Chunhui Lai; Lili Hu

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POTENTIALLY $K_m - G$-GRAPHICAL SEQUENCES: A SURVEY

Chunhui Lai, Lili Hu, Fujian

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Abstract. The set of all non-increasing nonnegative integer sequences $\pi = (d(v_1), d(v_2), \ldots, d(v_n))$ is denoted by NS_n. A sequence $\pi \in \mathrm{NS}_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. A graphical sequence π is potentially H-graphical if there is a realization of π containing H as a subgraph, while π is forcibly H-graphical if every realization of π contains H as a subgraph. Let K_k denote a complete graph on k vertices. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m). This paper summarizes briefly some recent results on potentially $K_m - G$-graphic sequences and give a useful classification for determining $\sigma(H, n)$.

Keywords: graph, degree sequence, potentially $K_m - G$-graphic sequences

MSC 2010: 05C07, 05C35

1. Introduction

The set of all non-increasing nonnegative integer sequences $\pi = (d(v_1), d(v_2), \ldots, d(v_n))$ is denoted by NS_n. A sequence $\pi \in \mathrm{NS}_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. A graphical sequence π is potentially H-graphical if there is a realization of π containing H as a subgraph, while π is forcibly H-graphical if every realization of π contains H as a subgraph. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1}-graphic. Let $\sigma(\pi) = d(v_1) + d(v_2) + \ldots + d(v_n)$, and $[x]$ denote the largest integer less than or equal to x. We denote by $G + H$ the graph

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with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy\colon\ x \in V(G), y \in V(H)\}$. Let K_k, C_k, T_k, and P_k denote a complete graph on k vertices, a cycle on k vertices, a tree on $k+1$ vertices, and a path on $k+1$ vertices, respectively. Let F_k denote the friendship graph on $2k+1$ vertices, that is, the graph of k triangles intersecting in a single vertex. For $0 \leqslant r \leqslant t$, denote the generalized friendship graph on $kt - kr + r$ vertices by $F_{t,r,k}$, where $F_{t,r,k}$ is the graph of k copies of K_t meeting in a common r set. We use the symbol Z_4 to denote $K_4 - P_2$. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m).

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geqslant l$ is forcibly H-graphical. Erdös, Jacobson and Lehel [13] first consider the following variant: determine the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geqslant l$ is potentially H-graphical. We denote this minimum l by $\sigma(H, n)$. Erdös, Jacobson and Lehel [13] showed that $\sigma(K_k, n) \geqslant (k-2)(2n-k+1)+2$ and conjectured that equality holds. They proved that if π does not contain zero terms, this conjecture is true for $k = 3$, $n \geqslant 6$. The conjecture was confirmed in [19] and [43]–[46]. Li et al. [46] and Mubayi [55] also independently determined the values $\sigma(K_r, 2k)$ for any $k \geqslant 3$. Li and Yin [51] further determined $\sigma(K_r, n)$ for $r \geqslant 7$ and $n \geqslant 2r + 1$. The problem of determining $\sigma(K_r, n)$ is completely solved.

