcal{S}_X$, then the subspace $Z(X, Y) \subset N$ defined with the use of φ is homeomorphic to $Z(X, Y) \subset N$ defined with the use of ψ.

Lemma 6. $\operatorname{Ind} Z(X,Y) \leqslant \max\{\operatorname{Ind} X + 1, \operatorname{Ind} Y\}$.

Proof. Suppose that $F_0, F_1 \subset Z = Z(X,Y)$ are disjoint closed sets. There are open sets $U_0, U_1 \subset Z$ such that $\operatorname{cl} U_0 \cap \operatorname{cl} U_1 = \emptyset$ and $F_i \subset U_i$ for $i = 0,1$. Write $F_i(\alpha) = H(\alpha) \cap \operatorname{cl} U_i$. Observe that $\pi_X \pi_2 F_i(\alpha) \subset \varphi\alpha$ for $\alpha \neq \mu$, and the set

$$A = \{\alpha \in A_{\mathfrak{m}} \setminus \{\mu\} \colon \ \pi_X \pi_2 F_0(\alpha) \cap \pi_X \pi_2 F_1(\alpha) \neq \emptyset\}$$

must be finite. Indeed, if there were a one-to-one sequence $(\alpha_n)_{n=1}^{\infty}$ of points in A, $(\alpha_n, x_n, y_{n,i}) \in F_i(\alpha_n)$, and $x \in X$ were a cluster point of the sequence $(x_n)_{n=1}^{\infty}$, then μ would be the limit of $(\alpha_n)_{n=1}^{\infty}$, and (μ, x) would be in $\operatorname{cl} U_0 \cap \operatorname{cl} U_1$. By Lemma 3, we have $\operatorname{Ind}(\varphi\alpha \times Y) = \operatorname{Ind} Y$. So, for each $\alpha \in A$ there exist disjoint open sets $V_i(\alpha) \subset H(\alpha)$ such that $F_i(\alpha) \subset V_i(\alpha)$ and $\operatorname{Ind} L(\alpha) < \operatorname{Ind} Y$, where $L(\alpha) = H(\alpha) \setminus [V_0(\alpha) \cup V_1(\alpha)]$. If $\mu \neq \alpha \notin A$, then there are analogous sets $V_i(\alpha)$ with $L(\alpha) = \emptyset$. When we set $V_i = U_i \cup \bigcup_{\alpha \neq \mu} V_i(\alpha)$, we obtain a partition

$$L = Z \setminus (V_0 \cup V_1) = [H(\mu) \setminus (U_0 \cup U_1)] \cup \bigcup_{\alpha \in A} L(\alpha)$$

in Z between F_0 and F_1. Since A is a finite set, $\operatorname{Ind} L < \max\{\operatorname{Ind} X + 1, \operatorname{Ind} Y\}$. □

The number $\operatorname{ind}_{b+} X \in \mathbb{N} \cup \{\infty\}$, defined below, is actually not necessary in our proof of Theorem 1. However, we will use Lemma 7 in the form with $\operatorname{ind}_{b+} X$ later (in Remark 4). Suppose that X is a regular space and $b \in X$. We put

$$\operatorname{ind}_{b+} X = \min\{n \colon \text{ there is a closed neighborhood } F \text{ of } b \text{ such that } \operatorname{ind} F \leqslant n\}$$

whenever the above set of n's is non-empty, and $\operatorname{ind}_{b+} X = \infty$ in the other case. Let us note that $\operatorname{ind} X \geqslant \operatorname{ind}_{b+} X \geqslant \operatorname{ind}_b X$.

Lemma 7. *If $B \subset X$ is a connected subspace that contains more than one point and X is a Fréchet space, then for every point $b_0 \in B$ we have*

$$\operatorname{ind}_{(\mu,b_0)} Z(X,Y) \geqslant \min\{\operatorname{ind} Y, \min\{\operatorname{ind}_{b+} X \colon \ b \in B\}\} + 1.$$

