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Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 2, 411–422

Czechoslovak Mathematical Journal, 60 (135) (2010), 411–422

COHOMOLOGY OF CONFIGURATION SPACES OF COMPLEX PROJECTIVE SPACES

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(Received November 30, 2008)

Abstract. In this paper we compute topological invariants for some configuration spaces of complex projective spaces. We shall describe Sullivan models for these configuration spaces.

Keywords: configuration spaces, cohomological algebra, complex projective spaces

MSC 2010: 57N65, 55T10

1. Introduction

Let X be a connected space. The topology of *ordered configuration spaces*:

$$F(X;n) = \{(x_1, x_2, \ldots, x_n) \in X^n \mid x_i \neq x_j\,;\ i \neq j\},$$

of n distinct labeled points in X has attracted considerable attention over the years. The cohomology rings $H^*(F(\mathbb{R}^2;n))$ have been described by Arnold [1]. In his 1972 thesis F. Cohen extended Arnold's computations to all Eulidean spaces; see [4]. For X an m-dimensional real oriented manifold, the Leray spectral sequence of the inclusion $F(X;n) \hookrightarrow X^n$ has been described by Cohen-Taylor [5] and further analyzed by Totaro [9]. With coefficient field $\mathbb{K}$, the above Cohen-Taylor spectral sequence converges multiplicatively to $H^*(F(X;n),\mathbb{K})$; it has the property $E_2 = E_m$; and the differential graded algebra (E_m, d_m) depends only on n and the cohomology algebra $H^*(X;\mathbb{K})$. Configuration spaces $\mathbb{R}^{n+1}$ and S^{n+1} are mostly studied in the literature, particularly the case of $n=1$ is in connection with classical braid theory, see [2].

Our aim in this paper is to find the Betti numbers and cohomology algebras of *configuration spaces of complex projective spaces* $\mathbb{CP}^m$ and those of *punctured complex projective spaces* $\mathring{\mathbb{CP}}^m = \mathbb{CP}^m \setminus \{pt\}$.

Given X, a topological space with finite *Betti* numbers $\beta_i(X)$, we denote its *Poincaré series* by

$$P_X(t) = \sum_{i \geqslant 0} \beta_i(X) t^i.$$

In all computations of cohomology algebras we will use complex coefficients: $H^*(X) = H^*(X; \mathbb{C})$.

In our proofs the basic tools are the Kriz model [8] for the configuration spaces of an algebric projective manifold and also the punctured model for these spaces [3]. In Section 2 we give a short presentation of these models. We shall compute Poincaré polynomials for configuration spaces with < 5 points and also describe their cohomology algebras.

Theorem 1. *The Poincaré polynomial of the configuration space $F(\mathbb{CP}^m; 2)$ is given by a product of cyclotomic polynomials:*

$$P_{F(\mathbb{CP}^m;2)}(t) = \prod_{\substack{d \mid m(m+1) \\ d \neq 1}} \varphi_d(t^2).$$

Theorem 2. *The multiplicative structure of the cohomological algebra of the configuration space $F(\mathbb{CP}^m; 2)$ is given by*

$$H^*_{(F(\mathbb{CP}^m;2))} \cong \frac{\mathbb{C}[a_1, a_2]}{\left\langle a_1^m + a_1^{m-1} a_2 + \ldots + a_2^m \ \ ; a_1^{m+1}; a_2^{m+1} \right\rangle},$$

where $\deg a_1 = \deg a_2 = 2$.

Proposition 1. *The multiplicative structure of cohomological algebra of the configuration space $F(\overset{\circ\circ}{\mathbb{CP}^m}, 1)$ is given by*

$$H^*(F(\overset{\circ\circ}{\mathbb{CP}^m}, 1)) \cong \frac{\mathbb{C}\langle x, z \rangle}{\langle x^m, xz \rangle},$$

where $\deg x = 2$ *and* $\deg z = 2m - 1$.

Proposition 2. *The multiplicative structure of the cohomological algebra of the configuration space $F(\overset{\circ}{\mathbb{CP}^m}, 2)$ is given by*

$$H^*(F(\overset{\circ}{\mathbb{CP}^m}, 2)) \cong \frac{\mathbb{C}\langle y, z, w \rangle}{\langle y^m, z^m, y^{m-1}z + \ldots + yz^{m-1}, yw, zw \rangle},$$

where $\deg y = \deg z = 2$, $\deg w = 4m - 3$.

