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BOUNDS ON LAPLACIAN EIGENVALUES RELATED TO TOTAL AND SIGNED DOMINATION OF GRAPHS

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Abstract. A total dominating set in a graph G is a subset X of $V(G)$ such that each vertex of $V(G)$ is adjacent to at least one vertex of X. The total domination number of G is the minimum cardinality of a total dominating set. A function $f\colon V(G) \to \{-1,1\}$ is a signed dominating function (SDF) if the sum of its function values over any closed neighborhood is at least one. The weight of an SDF is the sum of its function values over all vertices. The signed domination number of G is the minimum weight of an SDF on G. In this paper we present several upper bounds on the algebraic connectivity of a connected graph in terms of the total domination and signed domination numbers of the graph. Also, we give lower bounds on the Laplacian spectral radius of a connected graph in terms of the signed domination number of the graph.

Keywords: algebraic connectivity, Laplacian matrix, Laplacian spectral radius, signed domination, total domination

MSC 2010: 05C50, 05C69

1. Introduction

All graphs considered here are finite, undirected, and simple. For standard graph theory terminology not given here, we refer to [8]. Specifically, let $G=(V,E)$ be a graph with vertex set V and edge set E. The *order* of G is given by $n=|V|$. For a vertex v in V, the *open neighborhood* of v is $N(v)=\{u\in V;\ uv\in E\}$ and the *closed neighborhood* of v is $N[v]=\{v\}\cup N(v)$. For a subset $S\subseteq V$, the open neighborhood of S is $N(S)=\bigcup\limits_{v\in S} N(v)$ and the closed neighborhood of S is $N[S]=\bigcup\limits_{v\in S} N[v]$. The *degree* of v in G is denoted by $d(v)$, which equals to $|N(v)|$. If all vertices of G have the same degree k, then G is *k-regular*, or simply *regular*. For a subgraph H of G,

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let $N_H(v) = N(v) \cap V(H)$ and $|N_H(v)| = d_H(v)$. If $H = G$, then $N_H(v)$ and $d_H(v)$ are written as $N(v)$ and $d(v)$ respectively. Let $\Delta(G)$ and $\delta(G)$ be the maximum and minimum degree of vertices of G, respectively. Let $S \subseteq V$. Denote by $G[S]$ the subgraph of G induced by S. In the case of no confusion, we write $N_S(v)$ and $d_S(v)$ instead of $N_{G[S]}(v)$ and $d_{G[S]}(v)$, respectively. For disjoint subsets U and W of vertices of G, denote by $e(U, W)$ the number of edges between U and W.

A set $S \subseteq V$ is a *total dominating set* of a graph G if every vertex in V is adjacent to a vertex in S, that is $N(S) = V$. Every graph without isolated vertices has a total dominating set, since $S = V$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. Total domination in graphs was introduced by Cockayne *et al.* [2] and is now well studied in graph theory (see, for example, [8]).

For a real-valued function $f\colon V \to \mathbb{R}$, the weight of f is $w(f) = \sum\limits_{v \in V} f(v)$. For $S \subseteq V$, define $f(S) = \sum\limits_{v \in S} f(v)$, so that $w(f) = f(V)$. For a vertex v in V, denote $f(N(v))$ by $f[v]$ for notational convenience.

Dunbar et al. [3] defined the signed dominating function. Let $f\colon V \to \{-1, 1\}$ be a function which assigns to each vertex of a graph without isolated vertices an element in the set $\{-1, 1\}$. Then, f is called *signed dominating function* (SDF) if for every $v \in V$, $f(N[v]) \geqslant 1$. The *signed domination number*, denoted by $\gamma_s(G)$, of G is the minimum weight of an SDF on G.