Proof. Fix points $b_0 \neq b_1 \in B$. Take a partition L in $Z = Z(X,Y)$ between (μ, b_0) and (μ, b_1). There exist open sets $U_0, U_1 \subset N$ such that $(\mu, b_i) \in U_i$ for $i = 0,1$, $Z \cap U_0 \cap U_1 = \emptyset$ and $Z \setminus L = Z \cap (U_0 \cup U_1)$. Let $L' = L \cap H(\mu)$, $U_i' = U_i \cap H(\mu)$ for $i = 0,1$, and $B' = \{\mu\} \times B$. There are two cases. (1) If $B' \cap \operatorname{int}_{H(\mu)} L' \ni (\mu, b)$ for a point $b \in B$, then $\operatorname{ind} L \geqslant \operatorname{ind} L' \geqslant \operatorname{ind}_{b+} X$. (2) If $B' \cap \operatorname{int}_{H(\mu)} L' = \emptyset$, then

$B' \cap \operatorname{cl}_{H(\mu)} U'_0 \cap \operatorname{cl}_{H(\mu)} U'_1 \ni (\mu, b)$ since B is connected. As X is Fréchet, there are sequences $(b_i^n)_{n=1}^{\infty}$ convergent to b and such that $(\mu, b_i^n) \in U'_i$ for $n = 1, 2, \dots$ and $i = 0, 1$. Let $S = \{b\} \cup \{b_i^n \colon i = 1, 2,\ n = 1, 2, \dots\} \in \mathcal{S}_X$. Consider the projection $\pi_{X\times Y} \colon A_{\mathfrak{m}} \times X \times Y \to X \times Y$ and the sets $H_i = \pi_1^{-1} U_i$. By Lemma 1 there exists a set $A \subset A_{\mathfrak{m}}$ such that $\operatorname{card}(A_{\mathfrak{m}} \setminus A) < \mathfrak{m}$ and $A \times \pi_{X\times Y}(\pi_1^{-1} U'_i) \subset \pi_1^{-1} U_i$ for $i = 0, 1$. Since $\operatorname{card} \varphi^{-1} S = \mathfrak{m}$, there is an $\alpha \in A \setminus \{\mu\}$ such that $\varphi\alpha = S$. We have $\{(b_i^n, y) \colon n = 1, 2, \dots,\ y \in Y\} \subset \pi_{X\times Y}(\pi_1^{-1} U'_i)$, and hence,

$$\{\alpha\} \times \{b_i^n \colon n = 1, 2, \dots\} \times Y \subset H(\alpha) \cap \pi_1^{-1} U_i = H(\alpha) \cap U_i.$$

Consequently, $\{\alpha\} \times \{b\} \times Y \subset Z \setminus (U_0 \cup U_1) = L$ and $\operatorname{ind} L \geqslant \operatorname{ind} Y$. Therefore, in both cases $\operatorname{ind} L \geqslant \min\{\operatorname{ind} Y, \min\{\operatorname{ind}_{b+} X \colon b \in B\}\}$. □

P r o o f of Theorem 1. Fix $n \geqslant 1$. Using induction on m, we obtain compact spaces $X_{m,n}$ and arcs $B_m \subset X_{m,n}$ such that for every $m \geqslant n$ the following conditions hold:

(a) every component of $X_{m,n}$ is homeomorphic to I^n;
(b) $\operatorname{Ind} X_{m,n} \leqslant m$;
(c) $\operatorname{ind}_{b+} X_{m,n} \geqslant m$ for every $b \in B_m$; and
(d) $X_{m,n}$ is a Fréchet space.

For $m = n$, $X_{n,n}$ is the cube I^n and $B_n \subset I^n$ is any fixed arc. If $X_{m,n} \supset B_m$ with the properties (a)–(d) are defined, we take $\mathfrak{m} = \max\{(wX_{m,n})^+, \operatorname{card} \mathcal{S}_{X_{m,n}}\}$, where $\mathcal{S}_{X_{m,n}}$ is the family of all subsets of $X_{m,n}$ that are either finite or homeomorphic to $A_{\aleph_0}$, and put $X_{m+1,n} = Z(X_{m,n}, X_{m,n})$, $B_{m+1} = \{\mu\} \times B_m \subset X_{m+1,n} \subset N$. By Lemmas 4–7, the conditions (a)–(d) are true for $X_{m+1,n} \supset B_{m+1}$. □

2. Charalambous-Filippov-Ivanov dimension Ind_0

Recently, there is a growing interest in dimension functions ind_0 and Ind_0 defined in the 1970's by Charalambous [2] and Ivanov [12] (see Charalambous, Chatyrko [3] and the references in that paper). In this section we investigate the behavior of Ind_0 under our operation $Z(X, Y)$.

