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Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 1, 183–193

Persistent URL: http://dml.cz/dmlcz/140561

Czechoslovak Mathematical Journal, 60 (135) (2010), 183–193

SCHUR MULTIPLIER CHARACTERIZATION OF A CLASS OF INFINITE MATRICES

A. Marcoci, Bucharest, L. Marcoci, Bucharest, L. E. Persson, Luleå,
N. Popa, Bucharest

(Received September 9, 2008)

Abstract. Let $B_w(\ell^p)$ denote the space of infinite matrices A for which $A(x) \in \ell^p$ for all $x = \{x_k\}_{k=1}^{\infty} \in \ell^p$ with $|x_k| \searrow 0$. We characterize the upper triangular positive matrices from $B_w(\ell^p)$, $1 < p < \infty$, by using a special kind of Schur multipliers and the G. Bennett factorization technique. Also some related results are stated and discussed.

Keywords: infinite matrices, Schur multipliers, discrete Sawyer duality principle, Bennett factorization, Wiener algebra and Hardy type inequalities

MSC 2010: 15A48, 15A60, 47B35, 26D15

1. Introduction

In this paper we deal with infinite matrices A whose entries a_k^l, for $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_+$, are indexed with respect to the kth diagonal and with the lth place on this diagonal. In what follows, sometimes we describe an infinite matrix by $A = (a_k^l)_{k\in\mathbb{Z},l\in\mathbb{Z}_+}$, more precisely

$$A = \begin{pmatrix} a_0^1 & a_1^1 & a_2^1 & a_3^1 & \dots \\ a_{-1}^1 & a_0^2 & a_1^2 & a_2^2 & \ddots \\ a_{-2}^1 & a_{-1}^2 & a_0^3 & a_1^3 & \ddots \\ a_{-3}^1 & a_{-2}^2 & a_{-1}^3 & a_0^4 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The first two authors and the last author were partially supported by the CNCSIS grant ID-PCE 1905/2008.

We started our study motivated by the paper [13], where the first two authors introduced the space $B_w(\ell^2)$ of those infinite matrices A for which $A(x) \in \ell^2$ for all $x = \{x_k\}_{k=1}^{\infty} \in \ell^2$ with $|x_k| \searrow 0$.

This space is of interest because the matrix version of the Wiener algebra $A(\mathbb{T})$, denoted by $A(\ell^2)$, which consists of all infinite matrices $A = (a_k^l)_{k\in\mathbb{Z}, l\in\mathbb{Z}_+}$ such that $\sup\limits_{l\in\mathbb{Z}_+} \sum\limits_{k\in\mathbb{Z}} |a_k^l| < \infty$, is not contained in the matrix version $\mathcal{C}(\ell^2)$ of the space of all continuous functions $\mathcal{C}(\mathbb{T})$ (see [5] for the definition and the properties of $\mathcal{C}(\ell^2)$).

Such an example is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where in the $\frac{1}{2}n(n+1)$-column there are n entries equal to 1 placed on the $\frac{1}{2}n(n-1)+1, \dots, \frac{1}{2}n(n+1)$-rows and 0 otherwise. Clearly we have $\sup\limits_{l\in\mathbb{Z}_+} \sum\limits_{k\in\mathbb{Z}} |a_k^l| = 1$, hence $A \in A(\ell^2)$ and $|Ae_{\frac{1}{2}n(n+1)}|_2^2 = n$ for all $e_n = (\underbrace{0, \dots, 0}_{n-1}, 1, 0, \dots)$.