Theorem 3. *If $m \leqslant 4$ the Poincaré polynomial of the configuration space $F(\mathbb{CP}^m;3)$ is given by*

$$P_{F(\mathbb{CP}^m;3)}(t) = (1+t^2+t^4+\ldots+t^{2m-2})[(1+t^2+t^4+\ldots+t^{2m-2})^2+t^{4m-1}].$$

Theorem 4. *If $m \leqslant 4$ the multiplicative structure of the cohomological algebra of the configuration spaces $F(\mathbb{CP}^m;3)$ is given by*

$$H^*(F(\mathbb{CP}^m;3)) \cong \frac{\mathbb{C}\langle a, b_1, b_2, \eta\rangle}{\left\langle \sum_{i=1}^{m-1} b_1^{m-i} b_2^i, a^{m+1}, b_1^m, b_2^m, a^m\eta, b_1\eta, b_2\eta \right\rangle},$$

where $\deg a = 2$, *and* $\deg b_i = 2$, $i \in \{1,2\}$ *and* $\deg \eta = 4m-1$.

During the proofs we will describe completely the structure of the Serre spectral sequences of the natural fibrations associated to these spaces.

2. Sullivan models for algebraic configuration spaces

In this section we present two models for configuration spaces of algebraic projective manifolds. The first model was introduced by Fulton-MacPherson [7] and next a simplified version was given by Kriz [8].

Let M be a closed orientable manifold of dimension m with a fixed orientation class $\omega \in H^m(M)$. For an arbitrary homogenous basis $\{a_i\}$, $i = 1, 2, \ldots q$, in $H^*(M)$, take the dual basis $\{b_j\}_{j=1,2,\ldots q}$, $(a_i \cup b_j = \delta_{ij}\omega)$ and construct the *diagonal class* of $(M;\omega)$ by $\Delta = \sum_{i=0}^{q} a_i \otimes b_i \in H^*(M^2)$.

For $a \neq b \in \{1, 2, 3 \ldots, n\}$ let $p_a^*\colon H^*(M) \longrightarrow H^*(M^n)$ and $p_{ab}^*\colon H^*(M^2) \longrightarrow H^*(M^n)$ be the pullbacks of the projections maps

$$p_a\colon M^n \longrightarrow M, \quad p_a(x_1, \ldots x_a, \ldots x_n) = x_a,$$

and

$$p_{ab}\colon M^n \longrightarrow M^2, \quad p_{ab}(x_1, \ldots x_a, \ldots, x_b, \ldots x_n) = (x_a, x_b),$$

respectively.

Definition 1 [8]**.** Let H^* be a Poincaré duality algebra of dimension $2m$ and ω a fixed orientation class. Denote by $H^{*\otimes n}[G_{ab}]$ the algebra over $H^{\otimes n}$ with degree $2m-1$ exterior generators G_{ab}, $1 \leqslant a \neq b \leqslant n$. The Kriz model $E_n^*(H^*;\omega,d)$ is the differential graded algebra (DGA) given by the quotient of $H^{*\otimes n}[G_{ab}]$ modulo the following relations:

$$\begin{aligned} &1. \quad G_{ab} = G_{ba}, \\ &2. \quad p_a^*(x)G_{ab} = p_b^*(x)G_{ab}, \text{ for } x \in H^*; \\ &3. \quad G_{ab}G_{bc} + G_{bc}G_{ca} + G_{ca}G_{ab} = 0. \end{aligned}$$

The differential d of degree $+1$ is given by

$$d(p_a^*(x)) = 0$$

and

$$d(G_{ab}) = p_{ab}^*(\Delta).$$

Simplifying the model of Fulton-Macpherson, Kriz proved:

Theorem 5 [8]**.** *Let X be a complex projective manifold of dimension m with cohomology algebra $H = H^*(X;\mathbb{Q})$. Then the DGA $(E_n^*(H);d)$ is a rational model, in the sense of Sullivan, of the configuration space $F(X;n)$.*