Definition. For normal spaces X, the dimension $\operatorname{Ind}_0 X \in \{-1, 0, 1, 2, \dots, \infty\}$ is defined so that

(a) $\operatorname{Ind}_0 X = -1$ iff $X = \emptyset$;
(b) $\operatorname{Ind}_0 X \leqslant n \geqslant 0$ iff for every pair of disjoint closed sets $A, B \subset X$, between A and B there is a G_δ partition L such that $\operatorname{Ind}_0 L \leqslant n - 1$;
(c) $\operatorname{Ind}_0 X = n$ iff $\operatorname{Ind}_0 X \leqslant n$ and it is not true that $\operatorname{Ind}_0 X \leqslant n - 1$;
(d) $\operatorname{Ind}_0 X = \infty$ if for every $n \in \mathbb{N}$, it is not true that $\operatorname{Ind}_0 X \leqslant n$.

If we replace the set B in the above definition by a point, which arbitrarily runs over X, we obtain the definition of the dimension $\operatorname{ind}_0 X$. However, Charalambous and Ivanov's results [2, Propositions 15 and 16], [12, Theorem 3 and Corollary 2] readily yield

Lemma 8. $\operatorname{Ind}_0 X = \operatorname{ind}_0 X$ *and* $\operatorname{Ind}_0(X \times Y) \leqslant \operatorname{Ind}_0 X + \operatorname{Ind}_0 Y$ *for every pair of compact spaces* X *and* $Y \neq \emptyset$.

It is clear that $\operatorname{Ind} X \leqslant \operatorname{Ind}_0 X$ and $\operatorname{ind} X \leqslant \operatorname{ind}_0 X$ for every normal space X, and $\operatorname{Ind} X = \operatorname{Ind}_0 X$, $\operatorname{ind} X = \operatorname{ind}_0 X$ if X is perfectly normal.

Lemma 9. $\operatorname{Ind}_0 Z(X,Y) = \operatorname{Ind}_0 X + \operatorname{Ind}_0 Y$ *(if* X *and* Y *are non-empty compact spaces).*

P r o o f. We adopt the notation of Section 1. In virtue of Lemma 8, we can replace Ind_0 by ind_0. Since $\operatorname{ind}_0(\varphi\alpha \times Y) = \operatorname{ind}_0 Y$ for every $\alpha \neq \mu$ such that $\varphi\alpha \neq \emptyset$ (by Lemma 8), it suffices to evaluate ind_0 of $Z = Z(X,Y)$ only at points $(\mu, x) \in \{\mu\} \times X$. Set

$$\lambda(A,B) = Z \cap \pi_2^{-1}(A \times B),$$

where $A \subset A_{\mathfrak{m}}$ and $B \subset X$. Observe that all sets $\lambda(A,U)$, where $\mu \in A$, $A_{\mathfrak{m}} \setminus A$ are finite and $U \ni x$ are open in X, form a neighborhood base for (μ, x). Furthermore,

(∗) *if* $\mu \in A \subset A_{\mathfrak{m}}$*,* $\operatorname{card}(A_{\mathfrak{m}} \setminus A) < \mathfrak{m}$ *and* $L \subset X$ *is a non-empty closed subset, then* $\lambda(A,L) \subset Z$ *is homeomorphic to* $Z(L,Y)$*,*

where $Z(L,Y)$ is constructed with the use of the function $\varphi_L \colon A \setminus \{\mu\} \to \mathcal{S}_L$, $\varphi_L\alpha = L \cap \varphi\alpha$ for $\alpha \in A \setminus \{\mu\}$. Consequently, we infer that

(†) *if* $\mu \in A \subset A_{\mathfrak{m}}$*,* $A_{\mathfrak{m}} \setminus A$ *is finite and* $L \subset X$ *is a non-empty closed* G_δ*-set, then* $\lambda(A,L) \subset Z$ *is a* G_δ*-set homeomorphic to* $Z(L,Y)$*.*