This yields that $A(\ell^2) \subset B_w(\ell^2)$ (see Proposition 2 in [13]), where $B_w(\ell^2)$ is the Banach space with respect to the norm

$$\|A\|_{B_w(\ell^2)} = \sup_{\|x\|_2 \leqslant 1,\ |x_k| \searrow 0} \|A(x)\|_2.$$

We remark that $\ell^2_{\text{dec}} = \{x = (x_k) \searrow 0,\ x \in \ell^2\}$ is a cone and the solid hull of this cone, denoted by $\text{so}(\ell^2_{\text{dec}})$, coincides with the Banach space $d(2) = \left\{x;\ \sum\limits_{n=1}^{\infty} \sup\limits_{k\geqslant n} |x_k|^2 < \infty\right\}$. The spaces $d(p)$, $p \geqslant 1$ are introduced in [1], where it is described how they are connected to Hardy type inequalities (for historical information and results of this type we refer to the books [11] and [10]). Here $\text{so}(\ell^2_{\text{dec}}) = \{y = (y_k) \in \ell^2$ such that $|y_k| \leqslant x_k$ for all $k \in \mathbb{N}$, where $x_k \searrow 0$ in $\ell^2\}$.

Let A be a *positive* matrix, that is, such that all the elements of the sequence $A(x)$ are positive whenever $x = (x_j)_j$ is a sequence having only a finite number of nonzero positive elements. Clearly, if $A \in B_w(\ell^2)$, then $A \in B(d(2), \ell^2)$, that is, A is a bounded linear operator from $d(2)$ into ℓ^2.

The next lemma, which may be regarded as a discrete version of a special case of the Sawyer duality principle [16] (see also [10]) was obtained and applied in [13].

Lemma 1.1. *We have*

$$\sup_{|x_n|\searrow 0} \frac{\left|\sum_{n=1}^{\infty} a_n x_n\right|}{\left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}} = \sup_{|x_n|\searrow 0} \frac{\sum_{n=1}^{\infty} |a_n||x_n|}{\left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}} \approx \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|a_k|\right)^2\right)^{1/2},$$

where $(a_n)_n$ and $(x_n)_n$ are sequences of complex numbers.

For the investigations in this paper we need a corresponding (discrete Sawyer type) result for every $p > 1$ and not only for $p = 2$ as in Lemma 1.1 (see our Lemma 2.4).

In this paper we consider the space $B_w(\ell^p)$ consisting of infinite matrices A for which $A(x) \in \ell^p$ for all $x = \{x_k\}_{k=1}^{\infty} \in \ell^p$ with $|x_k| \searrow 0$ $(1 < p < \infty)$. In Theorem 2.1 we characterize the upper triangular positive matrices from $B_w(\ell^p)$ by using a special kind of Schur multipliers. Some related results are formulated in Section 2. The proofs can be found in Section 3. We pronounce that our proofs are heavily depending on various important factorization results by G. Bennett [1] and Lemma 2.4.

2. Main results

First let us recall the definition of Schur multipliers.

If $A = (a_{jk})$ and $B = (b_{jk})$ are matrices of the same size (finite or infinite) their Schur product (or Hadamard product) is defined to be the matrix of elementwise products

$$A * B = (a_{jk} b_{jk}).$$

There is, however, much justification for the term "Schur product" and we refer the reader to [2] and [20] for an historical discussion. This concept was first investigated by Schur in his paper [17] and has since appeared in several different areas of analysis: [15], [18], [19](complex function theory); [1], [12] (Banach spaces); [21], [14], [4] (operator theory); [5], [3] (matriceal harmonic analysis) and [20] (multivariate analysis).

If X and Y are two Banach spaces of matrices we define Schur multipliers from X to Y as the space $M(X, Y) = \{M\colon\ M * A \in Y \text{ for every } A \in X\}$, equipped with the natural norm

$$\|M\| = \sup_{\|A\|_X \leqslant 1} \|M * A\|_Y.$$

We use a matrix operation introduced in [6], which extends to general matrices, the usual product of a Toeplitz matrix A and a complex scalar c.

Namely, let $c = (c^1, c^2, \ldots)$ be a sequence of complex numbers. We denote by $[c]$ the matrix whose entries $[c]_k^l$ are equal to c^l for $l \geqslant 1$ and $k \in \mathbb{Z}$.