Let $A(G)$ be the adjacency matrix of G and $D(G) = \operatorname{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin's Theorem, it follows that its eigenvalues are nonnegative real numbers. The eigenvalues of an $n \times n$ matrix M are denoted by $\lambda_1(M), \lambda_2(M), \ldots, \lambda_n(M)$, while for a graph G, we will use $\lambda_i(G) = \lambda_i$ to denote $\lambda_i(L(G))$, $i = 1, 2, \ldots, n$ and assume that $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \ldots \geqslant \lambda_{n-1}(G) \geqslant \lambda_n(G)$. It is well known that $\lambda_n(G) = 0$ and the algebraic multiplicity of zero as an eigenvalue of $L(G)$ is exactly the number of connected components of G (see, [12]). In particular, the second smallest eigenvalue $\lambda_{n-1}(G) > 0$ if and only if G is connected. This leads Fiedler [6] to define it as the algebraic connectivity of G. The eigenvalue $\lambda_1(G)$ is called the Laplacian spectral radius of G. It is known that $\lambda_1(G) = \max \lambda_1(G_i)$, $i = 1, 2, \ldots, n$ if $G_1, G_2, \ldots, G_n$ are all components of G [6]. In recent years, the eigenvalues $\lambda_1(G)$ and $\lambda_{n-1}(G)$ have received a great deal of attention (see, for example, [12], [13]).

This paper is motivated by some recent papers (see, [5], [11], [13]) on graph eigenvalues involving domination of graphs. In this paper we further investigate the relationship among the Laplacian eigenvalues, total domination and signed domination in graphs. We give upper bounds on the algebraic connectivity $\lambda_{n-1}(G)$ for a connected graph G in terms of its total domination and signed domination numbers. Moreover, we establish lower bounds on the Laplacian spectral radius $\lambda_1(G)$ of G in terms of its the signed domination number.

2. Preliminary results

Let $G = (V, E)$ be a graph of order n and X a nonempty subset of V. The *edge density* of X is defined as

$$\varrho(X) = \frac{ne(X, X^c)}{|X||X^c|},$$

where $X^c = V - X$.

Fallat et al. [4] established the following relation between the edge density of X and the eigenvalue $\lambda_{n-1}(G)$.

Lemma 1 ([4]). *If $G = (V, E)$ is a graph of order n and X is nonempty subset of V, then*

$$\lambda_{n-1}(G) \leqslant \varrho(X) = \frac{ne(X, X^c)}{|X||X^c|}.$$

Lemma 2 ([6]). *For a graph G of order n, $\lambda_{n-1}(G) = n$ if and only if $G = K_n$.*

Lemma 3 ([7]). *If $G = (V, E)$ is a graph of order n and size $|E| > 0$, then $\lambda_1(G) \geqslant \Delta(G)+1$; if G is connected, then equality holds if and only if $\Delta(G) = n-1$.*

Proposition 4 ([2]). *If $G = (V, E)$ is a connected graph and X is a minimal total dominating set of G, then each vertex $v \in X$ has at least one of the following two properties:*
P_1: There exists a vertex $u \in V - X$ such that $N(u) \cap X = \{v\}$.
P_2: $G[X - \{v\}]$ contains an isolated vertex.

Lemma 5 ([10]). *If $G = (V, E)$ is a connected graph of order $n \geqslant 3$ and $G \neq K_n$, then G has a minimum total dominating set X in which every vertex has property P_1, or is adjacent to a vertex of degree 1 in $G[X]$ that has property P_1.*

Lemma 6 ([2]). *If G is a connected graph of order n and $\Delta(G) < n - 1$, then $\gamma_t(G) \leqslant n - \Delta(G)$.*

Mei Lu et al. [11] established the lower bound below on the Laplacian spectral radius $\lambda_1(G)$.

Lemma 7 ([11]). *Let $G=(V,E)$ be a connected graph of order n and G_1 be an induced subgraph of G with n_1 ($n_1<n$) vertices and average degree r_1 (i.e., $r_1=\sum\limits_{v\in V(G_1)} d_{G_1}(v)/n_1$). Set $d_1=\sum\limits_{v\in V(G_1)} d(v)/n_1$. Then*

$$\lambda_1(G)\geqslant\frac{n(d_1-r_1)}{n-n_1}.$$

Moreover, if the equality holds, then $d_{G_2}(u)=s$ for each vertex $u\in V(G_1)$ and $d_{G_1}(v)=t$ for each vertex $v\in V(G_2)$, where $G_2=G[V-V(G_1)]$.