On the other hand, for every G_δ-set $\Lambda \subset Z$ there is a G_δ-set $H \subset N$ such that $\Lambda = Z \cap H$. Write $L_\Lambda = \pi_X \pi_2[\Lambda \cap (\{\mu\} \times X)]$. Applying Lemma 1 to the G_δ-set $\pi_1^{-1}H \subset A_{\mathfrak{m}} \times X \times Y$, we obtain a set $A_\Lambda \subset A_{\mathfrak{m}}$ with $\mu \in A_\Lambda$, $\operatorname{card}(A_{\mathfrak{m}} \setminus A_\Lambda) < \mathfrak{m}$, $A_\Lambda \times L_\Lambda \times Y \subset \pi_1^{-1}H$. Hence, $\pi_2^{-1}(A_\Lambda \times L_\Lambda) \subset H$ and it follows that

(‡) *if* $\Lambda \subset Z$ *is a closed* G_δ*-set that meets* $\{\mu\} \times X$*, then* $\lambda(A_\Lambda, L_\Lambda) \subset \Lambda$ *is homeomorphic to* $Z(L_\Lambda, Y)$*.*

We will prove that $\operatorname{ind}_0 Z \leqslant \operatorname{ind}_0 X + \operatorname{ind}_0 Y$ by induction on $n = \operatorname{ind}_0 X$. If $n = 0$ and $x \in X$, then $\operatorname{ind}_{(\mu,x)} N = 0$, $\operatorname{ind}_{0(\mu,x)} Z = 0$, and $\operatorname{ind}_0 Z = \operatorname{ind}_0 Y$. Assume that the inequality is true for spaces X with $\operatorname{ind}_0 X \leqslant n$. Let $\operatorname{ind}_0 X = n+1$, consider an open neighborhood $\lambda(A,U) \ni (\mu, x)$, and take a G_δ partition L in X between x and $X \setminus U$, $\operatorname{ind}_0 L \leqslant n$. By the claim (†) and the induction hypothesis, $\lambda(A,L)$ is the needed partition in Z and $\operatorname{ind}_0 \lambda(A,L) \leqslant n + \operatorname{Ind}_0 Y$.

We shall show that *the inequality* $\operatorname{ind}_0 X \geqslant n$ *implies* $\operatorname{ind}_0 Z \geqslant n + \operatorname{ind}_0 Y$. This is obvious for $n = 0$. Assume that this is true for n. Let $\operatorname{ind}_0 X \geqslant n+1$. There is a point $x \in X$ and an open neighborhood $U \subset X$ of x such that every G_δ partition L in X between x and $X \setminus U$ has $\operatorname{ind}_0 L \geqslant n$. If $\Lambda \subset Z$ is a G_δ partition in Z between (μ, x) and $Z \setminus \lambda(A_{\mathfrak{m}}, U)$, then $\operatorname{ind}_0 L_\Lambda \geqslant n$, and by (‡) and the induction hypothesis we obtain $\lambda(A_\Lambda, L_\Lambda) \subset \Lambda$ with $\operatorname{ind}_0 \Lambda \geqslant \operatorname{ind}_0 \lambda(A_\Lambda, L_\Lambda) \geqslant n + \operatorname{ind}_0 Y$. Thus, $\operatorname{ind}_0 Z \geqslant \operatorname{ind}_{0(\mu,x)} Z \geqslant n + 1 + \operatorname{ind}_0 Y$. □

By induction we infer

Theorem 2. $\operatorname{Ind}_0 X_{m,n} = n2^{m-n}$ *for every pair of natural numbers* $m \geqslant n \geqslant 1$.

3. Remarks, generalizations, and an open problem

Let us note some more properties of spaces and maps constructed in Section 1.

Remark 1. In our construction, $X_{m+1,n} = Z(X_{m,n}, X_{m,n})$ is the disjoint union of two subspaces: $F_{m,n} = H(\mu)$ is closed and $G_{m,n} = X_{m+1,n} \setminus H(\mu)$ is the discrete sum of subspaces $H(\alpha)$, $\alpha \neq \mu$. Since $\operatorname{Ind} F_{m,n} = \operatorname{Ind} G_{m,n} = m$ and $\operatorname{Ind} X_{m+1,n} = m+1$, $X_{m+1,n}$ *is not hereditarily normal* by [9, Theorem 2.3.1]. Moreover, if $m = n$, then both the subspaces $F_{m,n}$ and $G_{m,n}$ are metrizable.