We observe that, for a Toeplitz matrix A and for a constant sequence $c = (c^1, c^1, \ldots)$, the matrix $[c] * A$ coincides with the usual product between the complex number c^1 and the matrix A. Hence we denoted in [6] the product $[c] * A$ by $c \odot A$ and considered it as an external product of a matrix and a sequence of complex numbers.

In what follows, using the results about multipliers from [1], we will characterize the upper triangular positive matrices from $B_w(\ell^p)$ by studying the behaviour of the matrix $[c]$. Here $B_w(\ell^p)$ denotes the space of those infinite matrices A for which $A(x) \in \ell^p$ for all $x = \{x_k\}_{k=1}^{\infty} \in \ell^p$ with $|x_k| \searrow 0$. It is clear that for $p > 1$ this is a Banach space with respect to the norm

$$\|A\|_{B_w(\ell^p)} = \sup_{\|x\|_p \leqslant 1,\ |x_k| \searrow 0} \|A(x)\|_p.$$

Here, as usual,

$$\ell^p = \left\{ x = \{x_k\}_{k=1}^{\infty} \colon \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} < \infty \right\}.$$

Moreover, let

$$d(p) = \left\{ x = \{x_k\}_{k=1}^{\infty} \colon \left(\sum_{k=1}^{\infty} \sup_{n \geqslant k} |x_n|^p \right)^{1/p} < \infty \right\}.$$

Our first result reads:

Theorem 2.1. *Let B be an upper triangular matrix. Then $B \in B(\ell^p, \operatorname{ces}(p))$, $1 < p < \infty$, if and only if $B * [c] \in B(\ell^p, \ell^1)$ for all $c \in d(p)$.*

Here

$$\operatorname{ces}(p) = \left\{ x = \{x_k\}_{k=1}^{\infty} \text{ with } \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}$$

denotes the Banach space equipped with the norm

$$\|x\|_{\operatorname{ces}(p)} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p}.$$

Now we can state our main result concerning the characterization of the matrices belonging to $B_w(\ell^p)$.

Theorem 2.2. *A lower triangular positive matrix A belongs to $B_w(\ell^p)$, $1 < p < \infty$, if and only if $A^* * [c] \in B(\ell^q, \ell^1)$, where $1/p + 1/q = 1$ for all $c \in d(p)$, where A^* is the usual adjoint of the matrix A.*

Besides the ces(p)-spaces who have already attracted a fair deal of attention in literature, an important role is played by ℓ^p, $d(p)$ and also $g(p)$, defined by

$$g(p) = \left\{ x = \{x_k\}_{k=1}^{\infty} : \sup_{n \geqslant 1} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p} < \infty \right\}.$$

Therefore we also state the following result where ces(p) in Theorem 2.2 is replaced by any of these spaces and $\ell^q \cdot d(p)$ is the sequence space of coordinatewise products (see [1] for further details).

Theorem 2.3. *Let $1 < p < \infty$, $1/p + 1/q = 1$ and let B be an upper triangular matrix. Then*

(1) *$B \in B(\ell^p, d(p))$ if and only if $B * [c] \in B(\ell^p, \ell^1)$ for all $c \in \mathrm{ces}(q)$;*
(2) *$B \in B(\ell^p, \ell^p)$ if and only if $B * [c] \in B(\ell^p, \ell^1)$ for all $c \in \ell^q$;*
(3) *$B \in B(\ell^p, g(p))$ if and only if $B * [c] \in B(\ell^p, \ell^1)$ for all $c \in \ell^q \cdot d(p)$.*

Our proof of Theorem 2.1 (and thus of Theorem 2.2) is heavily depending on the following extension of Lemma 1.1 which is of independent interest.

Lemma 2.4. *If $p > 1$, then*

$$\sup_{|x_n| \searrow 0} \frac{\left| \sum_{n=1}^{\infty} a_n x_n \right|}{\left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}} = \sup_{|x_n| \searrow 0} \frac{\sum_{n=1}^{\infty} |a_n||x_n|}{\left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}} \approx \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^q \right)^{1/q},$$

where $(a_n)_n$ and $(x_n)_n$ are sequences of complex numbers and $1/p + 1/q = 1$.