Let H be an even-dimensional Poincaré duality algebra, as before. We are going to consider another associated DGA, to be denoted by $E_n(\overset{\circ}{H})$. To begin with, let $\overset{\circ}{H}$ be the quotient algebra, $\overset{\circ}{H} = H/\mathbb{C}.\omega$, with multiplication induced from H. Note that, when $H = H^*(M;\mathbb{C})$, with M a closed oriented manifold, $\overset{\circ}{H}$ is nothing else but the cohomology algebra of the non-compact punctured manifold $\overset{\circ}{M} = M \setminus \{pt\}$. Denote by $\overset{\circ}{\Delta}$ the image of Δ in $\overset{\circ}{H}{}^{\otimes 2}$ and consider the induced differential $d\colon \overset{\circ}{H} \to \overset{\circ}{H}$.

Definition 2 [3]**.** The punctured Kriz-model is the differential graded algebra $E_n(\overset{\circ}{H}) = E_n^*(\overset{\circ}{H}, \overset{\circ}{\Delta})$, with the induced differential d.

Theorem 6 [3]**.** *Let X be a 1-connected complex projective manifold of dimension m with cohomology algebra $H = H^*(X;\mathbb{C})$. Then the DGA $(E_n^*(\overset{\circ}{H});d)$ is a complex model, in the sense of Sullivan, of the configuration space $F(\overset{\circ}{X};n)$.*

We use the standard notation E_n^d for the homogenous part of total degree d and $E_n^*[k]$ for the homogenous component of degree k in the exterior generators G_{ab}; for instance, $E_n[0] = H^{\otimes n}$ and $G_{ab} \in E_n^{2m-1}[1]$.

3. Two points configuration spaces

Let $X = \mathbb{CP}^m$ and let $E^*(n;m)$ be the Kriz model $(E_n(H^*(\mathbb{CP}^m); x^m))$ where x is a fixed generator of $H^2(\mathbb{CP}^m)$ and x^m is the orientation class. The cohomology algebra of X is given by $H^*(X) = \mathbb{C}[x]/\langle x^{m+1}\rangle$ with $\deg x = 2$; i.e. $H^{2i}(X;\mathbb{C})$ has x^i as a basis $(0 \leqslant i \leqslant m)$ and all other cohomology groups are zero.

First we construct the Kriz model $E(2;m)$. Using Künneth formula we find the canonical basis of $H^{2i}(X^2)$: $x^i \otimes 1,\ x^{i-1} \otimes x, \ldots, 1 \otimes x^i$. Now we add the exterior part: $(x^i \otimes 1)G_{12} = (1 \otimes x^i)G_{12}$ $(i = 1, \ldots, m)$, where the degree of G_{12} is $2m-1$. The differential is given by:

$$d(G_{12}) = p_{12}^*(\Delta) = x^m \otimes 1 + x^{m-1} \otimes x + \ldots + 1 \otimes x^m.$$

P r o o f of Theorem 1. Now we will calculate the cohomology of $(E(2;m), d)$. It is obvious that if $0 \leqslant k \leqslant m-1$, $H^k(F(X;2)) \cong E^k$ hence $\beta_{2k-1} = 0$ and $\beta_{2k} = k+1$. In higher degrees the sequence of differentials is given by

$$\begin{aligned} &E^{2m+2k-1} \xrightarrow{d} E^{2m+2k} \xrightarrow{d} 0,\\ &d((x^k \otimes 1)G_{12}) = x^m \otimes x^k + x^{m-1} \otimes x^{k+1} + \ldots + x^k \otimes x^m, \end{aligned}$$

so the first d is injective. As $\dim E^{2m+2k-1} = 1$, the even Betti numbers are $\beta_{2m+2k} = \dim E^{2m+2k} - 1 = (m-k+1) - 1 = (m-k)$, and $\beta_{2m+2k-1} = 0$.