Lemma 8 ([9]). *If G is a k-regular graph of order n, then*

$$\gamma_s(G)\geqslant\begin{cases}\dfrac{2n}{k+1} & \text{for } k \text{ odd},\\[2mm] \dfrac{n}{k+1} & \text{for } k \text{ even},\end{cases}$$

and these bounds are sharp.

By Lemma 8, we can determine $\gamma_s(K_n)$ for a complete graph K_n.

Lemma 9. *For a complete graph K_n, we have $\gamma_s(K_n)=1$ for n odd and $\gamma_s(K_n)=2$ for n even.*

Proof. By Lemma 5, clearly $\gamma_s(K_n)\geqslant 1$ for n odd and $\gamma_s(K_n)\geqslant 2$ for n even. On the other hand, we assign to $\frac{1}{2}(n+1)$ vertices in K_n the value 1, and to all other vertices -1 if n is odd, this produces a signed dominating function f with weight $w(f)=1$. Similarly, if n is even, we can produce a signed dominating function f with weight $w(f)=2$. So the equalities hold, as required. $\square$

3. Upper bounds on algebraic connectivity

Let $G=(V,E)$ be a connected graph and X a minimal total dominating set of G. For every vertex $v\in X$ satisfying Property P_1, we define $PN(v,X)=N(v)-N[X-\{v\}]$ which is called the *private neighbors* of v with respect to X.

First, we give an upper bound on the algebraic connectivity of a graph G in terms of its total domination number.

Theorem 10. *If G is a connected graph of order $n \geqslant 3$ and $G \neq K_n$, then*

$$\lambda_{n-1}(G) \leqslant \frac{n(n - \frac{1}{2}(3\gamma_t(G) - 1))}{n - \gamma_t(G)}.$$

P r o o f. Let X be a minimal total dominating set of G with $|X| = \gamma_t(G)$. Suppose that each vertex in X has Property P_1, then $\sum\limits_{v \in X} |PN(v, X)| \geqslant |X| = \gamma_t(G)$, hence

$$\begin{aligned} e(X, X^c) &\leqslant \sum_{v \in X} |PN(v, X)| + |X| \Big(n - |X| - \sum_{v \in X} |PN(v, X)| \Big) \\ &= \gamma_t(G)(n - \gamma_t(G)) - (\gamma_t(G) - 1) \sum_{v \in X} |PN(v, X)| \\ &\leqslant \gamma_t(G)(n - 2\gamma_t(G) + 1). \end{aligned}$$

Thus, by Lemma 1, we have

$$\lambda_{n-1}(G) \leqslant \frac{n(n - 2\gamma_t(G) + 1)}{n - \gamma_t(G)} \leqslant \frac{n(n - \frac{1}{2}(3\gamma_t(G) - 1))}{n - \gamma_t(G)}.$$

We may therefore assume that there exists at least one vertex in X that does not have Property P_1. Set

$$\begin{aligned} A &= \{v \in X;\ v \text{ has Property } P_1\}, \\ B &= X - A, \\ A_1 &= \{v \in A;\ d_X(v) = 1 \text{ and there exists a vertex } u \in B \text{ such that } uv \in E(G)\}, \\ A_2 &= A - A_1. \end{aligned}$$