Remark 2.5. If $1 < p < \infty$ and $1/p + 1/q = 1$, then

$$d(q)^{\times} = \mathrm{ces}(p).$$

Here $d(q)^{\times}$ is the associate space of $d(q)$, that is

$$d(q)^{\times} = \left\{ a = (a_n)_n; \text{ such that } \sum_{n=1}^{\infty} |a_n x_n| < \infty \text{ for all } (x_n)_n \in d(q) \right\}.$$

This result which gives us the Köthe dual of $d(p)$ has been obtained also by G. Bennett in [1] by using more technical methods, like factorization of some classical inequalities. This problem was first investigated by Jagers in 1974 in the paper [9].

Finally, we note that

$$\operatorname{so}(\ell^p_{\rm dec}) = d(p)$$

($\ell^p_{\rm dec}$ denotes the subspace of ℓ^p consisting of non-increasing sequences) and, hence, our results in particular imply Corollary 12.17 in paper [1] by G. Bennett.

3. Proofs

We first present a proof of the crucial Lemma 2.4, which is based on the following result of E. Sawyer [16]. For $p \leqslant 1$ a similar result has been proved by M. J. Carro and J. Soria in their paper [7].

Lemma 3.1. *Let* $w = \{w(n)\}_{n=1}^\infty$, $v = \{v(n)\}_{n=1}^\infty$ *be weights on* $\mathbb{N}^*$, *let*

$$S = \sup_{f\searrow} \frac{\sum\limits_{n=0}^{\infty} f(n)v(n)}{\Big(\sum\limits_{n=0}^{\infty} f(n)^p w(n)\Big)^{1/p}}$$

and $\tilde v = \sum\limits_{n=0}^{\infty} v(n)\chi_{[n,n+1)}$, $\tilde w = \sum\limits_{n=0}^{\infty} w(n)\chi_{[n,n+1)}$ *and* $\widetilde V(t) = \int_0^t \tilde v(s)\,\mathrm{d}s$, $\widetilde W(t) = \int_0^t \tilde w(s)\,\mathrm{d}s$.

If $1 < p < \infty$, *then*

$$S \approx \Big(\int_0^\infty \Big(\frac{\widetilde V(t)}{\widetilde W(t)}\Big)^{q-1} \tilde v(t)\,\mathrm{d}t\Big)^{1/q} \approx \Big(\int_0^\infty \Big(\frac{\widetilde V(t)}{\widetilde W(t)}\Big)^{q} \tilde w(t)\,\mathrm{d}t\Big)^{1/q} + \frac{\widetilde V(\infty)}{\widetilde W^{1/p}(\infty)}$$

where $1/p + 1/q = 1$.

Here, as usual, the relation $f \approx g$ means that there are two positive constants C_0 and C_1 such that $C_0 f(t) \leqslant g(t) \leqslant C_1 f(t)$, $t \in [0,\infty)$.

P r o o f of Lemma 2.4. We denote

$$S = \sup_{|x_n|\searrow 0} \frac{\sum\limits_{n=1}^{\infty} |a_n||x_n|}{\Big(\sum\limits_{n=1}^{\infty} |x_n|^p\Big)^{1/p}}.$$

According to Lemma 3.1 we have

$$S \approx \Big(\int\limits_0^\infty \Big(\frac{\widetilde V(t)}{\widetilde W(t)}\Big)^{q} \tilde w(t)\,\mathrm{d}t\Big)^{1/q} + \frac{\widetilde V(\infty)}{\widetilde W^{1/p}(\infty)},$$

where $v(n) = |a_n|$, $w(n) = 1$, $f(n) = |x_n|$ for every nonnegative integer n. In this case

$$\tilde{v} = \sum_{n=0}^{\infty} v(n)\chi_{[n,n+1)} = \sum_{n=0}^{\infty} |a_n|\chi_{[n,n+1)}, \text{ where } a_0 = 0.$$