So the Poincaré polynomial is

$$\begin{aligned} P_{F(X;2)}(t) &= 1 + 2t^2 + \ldots + mt^{2(m-1)} + mt^{2m} + (m-1)t^{2m+2} + \ldots + t^{2(2m-1)}\\ &= (1 + t^2 + \ldots + t^{2(m-1)})(1 + t^2 + \ldots + t^{2(m-1)} + t^{2m}). \end{aligned}$$

We denote by $\varphi_n(t)$ be the *nth cyclotomic* polynomial:

$$\varphi_n(t) = \prod_{1 \leqslant i \leqslant n, (i,n)=1} (t - \alpha^i),$$

where α is a primitive root of $t^n - 1$. We have, for every positive integer n,

$$t^n - 1 = \prod_{d|n} \varphi_d(t);$$

hence the Poincaré polynomial can be decomposed into irreducible factors in $\mathbb{Z}[t]$ as

$$P_{F(X;2)}(t) = \prod_{\substack{d|m(m+1)\\ d \neq 1}} \varphi_d(t^2).$$

□

P r o o f of Theorem 2. Introducing the cohomology classes $a_1 = [x \otimes 1]$ and $a_2 = [1 \otimes x]$ we have $[x^i \otimes x^j] = [x^i \otimes 1][1 \otimes x^i] = a_1^i a_2^j$, therefore a_1 and a_2 are generators of $H^*(E(2); d)$. Then we have to prove that $a_k^{m+1} = 0$ and $a_k^m \neq 0$, where $k = 1, 2$. The first one is clear because $x^{m+1} \otimes 1 = 0$ and so $[x^{m+1} \otimes 1] = 0$. But $x^m \neq 0$ because $d(\lambda G_{12}) \neq x^m \otimes 1$ where $\lambda \in \mathbb{C}$ and so $[x^m \otimes 1] \neq 0$. The equation $d(G_{12}) = x^m \otimes 1 + x^{m-1} \otimes x + \ldots + 1 \otimes x^m$ implies $[x^m \otimes 1 + x^{m-1} \otimes x + \ldots + 1 \otimes x^m] = 0$. So we obtain a relation $a_1^m + a_1^{m-1} a_2 + a_1^{m-2} a_2^2 + \ldots + a_2^m = 0$ and

$$H^*(F(\mathbb{CP}^m; 2)) \cong \frac{\mathbb{C}[a_1, a_2]}{\left\langle a_1^m + a_1^{m-1} a_2 + \ldots + a_2^m;\ a_1^{m+1};\ a_2^{m+1} \right\rangle},$$

where $\deg a_1 = a_2 = 2$. $\square$

4. Three points configuration spaces

In our approach to compute $H^*(F(\mathbb{CP}^m))$ we use two different fibrations, their associated spectral sequences, and comparing the possible results we shall use Kriz model in a unique dimension in order to remove the indeterminacy.

P r o o f of Proposition 1. Using the Mayer-Vietoris sequence for $X = \mathbb{CP}^m \setminus \{z_1\}$, $Y = \mathbb{CP}^m \setminus \{z_2\}$ and $X \cap Y = \mathbb{CP}^m \setminus \{z_1, z_2\}$, we obtain the Betti numbers of $\overset{\circ\circ}{\mathbb{CP}}{}^m$. Using the functoriality properties of the cup product we obtain the multiplicative structure of the cohomology algebra of $H^*(F(\overset{\circ\circ}{\mathbb{CP}}{}^m, 1))$. It is given by:

$$H^*(F(\overset{\circ\circ}{\mathbb{CP}}{}^m, 1)) \cong \frac{\mathbb{C}\langle x, z\rangle}{\langle x^m, xz\rangle},$$

where $\deg x = 2$ and $\deg z = 2m - 1$. $\square$

P r o o f of Proposition 2. In order to compute the cohomology algebra of $F(\overset{\circ}{\mathbb{CP}}{}^m, 2)$ the differential of the punctured Kriz model is given by:

$$d(G) = x^{m-1} \otimes x + x^{m-2} \otimes x^2 + \ldots + x \otimes x^{m-1}.$$

The non-zero Betti numbers are: $\beta_0 = 1$, $\beta_{2m} = m - 2$, $\beta_{4m-3} = 1$, and the cohomology algebra of the punctured model is generated by $y = [x \otimes 1]$, $z = [1 \otimes x]$, $w = [(x^{m-1} \otimes 1)G]$ and has the presentation:

$$H^*(\overset{\circ}{\mathbb{CP}}{}^m; 2) = \frac{\mathbb{C}\langle y, z, w\rangle}{\langle y^m, z^m, , y^{m-1}z + \ldots + yz^{m-1}, yw, zw\rangle}$$

where $\deg y = z = 2$ and $\deg w = 4m - 3$. $\square$

P r o o f of Theorem 3. For $m = 1, 2$ we obtain easily the results [1]:

$$P_{F(\mathbb{CP}^1;3)}(t) = 1 + t^3,$$
$$P_{F(\mathbb{CP}^2;3)}(t) = (1 + t^2)[(1 + t^2)^2 + t^7].$$

For $m \geqslant 3$ the direct computations in the Kriz model are too complicated. So we use the Leray-Serre spectral sequences for two fibrations; the first one is:

$$[2-3-1]^3\colon\ F(\overset{\circ}{\mathbb{CP}^3}, 2) \xrightarrow{i} F(\mathbb{CP}^3; 3) \xrightarrow{pr_1} F(\mathbb{CP}^3; 1).$$

Here $\pi_1(\mathbb{CP}^3) = 1$. By Proposition 2 the cohomology algebra of $F(\overset{\circ}{\mathbb{CP}^3}, 2)$ is generated by $y = [x \otimes 1]$, $z = [1 \otimes x]$, $w = [(x^2 \otimes 1)G_{12}]$ and presented by:

$$H^*(F(\overset{\circ}{\mathbb{CP}^3}, 2)) = \frac{\mathbb{C}\langle y, z, w\rangle}{\langle y^3, z^3, y^2z + yz^2, yw, zw\rangle}$$

and $\deg y = z = 2$, $\deg w = 9$.

Firstly we compute the Serre spectral sequence of this fibration. In the range $q \leqslant 6$ everything is concentrated in even degrees hence all the differentials are zero. It is obvious that $E_2 = E_3 = E_4$. For $d_4(w)$ we have two possibilities, $d_4(w) = 0$ or $d_4(w) \neq 0:$

C a s e 1: if $d_4(w) \neq 0$ the spectral sequence collapses at E_5, so the Poincaré polynomial is given by:

$$P_1(t) = 1 + 3t^2 + 6t^4 + 7t^6 + 6t^8 + 3t^{10} + t^{13} + t^{15}.$$

C a s e 2: if $d_4(w) = 0$ then we have two subcases $d_6(w) = 0$ or $d_6(w) \neq 0$.

C a s e 2.1: If $d_4(w) = 0$ and $d_6(w) \neq 0$ the spectral sequence collapses at E_7 and the Poincaré polynomial is given by:

$$P_2(t) = 1 + 3t^2 + 6t^4 + 7t^6 + 6t^8 + 3t^{10} + t^{11} + t^{12} + t^{13} + t^{15}.$$

C a s e 2.2: If $d_4(w) = 0$ and $d_6(w) = 0$ the spectral sequence collapses at E_2 (see Figs. 1 and 2) and the Poincaré polynomial is given by:

$$\begin{aligned} P_3(t) &= (1 + t^2 + t^4 + t^6)(1 + 2t^2 + 3t^4 + t^6 + t^9) \\ &= 1 + 3t^2 + 6t^4 + 7t^6 + 6t^8 + t^9 + 4t^{10} + t^{11} + t^{12} + t^{13} + t^{15}. \end{aligned}$$

Next we will calculate the Poincaré polynomial of $F(\mathbb{CP}^3; 3)$ by using the second fibration:

$$[1-3-2]^3\colon\ F(\overset{\circ\circ}{\mathbb{CP}^3}; 1) \xrightarrow{i} F(\mathbb{CP}^3; 3) \xrightarrow{pr_{12}} F(\mathbb{CP}^3; 2)$$