Example 2. Consider the map h defined before Lemma 4 and put $X = Y = I^n$. Then $h\colon Z(I^n, I^n) = X_{n+1,n} \to A_{\mathfrak{m}}$ is not an onto map (as $H(\alpha) = \emptyset$ if $\varphi\alpha = \emptyset$), but the image $hX_{n+1,n}$ is homeomorphic to $A_{\mathfrak{m}}$. Observe that *every point-inverse* $h^{-1}\alpha = H(\alpha)$ *is metrizable, and* h *is a counter-example to the theorem on dimension-lowering maps in all the three cases of* ind, Ind, *and* Ind_0. Indeed,

$$\begin{aligned} \operatorname{Ind}_0 X_{n+1,n} &= 2n \geqslant n+1 = \operatorname{Ind} X_{n+1,n} = \operatorname{ind} X_{n+1,n} \\ &> n = \operatorname{Ind}_0 hX_{n+1,n} + \operatorname{Ind}_0 h = \operatorname{Ind} hX_{n+1,n} + \operatorname{Ind} h \\ &= \operatorname{ind} hX_{n+1,n} + \operatorname{ind} h. \end{aligned}$$

A theorem on inductive-dimension-lowering maps holds in the following circumstances. A map $f\colon X \to Y$ between compact spaces X and Y is said to be *fully closed*[4] if for every pair of disjoint closed sets $F, G \subset X$ the intersection $fF \cap fG$ is finite. It immediately results from [13, Theorem 2.3] that, *if* f *is a fully closed map from a compact space* X *to a first countable space, then* $\operatorname{Ind}_0 X \leqslant \operatorname{Ind}_0 fX + \operatorname{Ind}_0 f$.

[4] Fully closed maps are usually investigated in much more general setting, cf. Fedorchuk [10] (an extensive survey). See [10, Section II.1] for equivalent definitions of this class of maps.

When in Theorem 3 below we consider the map $f\colon X \to X/\mathcal{D}$ that collapses every component of X to a point, then f is fully closed by (c), and consequently, we obtain

Theorem 3. *If X is a compact space such that*

(a) $\operatorname{ind} X < \infty$,

(b) *every component of X is a perfectly normal G_δ subspace, and*

(c) *for every pair of disjoint closed sets $F, G \subset X$ there is only a finite number of components P of X with $P \cap F \neq \emptyset \neq P \cap G$,*

then there is a component P of X such that $\operatorname{ind} P = \operatorname{ind} X = \operatorname{Ind} X = \operatorname{Ind}_0 X$.

At the end, we sketch a few modifications of our constructions. Our attention is now directed to the dimension Ind_0.

Remark 2. If we replace the family $\mathcal{S}_X$ by another one, $\mathcal{S}_X{}^{\leqslant 1}$, which consists of the empty set and all one-point subsets of X, we can repeat our construction in the same way and obtain a compact space $Z^{\leqslant 1}(X,Y)$ instead of $Z(X,Y)$. It is easily checked that Lemmas 4, 5, and 9 remain true if $Z(X,Y)$ is replaced by $Z^{\leqslant 1}(X,Y)$.

Observe that, *if Y is a non-empty compact space, then*

$$\operatorname{ind} Z^{\leqslant 1}(I,Y) = \max\{1, \operatorname{ind} Y\} \quad \text{and} \quad \operatorname{Ind} Z^{\leqslant 1}(I,Y) = \max\{1, \operatorname{Ind} Y\}.$$