Therefore, for $t \in (j, j+1)$, this yields that

$$\begin{aligned}
\widetilde{V}(t) &= \int_0^t \tilde{v}(s)\,\mathrm{d}s = \int_0^j \tilde{v}(s)\,\mathrm{d}s + \int_j^t \tilde{v}(s)\,\mathrm{d}s \\
&= \sum_{m=0}^{j-1} \int_m^{m+1} \tilde{v}(s)\,\mathrm{d}s + \int_j^t \tilde{v}(s)\,\mathrm{d}s = \sum_{m=0}^{j-1} |a_m| + |a_j|(t-j), \\
\widetilde{V}(\infty) &= \int_0^\infty \tilde{v}(s)\,\mathrm{d}s = \int_0^\infty \sum_{n=0}^{\infty} |a_n|\chi_{[n,n+1)}(s)\,\mathrm{d}s = \sum_{n=1}^{\infty} |a_n|
\end{aligned}$$

and

$$\widetilde{W}(\infty) = \int_0^\infty \tilde{w}(s)\,\mathrm{d}s = \infty,$$

since $\tilde{w}(s) = \sum\limits_{n=0}^{\infty} \chi_{[n,n+1)}(s)$. Letting $\tilde{v}_M = \sum\limits_{n=0}^{M} |a_n|\chi_{[n,n+1)}$, $\widetilde{V}_M = \int\limits_0^\infty \tilde{v}_M(s)\,\mathrm{d}s = \int\limits_0^\infty \sum\limits_{n=0}^{M} |a_n|\chi_{[n,n+1)}(s)\,\mathrm{d}s = \sum\limits_{n=1}^{M} |a_n| < \infty$, we get

$$\begin{aligned}
S &= \sup_{|x_n| \searrow 0} \frac{\sum\limits_{n=1}^{\infty} |a_n||x_n|}{\left(\sum\limits_{n=1}^{\infty} |x_n|^p\right)^{1/p}} \\
&\approx \sup_M \left[\left(\int_0^\infty \left(\frac{\widetilde{V}_M(t)}{t}\right)^q \mathrm{d}t\right)^{1/q} + \frac{\widetilde{V}_M(\infty)}{\widetilde{W}^{1/q}(\infty)}\right] \approx \left(\int_0^\infty \left(\frac{\widetilde{V}(t)}{t}\right)^q \mathrm{d}t\right)^{1/q}.
\end{aligned}$$

But

$$\begin{aligned}
\int_0^\infty \left(\frac{\widetilde{V}(t)}{t}\right)^q \mathrm{d}t &= \sum_{j=1}^{\infty} \int_j^{j+1} \left(\frac{\sum\limits_{m=0}^{j-1} |a_m| + |a_j|(t-j)}{t}\right)^q \\
&\approx \sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{m=1}^{j} |a_m|\right)^q,
\end{aligned}$$

which implies that

$$S \approx \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |a_k|\right)^q\right)^{1/q}.$$

The proof is complete. □

Proof of Theorem 2.1. For clearness we first prove the theorem for the special case $p=q=2$. Note that A^* is an upper triangular matrix.

Let $C \stackrel{\text{def}}{=} \{c\}$ be the upper triangular matrix obtained from $[c]$ by taking the triangular projection P_T, which acts as follows:

$$P_T(A) = \begin{cases} a_{ij} & \text{if } i \leqslant j, \\ 0 & \text{otherwise.} \end{cases}$$

(See [6].)