Then $B \neq \emptyset$ and $|X| = |A| + |B|$. By Lemma 5, $|B| \leqslant |A_1| \leqslant |A|$. This implies that $\gamma_t(G) = |X| \leqslant 2|A|$. Hence $\frac{1}{2}\gamma_t(G) \leqslant |A| \leqslant \sum\limits_{v \in A} |PN(v, X)|$. By estimating the number of edges between X and X^c, we have

$$\begin{aligned} e(X, X^c) &\leqslant \sum_{v \in A} |PN(v, X)| + |X| \Big(n - |X| - \sum_{v \in A} |PN(v, X)| \Big) \\ &= \gamma_t(G)(n - \gamma_t(G)) - (\gamma_t(G) - 1) \sum_{v \in A} |PN(v, X)| \\ &\leqslant \gamma_t(G)(n - \gamma_t(G)) - \frac{1}{2}\gamma_t(G)(\gamma_t(G) - 1) \\ &= \gamma_t(G)\Big(n - \frac{3}{2}\gamma_t(G) + \frac{1}{2}\Big). \end{aligned}$$

Thus, by Lemma 1, we have

$$\lambda_{n-1}(G) \leqslant \frac{n(n-\frac{1}{2}(3\gamma_t(G)-1))}{n-\gamma_t(G)},$$

and the desired result follows. □

Note that if G is connected, then $\lambda_{n-1}(G) > 0$. Hence, by Theorem 10, we have $\gamma_t(G) < \frac{1}{3}(2n+1)$. This implies that $\gamma_t(G) \leqslant \frac{2}{3}n$ as $\gamma_t(G)$ is an integer. Moreover, every complete graph K_n ($n \geqslant 3$) clearly satisfies the formulation, so we obtain the following result due to Cockayne et al. [2].

Corollary 11 ([2]). *If G is a connected graph of order $n \geqslant 3$, then $\gamma_t(G) \leqslant \frac{2}{3}n$.*

Next we present another upper bound on the algebraic connectivity $\lambda_{n-1}(G)$ for a connected graph G of order $n \geqslant 3$ that is not isomorphic to a complete graph.

Theorem 12. *If $G = (V, E)$ is a connected graph of order $n \geqslant 3$ and $G \neq K_n$, then*

$$\lambda_{n-1}(G) \leqslant n - \gamma_t(G) \tag{1}$$

where equality holds if and only if G is either the complement of a graph consisting of some K_2's or the complement of a graph consisting of some K_2's and isolated vertices.

Proof. Let X be a minimal total dominating set of G with $|X| = \gamma_t(G)$. We claim that $\delta(G) \leqslant n - \gamma_t(G)$. Indeed, by Lemma 7, there exists at least a vertex $u \in V - X$ that is a private neighbor of some vertex v in X, i.e., $N(u) \cap X = \{v\}$, so $\delta(G) \leqslant d(u) \leqslant (n-1) - |X - \{v\}| = n - \gamma_t(G)$. According to our assumption $G \neq K_n$, this means that $|E(\overline{G})| \geqslant 1$. By Lemma 3, we have

$$\lambda_{n-1}(G) = n - \lambda_1(\overline{G}) \leqslant n - \Delta(\overline{G}) - 1 = n - (n - 1 - \delta(G)) - 1 \leqslant n - \gamma_t(G).$$

Suppose now that the equality holds in (1), this implies that $\delta(G) = n - \gamma_t(G)$ and $\lambda_1(\overline{G}) = \Delta(\overline{G}) + 1 = \gamma_t(G)$.

If $\Delta(G) < n-1$, then, by Lemma 6, we have $\delta(G) = \Delta(G)$, that is, G is a regular graph. Thus $\overline{G}$ is also a regular graph. We claim that $\overline{G}$ is disconnected. Otherwise, by Lemma 3, we have $\Delta(\overline{G}) = |V(\overline{G})| - 1 = n - 1$. But since $\Delta(\overline{G}) = \gamma_t(G) - 1$, we have $\gamma_t(G) = n$, which contradicts the fact that G is connected and $n \geqslant 3$. This implies that $\gamma_t(G) = 2$. Then $\Delta(\overline{G}) + 1 = \gamma_t(G) = 2$, and so $\Delta(\overline{G}) = 1$. The regularity of $\overline{G}$ implies that $\overline{G}$ consists of some K_2's.