Indeed, write $Z = Z^{\leqslant 1}(I,Y)$. If $\alpha \neq \mu$ and $\varphi\alpha \neq \emptyset$, then $h^{-1}\alpha = H(\alpha)$ is homeomorphic to Y, and $\operatorname{ind}_p Z \leqslant \operatorname{ind} Y$ for every $p \in H(\alpha)$. If $0 \leqslant t < s \leqslant 1$, $\mu \in A \subset A_{\mathfrak{m}}$, and $A_{\mathfrak{m}} \setminus A$ is finite, then the closed set $\Phi = Z \cap \pi_2^{-1}(A \times [t,s])$ has a finite boundary, $\operatorname{bd} \Phi = \{(\mu,s),(\mu,t)\} \setminus \{(\mu,0),(\mu,1)\} \subset H(\mu)$. Every point $p = (\mu,x) \in H(\mu)$ has arbitrarily small closed neighborhoods of the form Φ, and so, $\operatorname{ind}_p Z = 1$. The proof of the first equality is complete. Now, it suffices to show that $\operatorname{Ind} Z \leqslant \max\{1, \operatorname{Ind} Y\}$. Assume that $\operatorname{Ind} Y = n < \infty$, and take disjoint closed sets $F_0, F_1 \subset Z$. By an argument similar to that in our proof of Lemma 6, we infer that the set

$$A = \{\alpha \in A_{\mathfrak{m}} \setminus \{\mu\} \colon F_0 \cap H(\alpha) \neq \emptyset \neq F_1 \cap H(\alpha)\}$$

is finite. The pre-image $h^{-1}A$ is clopen in Z, and there exists a partition L in $h^{-1}A$ between $F_0 \cap h^{-1}A$ and $F_1 \cap h^{-1}A$, $\operatorname{Ind} L \leqslant n-1$. Every point $p \in F_0 \setminus h^{-1}A$ has an open neighborhood $U_p \subset \operatorname{cl} U_p \subset Z \setminus (F_1 \cup h^{-1}A)$ such that $\operatorname{bd} U_p$ has at most two elements. There are points $p_1, \ldots, p_k \in F_0 \setminus h^{-1}A$ with $F_0 \setminus h^{-1}A \subset V = U_{p_1} \cup \ldots \cup U_{p_k}$. $L \cup \operatorname{bd} V$ is a partition in Z between F_0 and F_1, and $\operatorname{Ind}(L \cup \operatorname{bd} V) \leqslant \max\{0, n-1\}$ as $\operatorname{bd} V$ is finite. Therefore, $\operatorname{Ind} Z \leqslant \max\{1,n\}$ and the second equality is true.

Let us define spaces by induction: $Y_{1,1,1} = I$ and $Y_{n+1,1,1} = Z^{\leqslant 1}(I, Y_{n,1,1})$ for $n \geqslant 1$. *Every $Y_{n,1,1}$ is a compact Fréchet space,* $\dim Y_{n,1,1} = \operatorname{ind} Y_{n,1,1} = \operatorname{Ind} Y_{n,1,1} = 1$, *and* $\operatorname{Ind}_0 Y_{n,1,1} = n$ (the last equality follows from the $Z^{\leqslant 1}$ analogue of Lemma 9).

If $n > 1$, then the map $f\colon Y_{n,1,1} \to Y_{n,1,1}/\mathcal{D}$ that collapses every component of $Y_{n,1,1}$ to a point has $\operatorname{Ind}_0 Y_{n,1,1} = n > 1 = \operatorname{Ind}_0 Y_{n,1,1}/\mathcal{D} + \operatorname{Ind}_0 f$, and every point-inverse of f is homeomorphic to $[0,1]$.

Chatyrko [4] constructed certain first countable compact spaces I_m with $\dim I_m = 1$ and $\operatorname{ind} I_m = m$. It appears that the spaces also have $\operatorname{Ind} I_m = \operatorname{Ind}_0 I_m = m$ (Krzempek [13, Corollary 2.7]). When we use the examples of Remark 2, Chatyrko's spaces I_m, n-dimensional cubes I^n, and take disjoint unions $Y_{k,m,n} = Y_{k,1,1} \oplus I_m \oplus I^n$, we obtain

Theorem 4. *For every triple of natural numbers $k \geqslant m \geqslant n \geqslant 1$ there exists a compact Fréchet space $Y_{k,m,n}$ such that $\dim Y_{k,m,n} = n$, $\operatorname{ind} Y_{k,m,n} = \operatorname{Ind} Y_{k,m,n} = m$, and $\operatorname{Ind}_0 Y_{k,m,n} = k$.*

Further modifications are directed towards other topological types of components as well as transfinite dimensions trind and trInd (see [9, Section 7.1] for definitions).