Let B be an upper triangular matrix from $B(\ell^2, \operatorname{ces}(2))$. We have $B(x) = \Big(\sum\limits_{j=1}^{\infty} b_{ij}x_j\Big)_{i=1}^{\infty} \in \operatorname{ces}(2)$ for all $x=(x_j)_{j=1}^{\infty} \in \ell^2$. But $(B*C)(x) = \Big(\Big(\sum\limits_{j=1}^{\infty} b_{ij}x_j\Big)c^i\Big)_{i=1}^{\infty}$ is the product of two sequences, one from $\operatorname{ces}(2)$, and the other one completely arbitrary. By Proposition 15.4 in [1] we have that

$$d(2) = I(2,2) \stackrel{\text{def}}{=} \Big\{m\colon \sum_{k=1}^{\infty}(i_k - i_{k-1})|m_{i_k}|^2 < \infty \text{ for each sequence } i \text{ of integers with } i_0 = 0 < i_1 < i_2 < \ldots\Big\}.$$

Then, by using the table 29 on page 70 in [1], we get that $(B*C)(x) \in \ell^1$, where $c \in d(2) = I(2,2)$ and $x \in \ell^2$. Hence $B*C \in B(\ell^2, \ell^1)$.

Conversely, let $B*C \in B(\ell^2, \ell^1)$ for each $c \in d(2)$. By Hölder's inequality we have that $\ell^1 = \ell^2 \cdot \ell^2$, and, in view of Theorem 3.8 in [1], it follows that $\ell^2 = g(2) \cdot d(2)$, where

$$g(2) = \Big\{x;\ \sup_n \frac{1}{n}\sum_{k=1}^{n} |x_k|^2 < \infty\Big\}.$$

Hence $\ell^1 = (\ell^2 \cdot g(2)) \cdot d(2)$ and, according to Theorem 4.5 in [1], this yields that $\ell^1 = \operatorname{ces}(2) \cdot d(2)$. On the other hand, by Proposition 14.5 in [1] $\operatorname{ces}(2)$ has the $d(2)$-cancellation property, that is, the inclusion $y \cdot d(2) \subset \operatorname{ces}(2) \cdot d(2)$ implies that $y \in \operatorname{ces}(2)$.

Now, by hypotheses, for each $x \in \ell^2$ we have

$$(B*C)(x) = \Big(\Big(\sum_{j=1}^{\infty} b_{ij}x_j\Big)c^i\Big)_i \in \ell^1 = \operatorname{ces}(2) \cdot d(2)$$

for all $c \in d(2)$. By the cancellation property it follows

$$B(x) = \Big(\sum_{j=1}^{\infty} b_{ij}x_j\Big)_i \in \operatorname{ces}(2),$$

that is, by the closed graph theorem, $B \in B(\ell^2, \operatorname{ces}(2))$.

Now we consider the case $p \neq 2$.

If $B \in B(\ell^p, \operatorname{ces}(p))$, $c \in d(q)$, $q > 1$, and $1/p + 1/q = 1$, then as in the proof of the case $p = q = 2$ we have that $d(q) = I(q,q)$ and, in view of the table on page 70 in [1], it follows that

$$(B * C)(x) \in \ell^1 \text{ for all } x \in \ell^p,$$

that is, $B * C \in B(\ell^p, \ell^1)$.

Conversely, let $B * C \in B(\ell^p, \ell^1)$ for all $c \in d(q)$. Then, similarly to the proof of the case $p = q = 2$ we find that

$$\ell^1 = \ell^p \cdot \ell^q = \ell^p \cdot g(q) \cdot d(q) = (\text{by Theorem 4.5 in [1]}) = \operatorname{ces}(p) \cdot d(q).$$

Since $\operatorname{ces}(p)$ has the $d(q)$-cancellation property (see Proposition 14.5 in [1]) it follows that $B \in B(\ell^p, \operatorname{ces}(p))$.