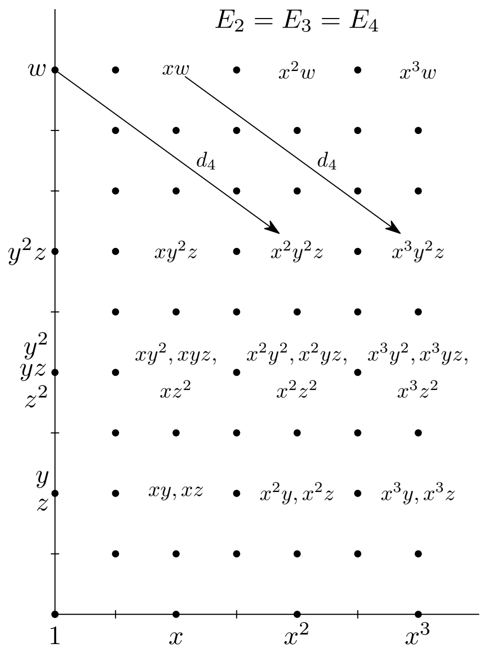

Fig. 1

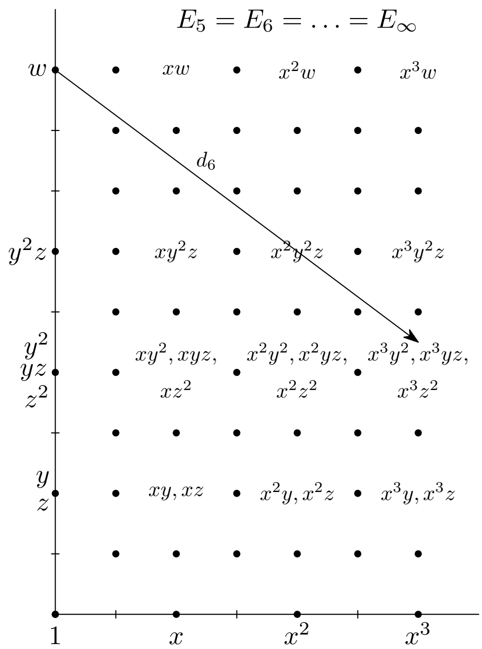

Fig. 2

where $\overset{\circ\circ}{\mathbb{CP}}{}^3 = F(\mathbb{CP}^3 \setminus \{z_1, z_2\})$. The cohomology of the fiber space can be calculated by using Proposition 1 and the cohomology algebra is given by: $H^*(F(\overset{\circ\circ}{\mathbb{CP}}{}^3; 1) = \mathbb{C}\left\langle x, z\right\rangle / \left\langle x^3, xz\right\rangle$; where deg $x = 2$, $z = 5$. For the differential $d_2(z)$ there are two possibilities: $d_2(z) = 0$ or $d_2(z) \neq 0$

Case 1.1: if $d_2(z) \neq 0$ and $d_4(w_1) = 0$, the sequence collapses at E_3 and the Poincaré polynomial is given by:

$$Q_1(t) = 1 + 3t^2 + 6t^4 + 7t^6 + 6t^8 + 3t^{10} + t^{11} + t^{12} + t^{13} + t^{15}.$$

Case 1.2: if $d_2(z) \neq 0$ and $d_4(w_1) \neq 0$ the sequence collapses at E_5 and the Poincaré polynomial is given by:

$$Q_2(t) = 1 + 3t^2 + 6t^4 + 7t^6 + 6t^8 + 3t^{10} + t^{13} + t^{15}.$$

Case 2.1: if $d_2(z) = 0$ and $d_4(z) \neq 0$, $d_6(y_1) = 0$ the sequence collapses at E_5 and the Poincaré polynomial is given by:

$$Q_3(t) = 1 + 3t^2 + 6t^4 + 7t^6 + 6t^8 + 4t^{10} + 2t^{11} + 2t^{13} + t^{14} + t^{15}.$$

Case 2.2: if $d_2(z) = 0$ and $d_4(z) \neq 0$ and $d_6(y_1) \neq 0$ the sequence collapses at E_7, so the Poincaré polynomial is given by:

$$Q_4(t) = 1 + 3t^2 + 6t^4 + 7t^6 + 6t^8 + 3t^{10} + 2t^{11} + 2t^{12} + 2t^{13} + t^{14} + t^{15}.$$