Remark 3. Suppose that K is a non-degenerate metric continuum (=connected compact space) with $\dim K = n < \infty$. The set $\{x \in K\colon \operatorname{ind}_x K = n\}$ is F_σ and n-dimensional (see [9, Exercise 1.5.H]). It follows from [9, Theorems 1.3.1 and 1.4.5] that the set contains a non-degenerate continuum B. It is easily checked that in our proof of Theorem 1, one can replace I^n and the arc B_n by K and the continuum B, respectively (since $\operatorname{ind}_{x+} K = n$ for $x \in B$). In this way, for $m \geqslant \dim K$ one obtains *compact Fréchet spaces $X_{m,K}$ such that* $\operatorname{ind} X_{m,K} = \operatorname{Ind} X_{m,K} = m$ *and every component of $X_{m,K}$ is homeomorphic to K.*

Remark 4. Define $\operatorname{trind}_{b+} X$ in the way similar to $\operatorname{ind}_{b+} X$ (see p. 5). One easily checks that Lemmas 6 and 7 remain true if Ind, ind, ind_b, and ind_{b+} are replaced by trInd, trind, trind_b, and $\operatorname{trind}_{b+}$, respectively. So, if we want to prove a transfinite analogue of Theorem 1, a successor step of transfinite induction can be taken.

Let K be a finite dimensional metric non-degenerate continuum, and let $\gamma \geqslant n = \dim K$ be a limit ordinal. Assume that for every ordinal δ, $n \leqslant \delta < \gamma$, there is a compact Fréchet space $X_{\delta,K}$ such that $\operatorname{trind} X_{\delta,K} = \operatorname{trInd} X_{\delta,K} = \delta$ and every component of $X_{\delta,K}$ is homeomorphic to K. We shall define $X_{\gamma,K}$ and $B_\gamma \subset X_{\gamma,K}$ so that the transfinite analogues of conditions (a)–(d) in the proof of Theorem 1 be satisfied. Consider the one-point compactification of the discrete sum $\bigoplus_{n\leqslant\delta<\gamma} X_{\delta,K}$, and join a homeomorphic copy of K to the compactification at the one-point remainder so as to obtain a compact space X_0 whose every component is homeomorphic to K. X_0 is Fréchet and $\operatorname{trind} X_0 = \operatorname{trInd} X_0 = \gamma$. Let $X_{\gamma,K} = Z(K, X_0)$. The trInd analogue of Lemma 6 implies that $\operatorname{trInd} X_{\gamma,K} \leqslant \gamma$. It is easily seen that $\operatorname{trind}_{(\mu,b)+} X_{\gamma,K} \geqslant \gamma$ for every point $(\mu, b) \in B_\gamma = H(\mu)$. By virtue of Lemmas 4–5, every component of $X_{\gamma,K}$ is homeomorphic to K, and $X_{\gamma,K}$ is Fréchet. Therefore, also the limit γth step of induction can be taken.

By transfinite induction and Remarks 3–4 we obtain

Theorem 5. *If K is a finite dimensional non-degenerate metric continuum and $\gamma \geqslant \dim K$ is an ordinal number, then there is a compact Fréchet space $X_{\gamma,K}$ such that*
(a) $\dim X_{\gamma,K} = \dim K$, $\operatorname{trind} X_{\gamma,K} = \operatorname{trInd} X_{\gamma,K} = \gamma$, *and*
(b) *every component of $X_{\gamma,K}$ is homeomorphic to K.*

We conclude this paper with a collection of questions (in fact, these are seven questions as ind = Ind for perfectly normal compact spaces).

Problem. Suppose that $\mathcal{K}$ is one of the following four classes of compact spaces: hereditarily normal compact spaces, first countable compact spaces, compact spaces whose every component is a G_δ-set, perfectly normal compact spaces. Then, does there exist a space $X \in \mathcal{K}$ whose every component P has $\operatorname{ind} P < \operatorname{ind} X < \infty$ ($\operatorname{Ind} P < \operatorname{Ind} X < \infty$)?

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