Gould, Jacobson and Lehel [19] also proved that $\sigma(pK_2, n) = (p-1)(2n-p)+2$ for $p \geqslant 2$; $\sigma(C_4, n) = 2[(1/2)(3n-1)]$ for $n \geqslant 4$. Lai [29] gave a lower bound of $\sigma(C_k, n)$ and proved that $\sigma(C_5, n) = 4n - 4$ for $n \geqslant 5$ and $\sigma(C_6, n) = 4n - 2$ for $n \geqslant 7$. Lai [32] proved that $\sigma(C_{2m+1}, n) = m(2n-m-1)+2$, for $m \geqslant 2$, $n \geqslant 3m$; and $\sigma(C_{2m+2}, n) = m(2n-m-1)+4$, for $m \geqslant 2$, $n \geqslant 5m-2$. Li and Luo [41] gave a lower bound for $\sigma({}_3C_l, n)$ and determined $\sigma({}_3C_l, n)$, $4 \leqslant l \leqslant 6$, $n \geqslant l$. Li, Luo and Liu [42] determined $\sigma({}_3C_l, n)$ for $3 \leqslant l \leqslant 8$, and $n \geqslant l$. and $\sigma({}_3C_9, n)$ for $n \geqslant 12$. Li and Yin [48] determined $\sigma({}_3C_l, n)$ for n sufficiently large. Yin, Li and Chen [68] determined $\sigma({}_kC_l, n)$, $l \geqslant 7$, $3 \leqslant k \leqslant l$. Chen and Yin [9] determined the values $\sigma(W_5, n)$ for $n \geqslant 11$, where W_r is a wheel graph on r vertices. For $r \times s$ complete bipartite graph $K_{r,s}$, Gould, Jacobson and Lehel [19] determined $\sigma(K_{2,2}, n)$. Yin et al. [63], [65], [69], [70] determined $\sigma(K_{r,s}, n)$ for $s \geqslant r \geqslant 2$ and sufficiently large n. For $r \times s \times t$ complete 3-partite graph $K_{r,s,t}$, Erdös, Jacobson and Lehel [13] determined $\sigma(K_{1,1,1}, n)$. Lai [30] determined $\sigma(K_{1,1,2}, n)$. Yin [58] and Lai [34] independently determined $\sigma(K_{1,1,3}, n)$. Chen [7] determined $\sigma(K_{1,1,t}, n)$ for $t \geqslant 3$, $n \geqslant 2[\frac{1}{4}(t+5)^2]+3$. Chen[5] determined $\sigma(K_{1,2,2}, n)$ for $5 \leqslant n \leqslant 8$ and $\sigma(K_{2,2,2}, n)$

for $n \geqslant 6$. Let K_s^t denote the complete t partite graph such that each partite set has exactly s vertices. Guantao Chen, Michael Ferrara, Ronald J. Gould and John R. Schmitt [11] showed that $\sigma(K_s^t, n) = \pi(K_{(t-2)s} + K_{s,s}, n)$ and obtained the exact value of $\sigma(K_j + K_{s,s}, n)$ for n sufficiently large. Consequently, they obtained the exact value of $\sigma(K_s^t, n)$ for n sufficiently large. For $n \geqslant 5$, Ferrara, Jacobson and Schmitt [17] determined $\sigma(F_k, n)$ where F_k denotes the graph of k triangles intersecting at exactly one common vertex. In [16], Ferrara, Gould and Schmitt determined a lower bound for $\sigma(K_s^t, n)$, where K_s^t denotes the complete multipartite graph with t partite sets each of size s, and proved equality in the case $s = 2$. They also provided a graph theoretic proof for the value of $\sigma(K_t, n)$. Michael J. Ferrara [15] determined $\sigma(H, n)$ for the graph $H = K_{m_1} \cup K_{m_2} \cup \ldots \cup K_{m_k}$, where n is sufficiently large integer. Ferrara, M., Jacobson, M., Schmitt, J. and Siggers M. [18] determined $\sigma(K_{s,t}, m, n)$, $\sigma(P_t, m, n)$ and $\sigma(C_{2t}, m, n)$ where $\sigma(H, m, n)$ is the minimum integer k such that every bigraphic pair $S = (A, B)$ with $|A| = m$, $|B| = n$ and $\sigma(S) \geqslant k$ is potentially H-bigraphic. For an arbitrarily chosen H, Schmitt, J. R. and Ferrara, M. [56] gave a good lower bound for $\sigma(H, n)$. Yin and Li [67] determined $\sigma(K_{r_1, r_2, \ldots, r_l, r, s}, n)$ for sufficiently large n. Moreover, Yin, Chen and Schmitt [62] determined $\sigma(F_{t,r,k}, n)$ for $k \geqslant 2, t \geqslant 3, 1 \leqslant r \leqslant t-2$ and sufficiently large, where $F_{t,r,k}$ denotes the graph of k copies of K_t meeting in a common r set. Gupta, Joshi and Tripathi [20] gave a necessary and sufficient condition for the existence of a tree of order n with a given degree set. Yin [59] gave a new necessary and sufficient condition for π to be potentially K_{r+1}-graphic. Jiong-sheng Li and Jianhua Yin [50] gave a survey on graphical sequences.