If $\Delta(G) = n-1$, then $\gamma_t(G) = 2$ and $\delta(G) = n-2$, it follows that $\Delta(\overline{G}) = n-1-\delta(G) = 1$ and $\delta(\overline{G}) = n-1-\Delta(G) = 0$. This immediately implies that $\overline{G}$ consists of some K_2's and isolated vertices.

Conversely, let $G = \overline{\frac{1}{2}nK_2}$ or G is the complement of a graph which is constructed by some K_2's and isolated vertices. Clearly, in either case above, we have $\gamma_t(G) = 2$ and $\lambda_{n-1}(G) = n - \lambda_1(\overline{G}) = n-2$. So the equality holds. □

To compare this bound in Theorem 10 with that of Theorem 12, we rewrite the bound in Theorem 10 as

$$\lambda_{n-1}(G) \leqslant n - \gamma_t(G) + \frac{n(\gamma_t(G)+1) - 2\gamma_t^2(G)}{2(n-\gamma_t(G))}.$$

Archdeacon et al. [1] showed a well-known result on the total domination as follows: If G is a graph of order n with $\delta(G) \geqslant 3$, then $\gamma_t(G) \leqslant \lfloor \frac{1}{2}n \rfloor$. Then, by this result, for a graph G with $\delta(G) \geqslant 3$ satisfying the conditions in Theorems 10 and 12, we have $2\gamma_t(G) \leqslant n$. Hence $n(\gamma_t(G)+1) \geqslant 2\gamma_t(G)(\gamma_t(G)+1)$. This implies that $\frac{1}{2}\left(n(\gamma_t(G)+1) - 2\gamma_t^2(G)\right)/(n-\gamma_t(G)) > 0$. Therefore, when $\delta(G) \geqslant 3$, the bound in Theorem 12 is better than that of Theorem 10.

Now we present the third upper bound on algebraic connectivity $\lambda_{n-1}(G)$ of a graph G in terms of its signed domination number.

Theorem 13. *If G is a connected graph of order $n \geqslant 2$, then*

$$\lambda_{n-1}(G) \leqslant \frac{n(\gamma_s(G)+n-2)}{n-\gamma_s(G)}$$

and this bound is sharp.

Proof. Let f be an SDF of G for which $\omega(f) = \gamma_s(G)$, and let

$$P = \{v \in V;\ f(v) = 1\},$$
$$M = \{v \in V;\ f(v) = -1\}.$$

Then, $|P| + |M| = n$. Let $|P| = p$. Since $f(N[v]) \geqslant 1$ for each vertex $v \in V$, it follows that for any vertex $v \in P$, we have $d_P(v) \geqslant d_M(v)$. So

$$e(P,M) = \sum_{v\in P} d_M(v) \leqslant \sum_{v\in P} d_P(v) \leqslant p(p-1).$$

Thus, by Lemma 1, we have

$$\lambda_{n-1}(G) \leqslant \frac{n(p-1)}{n-p} = \frac{n(\gamma_s(G)+n-2)}{n-\gamma_s(G)}.$$

That the bound is sharp may be seen as follows. Let $G = K_n$, where n is odd, then, by Lemma 9, we have that $\gamma_s(K_n) = 1$. Hence, the equality in Theorem 13 follows from Lemma 2. □

4. Lower bounds on Laplacian spectral radius

In this section we turn our attention to the Laplacian spectral radius λ_1 in graphs. We will investigate lower bounds on λ_1 of a graph G in terms of its signed domination number.

We begin by giving a lower bound on the Laplacian spectral radius for a general graph G.

Theorem 14. *If $G=(V,E)$ is a connected graph of order $n\geqslant 2$, then*

$$\lambda_1(G)\geqslant \frac{4n}{\gamma_s(G)+n}.$$

Moreover, the equality holds if and only if $G=K_3$.