Case 2.3: if $d_2(z) = d_4(z) = 0$ but $d_6(z) \neq 0$ the sequence collapses at E_7, so the Poincaré polynomial is given by:

$$Q_5(t) = 1 + 3t^2 + 6t^4 + 7t^6 + 6t^8 + 2t^9 + 5t^{10} + 3t^{11} + 3t^{12} + 2t^{13} + t^{14} + t^{15}.$$

Case 2.4: when $d_2(z) = d_4(z) = d_6(z) = 0$ the sequence collapses at E_2, the Poincaré polynomial is given by:

$$Q_6(t) = (1 + t^2 + t^4 + t^5)(1 + 2t^2 + 3t^4 + 3t^6 + 2t^8 + t^{10}).$$

One can see that between two sets of Poincaré polynomials there are only two matches, $P_1(t) = Q_2(t)$ and $P_2(t) = Q_1(t)$. Looking at the Kriz model in dimension 11 we have to decide whether β_{11} is 0 or 1. The differential

$$d\colon\ E^{11}[1] \longrightarrow E^{12}[0]$$

has a matrix representation: $A = \begin{pmatrix} I_3 & * & * & * \\ 0 & I_3 & * & * \\ 0 & 0 & B & C \end{pmatrix}$, where $B = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So the rank of A is 9 therefore $H^{11}(\mathbb{CP}^3; 3) \neq 0$ and the Poincaré polynomial of $F(\mathbb{CP}^3; 3)$ is

$$P(t) = 1 + 3t^2 + 6t^4 + 7t^6 + 6t^8 + 3t^{10} + t^{11} + t^{12} + t^{13} + t^{15},$$
$$P(t) = (1 + t^2 + t^4)[(1 + t^2 + t^4)^2 + t^{11}].$$

□

Corollary 1. *In the Leray- Serre spectral sequence of the first fibraton*

$$[2-3-1]^3\colon\ F(\overset{\circ}{\mathbb{CP}^3}; 2) \overset{i}{\longrightarrow} F(\mathbb{CP}^3, 3) \overset{pr_1}{\longrightarrow} F(\mathbb{CP}^3; 1)$$

$d_4(w) = d_5(w) = 0$ but $d_6(w) \neq 0$. Thus the spectral sequence start with $E_2 = E_3 = \ldots = E_6$ and collapses at E_7.

Corollary 2. *In the Leray- Serre spectral sequence of the second fibraton*

$$[1-3-2]^3\colon\ F(\overset{\circ\circ}{\mathbb{CP}^3};1) \xrightarrow{\ i\ } F(\mathbb{CP}^3;3) \xrightarrow{pr_{12}} F(\mathbb{CP}^3;2)$$

$d_2(z) \neq 0$ *and* $d_4(w_1) = 0$*. Thus the spectral sequence collapses at* E_3.

For $m = 4$, with the same method, we use the Leray-Serre spectral sequences of the fibrations

$$[2-3-1]^4\colon\ F(\overset{\circ}{\mathbb{CP}^4};2) \xrightarrow{\ i\ } F(\mathbb{CP}^4,3) \xrightarrow{pr_1} F(\mathbb{CP}^4;1)$$

and

$$[1-3-2]^4\colon\ F(\overset{\circ\circ}{\mathbb{CP}^4};1) \xrightarrow{\ i\ } F(\mathbb{CP}^4;3) \xrightarrow{pr_{12}} F(\mathbb{CP}^4;2).$$

Here $\pi_1(\mathbb{CP}^4) = 1$. By looking at all possible cases we have only three matches and these have different β_{15}. The Kriz model in dimension 15 has the differential

$$d\colon\ E^{15}[1] \longrightarrow E^{16}[0]$$

given by a matrix representation: $A = \begin{pmatrix} I_3 & * & * & * & * \\ 0 & I_3 & * & * & * \\ 0 & 0 & 0 & 0 & B \\ 0 & 0 & C & I_3 & I_3 \\ 0 & 0 & I_3 & C & 0 \end{pmatrix}$, where $B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. So the rank of A is 12, therefore $H^{15}(\mathbb{CP}^4;$ $3 \neq 0$ and the Poincaré polynomial of $F(\mathbb{CP}^4;3)$ is

$$\begin{aligned} P(t) &= 1 + 3t^2 + 6t^4 + 10t^6 + 12t^8 + 12t^{10} + 10t^{12} \\ &\quad + 6t^{14} + t^{15} + 3t^{16} + t^{17} + t^{18} + t^{19} + t^{21}. \\ P(t) &= (1 + t^2 + t^4 + t^6)[(1 + t^2 + t^4 + t^6)^2 + t^{15}]. \end{aligned}$$