2. Potentially $K_m - G$-graphical sequences

Let H be a graph with m vertices, then $H = K_m - (K_m - H)$. Let $G = K_m - H$, then $\sigma(H, n) = \sigma(K_m - G, n)$. If Problems 1–5 in the Open Problems section are solved, then the problem of determining $\sigma(H, n)$ is completely solved. We think Problems 3 and 4 are a useful classification for determining $\sigma(H, n)$.

Gould, Jacobson and Lehel [19] pointed out that it would be nice to see where in the range from $3n-2$ to $4n-4$ the value $\sigma(K_4 - e, n)$ lies. Later, Lai [30] proved that

Theorem 1. *For $n = 4, 5$ and $n \geqslant 7$*

$$\sigma(K_4 - e, n) = \begin{cases} 3n-1 & \text{if } n \text{ is odd,} \\ 3n-2 & \text{if } n \text{ is even.} \end{cases}$$

For $n = 6$, if S is a 6-term graphical sequence with $\sigma(S) \geqslant 16$, then either there is a realization of S containing $K_4 - e$ or $S = (3^6)$. (Thus $\sigma(K_4 - e, 6) = 20$.)

Huang [26] gave a lower bound of $\sigma(K_m - e, n)$. Yin, Li and Mao [71] and Huang [27] independently determined the values $\sigma(K_5 - e, n)$ as follows.

Theorem 2. *If $n \geqslant 5$, then*

$$\sigma(K_5 - e, n) = \begin{cases} 5n - 7, & \text{if } n \text{ is odd,} \\ 5n - 6, & \text{if } n \text{ is even.} \end{cases}$$

Lai [35]–[36] determined $\sigma(K_5 - C_4, n)$, $\sigma(K_5 - P_3, n)$ and $\sigma(K_5 - P_4, n)$.

Theorem 3. *For $n \geqslant 5$, $\sigma(K_5 - C_4, n) = \sigma(K_5 - P_3, n) = \sigma(K_5 - P_4, n) = 4n - 4$.*

Yin and Li [66] gave a good method for determining the values $\sigma(K_{r+1} - e, n)$ (in fact, Yin and Li [66] also determined the values $\sigma(K_{r+1} - ke, n)$ for $r \geqslant 2$ and $n \geqslant 3r^2 - r - 1$).

Theorem 4. *Let $n \geqslant r + 1$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ with $d_{r+1} \geqslant r$. If $d_i \geqslant 2r - i$ for $i = 1, 2, \ldots, r - 1$, then π is potentially A_{r+1}-graphic.*

Theorem 5. *Let $n \geqslant 2r + 2$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ with $d_{r+1} \geqslant r$. If $d_{2r+2} \geqslant r - 1$, then π is potentially A_{r+1}-graphic.*

Theorem 6. *Let $n \geqslant r + 1$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ with $d_{r+1} \geqslant r - 1$. If $d_i \geqslant 2r - i$ for $i = 1, 2, \ldots, r - 1$, then π is potentially $K_{r+1} - e$-graphic.*

Theorem 7. *Let $n \geqslant 2r + 2$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ with $d_{r-1} \geqslant r$. If $d_{2r+2} \geqslant r - 1$, then π is potentially $K_{r+1} - e$-graphic.*

Theorem 8. *If $r \geqslant 2$ and $n \geqslant 3r^2 - r - 1$, then*

$$\sigma(K_{r+1} - ke, n) = \begin{cases} (r-1)(2n-r) - (n-r) + 1 & \text{if } n - r \text{ is odd,} \\ (r-1)(2n-r) - (n-r) + 2 & \text{if } n - r \text{ is even.} \end{cases}$$

After reading [66], Yin [72] determined the values $\sigma(K_{r+1} - K_3, n)$ for $r \geqslant 3$, $n \geqslant 3r + 5$.