Proof. Let f be an SDF of G for which $\omega(f)=\gamma_s(G)$, and P and M be defined as in Theorem 13. Let $|P|=p$. By the definition of the signed dominating function, each vertex in M is adjacent to at least two vertices in P, so we have

$$|e(P,M)|\geqslant 2|M|\geqslant 2(n-p). \tag{2}$$

Thus, by Lemma 7, we have

$$\lambda_1(G)\geqslant \frac{n(d_1-r_1)}{n-p},$$

where $d_1=p^{-1}\sum\limits_{v\in P}d(v)$ and $r_1=p^{-1}\sum\limits_{v\in P}d_P(v)$. By (2), we have that

$$\begin{aligned}\lambda_1(G)&\geqslant \frac{n}{p(n-p)}\Bigl(\sum_{v\in P}d(v)-\sum_{v\in P}d_P(v)\Bigr)=\frac{n}{p(n-p)}|e(P,M)|\\&\geqslant\frac{2n}{p}=\frac{4n}{\gamma_s(G)+n}.\end{aligned}$$

Suppose that $\lambda_1(G)=4n/(\gamma_s(G)+n)$. Then all equalities in the above inequality chain hold. By Lemma 7, we have $d_M(u)=s$ for each vertex $u\in P$ and $d_P(v)=t$ for each vertex $v\in M$. The equality in (2) implies that $t=2$. We claim that $\Delta(G)=n-1$. Suppose to the contrary that $\Delta(G)<n-1$. Then, by Lemma 3, $\lambda_1(G)>\Delta(G)+1$, hence we have

$$\gamma_s(G)=\frac{4n}{\lambda_1(G)}-n<\frac{4n}{\Delta(G)+1}-n.$$

Note that $\gamma_s(G) = 2p - n$, $n = |P| + |M|$ and $s|P| = 2|M|$. Hence $2p - n < 4n/(\Delta(G)+1) - n$, and so $p < (2+s)p/(\Delta(G)+1)$. This means that $s+1 > \Delta(G)$, thus $s = \Delta(G)$, a contradiction. So $\Delta(G) = n-1$, i.e., G is complete. It follows from Lemma 3 that $\lambda_1(G) = \Delta(G)+1$. This implies that $p = (2+s)p/(\Delta(G)+1)$, i.e., $s = \Delta(G) - 1 = n-2$. Note that $t = 2$. So $G = K_3$.

Conversely, let $G = K_3$. Then the equality immediately follows from Lemmas 3 and 9. □

Theorem 15. *If G is a k-regular graph of order n, then*

$$\lambda_1(G) \geqslant \begin{cases} \dfrac{n(k+3)}{\gamma_s(G)+n} & \text{for } k \text{ odd,} \\ \dfrac{n(k+2)}{\gamma_s(G)+n} & \text{for } k \text{ even.} \end{cases}$$

Moreover, the equality holds if and only if $G = K_n$.

Proof. Let f be an SDF of G for which $\omega(f) = \gamma_s(G)$, and P and M be defined as in Theorem 13. Let $|P| = p$, $|M| = m$. Then $n = m+p$. We distinguish two cases depending on the parity of k.

Case 1. k is odd. For each $u \in P$, let u be adjacent to s ($\geqslant 0$) vertices in M. Then u is adjacent to $k-s$ vertices in P. By the definition of SDF, we have $f(N[u]) = k - 2s + 1 \geqslant 2$, so $s \leqslant \frac{1}{2}(k-1)$. For each $v \in M$, let v be adjacent to t vertices in P. Then $t \geqslant 2$ and v is adjacent to $k-t$ vertices in M. By the definition of SDF, we have $f(N[v]) = 2t - k - 1 \geqslant 2$, so

$$t \geqslant \frac{1}{2}(k+3). \tag{3}$$

By counting the number of edges between P and M, we get that $e(P,M) \geqslant \frac{1}{2}m(k+3)$. Note that $2p = \gamma_s(G) + n$. Then, by Lemma 7, we have

$$\lambda_1(G) \geqslant \frac{n}{p(n-p)}|e(P,M)| \geqslant \frac{nm(k+3)}{2p(n-p)} = \frac{n(k+3)}{\gamma_s(G)+n}. \tag{4}$$