The proof is complete. □

P r o o f of Theorem 2.2. First let us note that by Lemma 2.4 it follows that $A \in B_w(\ell^p)$ if and only if $A^* \in B(\ell^q, \operatorname{ces}(q))$ for $1/p + 1/q = 1$. It remains to apply Theorem 2.1. □

P r o o f of Theorem 2.3. (1) If $B \in B(\ell^p, d(p))$, $c \in \operatorname{ces}(q)$ with $1/p + 1/q = 1$ and $x \in \ell^p$ we have

$$\begin{aligned}(B * [c])(x) &= \left(\left(\sum_{j=1}^{\infty} b_{ij} x_j\right) c^i\right)_{i=1}^{\infty} = \left(y_i c^i\right)_{i=1}^{\infty} \in d(p) \cdot \operatorname{ces}(q) \\ &= (\text{by Corollary 12.17 in [1]}) = d(p) \cdot d(p)^* \subset \ell^1.\end{aligned}$$

Hence, by the closed graph theorem, this yields that

$$B * [c] \in B(\ell^p, \ell^1).$$

Conversely, if $B * [c] \in B(\ell^p, \ell^1)$ for all $c \in \operatorname{ces}(q)$, then, denoting $(y_i)_i = \left(\sum_{j=1}^{\infty} b_{ij} x_j\right)_{i \in \mathbb{N}}$, we have that $(y_i c^i)_i \in \ell^1$ for all $c \in \operatorname{ces}(q)$. Thus

$$(y_i)_i \in \operatorname{ces}(q)^* = (\text{by Corollary 12.17 in [1]}) = d(p),$$

that is,

$$B \in B(\ell^p, d(p)).$$

(2) If $B \in B(\ell^p, \ell^p)$, $x \in \ell^p$, $c \in \ell^q$ with $1/p + 1/q = 1$, then, by Hölder's inequality, $B * [c] \in B(\ell^p, \ell^1)$.

Conversely, let $(y_i c^i)_i \in \ell^1$ for all $c \in \ell^q$. Then $(y_i)_i \in \ell^p$ and, consequently,

$$B \in B(\ell^p, \ell^p).$$

(3) If $B \in B(\ell^p, g(p))$ and $c \in \ell^q \cdot d(p)$, then, using the previous notation, we find that

$$(y_i c^i)_i \in g(p) \cdot \ell^q \cdot d(p) = (\text{by Theorem 3.8 in [1]}) = \ell^p \cdot \ell^q \subset \ell^1.$$

Conversely, let $(y_i)_i \cdot \ell^q \cdot d(p) \in \ell_1 = (\text{by Theorem 3.8 in [1]}) = g(p) \cdot d(p) \cdot \ell^q$. Consequently, we have to show that

$$(y_i)_i \in g(p).$$

This fact follows clearly if $g(p)$ has the $d(p) \cdot \ell^q$-cancellation property.

We note that using Proposition 14.5 in [1] we get that $(y_i)_i \cdot d(p) \in g(p) \cdot d(p) \cdot \ell^p$.

Indeed, let $(z_i)_i \in d(p)$ be fixed. Then $(y_i z_i)_i \cdot \ell^q \in \ell^p \cdot \ell^q$. Since ℓ^p has the ℓ^q-cancellation property (see Proposition 14.5 in [1]) it follows that

$$(y_i z_i)_i \in \ell^p = g(p) \cdot d(p) \text{ for all } (z_i)_i \in d(p);$$

in other words $(y_i)_i \cdot d(p) \in g(p) \cdot d(p)$. Using now the fact that $g(p)$ has the $d(p)$-cancellation property, it follows that $(y_i)_i \in g(p)$. The proof is complete. □

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Authors' addresses: A. Marcoci, L. Marcoci, Department of Mathematics and Informatics, Technical University of Civil Engineering Bucharest, 124 Lacul Tei Boulevard, Bucharest 020396, Romania, e-mails: anca_marcoci@yahoo.com, liviu_marcoci@yahoo.com; L. E. Persson, Department of Mathematics, Luleå University of Technology, SE-97 187 Luleå, Sweden, e-mail: larserik@sm.luth.se; N. Popa, Department of Mathematics, University of Bucharest and Institute of Mathematics of Romanian Academy, P.O. BOX 1-764, RO-014700 Bucharest, Romania, e-mail: npopa@imar.ro.