P r o o f of Theorem 4. We give complete details of the proof for the case $m = 3$. Introduce $a = [x]$, $b_1 = [y]$ and $b_2 = [z]$, generators of degree 2. Also take $\eta = [xw]$, as an exterior generator of degree 11. In the cohomology of the basis $x^4 = 0$, hence $a^4 = 0$; and $a^3 \neq 0$ because no differential is pointing towards x^3. It is also clear that $b_i^3 = 0$, $i = 1, 2$ and $b_i^2 \neq 0$ because all the incident differentials are zero. From degree reason, we find $\eta^2 = 0$, $b_i\eta = 0$ if $i = 1, 3$. Because $[x^k w] = [x^{k-1}][xw]$ $(k = 1, 2, 3)$, we find $a^k\eta \neq 0$, if $k = 1, 2$, and $a^3\eta = 0$.

$$H^*(F(\mathbb{CP}^3; 3)) \cong \frac{\mathbb{C}\langle a, b_1, b_2, \eta\rangle}{\langle b_1 b_2^2 + b_1^2 b_2, a^3, b_1^3, b_2^3, a^3\eta, b_1\eta, b_2\eta\rangle}$$

where $\deg a = 2$, $\deg b_i = 2$, $i \in \{1, 2\}$, and $\deg \eta = 11$.

Similarly we can prove the case $m = 4$. □

Our computations suggest the following conjectures:

Conjecture 1. *The Poincaré polynomial of the configuration space $F(\mathbb{CP}^m; 3)$ is given by*

$$P_{F(\mathbb{CP}^m;3)}(t) = (1 + t^2 + t^4 + \ldots + t^{2m-2})[(1 + t^2 + t^4 + \ldots + t^{2m-2})^2 + t^{4m-1}].$$

Conjecture 2. *The multiplicative structure of the cohomological algebra of the configuration spaces $F(\mathbb{CP}^m; 3)$ is given by*

$$H^*(F(\mathbb{CP}^m; 3)) \cong \frac{\mathbb{C}\langle a, b_1, b_2, \eta\rangle}{\left\langle \sum_{i=1}^{m-1} b_1^{m-i} b_2^i, a^{m+1}, b_1^m, b_2^m, a^m\eta, b_1\eta, b_2\eta \right\rangle},$$

where $\deg a = 2$, $\deg b_i = 2$, $i \in \{1, 2\}$, *and* $\deg \eta = 4m - 1$.

Conjecture 3. *In the Leray-Serre spectral sequence of the first fibration*

$$[2-3-1]^3\colon\ F(\overset{\circ}{\mathbb{CP}}{}^m; 2) \xrightarrow{i} F(\mathbb{CP}^m, 3) \xrightarrow{pr_1} F(\mathbb{CP}^m; 1)$$

$d_4(w) = d_6(w) = \ldots = d_{2m-2}(w) = 0$ *but* $d_{2m}(w) \neq 0$. *Thus the spectral sequence start with* $E_2 = E_3 = \ldots = E_{2m}$ *and collapses at* E_{2m+1}.

Conjecture 4. *In the Leray-Serre spectral sequence of the second fibration*

$$[1-3-2]^3\colon\ F(\overset{\circ\circ}{\mathbb{CP}}{}^m; 1) \xrightarrow{i} F(\mathbb{CP}^m; 3) \xrightarrow{pr_{12}} F(\mathbb{CP}^m; 2)$$

$d_2(z) \neq 0$ *and* $d_4(w_1) = d_5(w_1) = d_6(w_1) = \ldots = d_{2m-2}(w_1) = 0$. *Thus the spectral sequence collapses at* E_3.

Acknowledgement. Tanweer Sohail would like to thank Dr. B. Berceanu for many valuable discussions. This research was partly supported by the Higher Education Commission of Pakistan.

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