Suppose that the above equality holds. Then the inequalities in (3) and (4) are equalities. Hence $t = \frac{1}{2}(k+3)$. By Lemma 7, we have $d_M(u) = s$ for each vertex $u \in P$ and $d_P(v) = t$ for each vertex $v \in M$. So $sp = tm$, and thus $n = p + m = p + sp/t = p(1 + 2s/(k+3))$. We claim that $k = n-1$. Suppose to the contrary that $k < n-1$. Then, by Lemma 3, $\lambda_1(G) > k+1$. By (4), we have

$$\gamma_s(G) = \frac{n(k+3)}{\lambda_1(G)} - n < \frac{n(k+3)}{k+1} - n.$$

Observing that $\gamma_s(G) = 2p - n$, we have

$$2p - n < \frac{n(k+3)}{k+1} - n,$$

or equivalently

$$2p < \frac{n(k+3)}{k+1}.$$

By substituting $p(1 + 2s/(k+3))$ for n, we have $2p < (k+3+2s)p/(k+1)$. This implies that $s > \frac{1}{2}(k-1)$, contradicting the fact that $s \leqslant \frac{1}{2}(k-1)$. So $k = n-1$, which implies that $G = K_n$.

Conversely, let $G = K_n$. The equality in (4) immediately follows from Lemmas 3 and 9.

Case 2. k is even. For each $u \in P$, let u be adjacent to s vertices in M. Then u is adjacent to $k-s$ vertices in P. By the definition of SDF, we have $f(N[u]) = k - 2s + 1 \geqslant 1$, hence $s \leqslant \frac{1}{2}k$. For each $v \in M$, let v be adjacent to t vertices in P. Then v is adjacent to $k-t$ vertices in M, and $f(N[v]) = 2t - k - 1 \geqslant 1$. Hence

$$t \geqslant \frac{1}{2}(k+2). \tag{5}$$

By counting the number of edges between P and M, we have $e(P,M) \geqslant \frac{1}{2}m(k+2)$. Then, by Lemma 2, we have

$$\lambda_1(G) \geqslant \frac{n}{p(n-p)}|e(P,M)| \geqslant \frac{nm(k+2)}{2p(n-p)} = \frac{n}{2p}(k+2).$$

The formula $\gamma_s(G) = 2p - n$ implies that $\lambda_1(G) \geqslant n(k+2)/(\gamma_s(G)+n)$, as required.

Suppose that $\lambda_1(G) = n(k+2)/(\gamma_s(G)+n)$. Then the equalities $t = \frac{1}{2}(k+2)$, $e(P,M) = m\frac{1}{2}(k+2)$ and $\lambda_1(G) = n(k+2)/(\gamma_s(G)+n)$ hold. By Lemma 2, we have $d_M(u) = s$ for each vertex $u \in P$ and $d_P(v) = t$ for each vertex $v \in M$. So $sp = tm$, and thus $n = p + m = p(1 + 2s/(k+2))$. We show that $G = K_n$. If this is not the case, then $k < n-1$. By Lemma 3, we have $\lambda_1(G) > k+1$. Hence

$$2p - n = \gamma_s(G) = \frac{n(k+2)}{\lambda_1(G)} - n < \frac{n(k+2)}{k+1} - n,$$

that is, $2p < n(k+2)/(k+1)$. Thus

$$2p < \frac{(k+2+2s)p}{k+1}.$$

But then $s > \frac{1}{2}k$, contradicting the fact $s \leqslant \frac{1}{2}k$. This implies $k = n-1$. So $G = K_n$.

Conversely, let $G = K_n$. Then the equality immediately follows from Lemmas 3 and 9. $\square$

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