Theorem 9. *If $r \geqslant 3$ and $n \geqslant 3r+5$, then $\sigma(K_{r+1}-K_3,n)=(r-1)(2n-r)-2(n-r)+2$.*

Determining $\sigma(K_{r+1}-H,n)$, where H is a tree on 4 vertices, is more useful than a cycle on 4 vertices (for example, $C_4 \not\subset C_i$, but $P_3 \subset C_i$ for $i \geqslant 5$). So, after reading [66] and [72], Lai and Hu [38] determined $\sigma(K_{r+1}-H,n)$ for $n \geqslant 4r+10$, $r \geqslant 3$, $r+1 \geqslant k \geqslant 4$ and H a graph on k vertices which containing a tree on 4 vertices but does not contain a cycle on 3 vertices and $\sigma(K_{r+1}-P_2,n)$ for $n \geqslant 4r+8$, $r \geqslant 3$.

Theorem 10. *If $r \geqslant 3$ and $n \geqslant 4r+8$, then $\sigma(K_{r+1}-P_2,n)=(r-1)(2n-r)-2(n-r)+2$.*

Theorem 11. *If $r \geqslant 3$, $r+1 \geqslant k \geqslant 4$ and $n \geqslant 4r+10$, then $\sigma(K_{r+1}-H,n)=(r-1)(2n-r)-2(n-r)$, where H is a graph on k vertices which contains a tree on 4 vertices but not contains a cycle on 3 vertices.*

There are a number of graphs on k vertices which contain a tree on 4 vertices but do not containing a cycle on 3 vertices (for example, the cycle on k vertices, the tree on k vertices, and the complete 2-partite graph on k vertices, etc).

Lai and Sun [39] determined $\sigma(K_{r+1}-(kP_2 \cup tK_2),n)$ for $n \geqslant 4r+10$, $r+1 \geqslant 3k+2t$, $k+t \geqslant 2$, $k \geqslant 1$, $t \geqslant 0$.

Theorem 12. *If $n \geqslant 4r+10$, $r+1 \geqslant 3k+2t$, $k+t \geqslant 2$, $k \geqslant 1$, $t \geqslant 0$, then $\sigma(K_{r+1}-(kP_2 \cup tK_2),n)=(r-1)(2n-r)-2(n-r)$.*

As yet, the problem of determining $\sigma(K_{r+1}-H,n)$ for H not containing a cycle on 3 vertices and n sufficiently large has not been solved.

Lai [37] determined $\sigma(K_{r+1}-Z,n)$ for $n \geqslant 5r+19$, $r+1 \geqslant k \geqslant 5$, $j \geqslant 5$ and Z a graph on k vertices and j edges which contains a graph Z_4 but does not contain a cycle on 4 vertices. In the same paper, the author also determined the values of $\sigma(K_{r+1}-Z_4,n)$, $\sigma(K_{r+1}-(K_4-e),n)$ and $\sigma(K_{r+1}-K_4,n)$ for $n \geqslant 5r+16$, $r \geqslant 4$.

Theorem 13. *If $r \geqslant 4$ and $n \geqslant 5r+16$, then*

$$\sigma(K_{r+1}-K_4,n)=\sigma(K_{r+1}-(K_4-e),n)=$$

$$\sigma(K_{r+1}-Z_4,n)=\begin{cases}(r-1)(2n-r)-3(n-r)+1 & \text{if } n-r \text{ is odd,}\\ (r-1)(2n-r)-3(n-r)+2 & \text{if } n-r \text{ is even.}\end{cases}$$

Theorem 14. *If $n \geqslant 5r+19$, $r+1 \geqslant k \geqslant 5$, and $j \geqslant 5$, then*

$$\sigma(K_{r+1}-Z,n)=\begin{cases}(r-1)(2n-r)-3(n-r)-1 & \text{if } n-r \text{ is odd,}\\ (r-1)(2n-r)-3(n-r)-2 & \text{if } n-r \text{ is even}\end{cases}$$

where Z is a graph on k vertices and j edges which contains a graph Z_4 but does not contain a cycle on 4 vertices.

There are a number of graphs on k vertices and j edges which contain a graph Z_4 but do not contain a cycle on 4 vertices. (For example, the graph obtained by $C_3, C_{i_1}, C_{i_2}, \ldots, C_{i_p}$ intersecting in a single vertex ($i_j \neq 4, j = 1,2,3,\ldots,p$) (if $i_j = 3$, $j = 1,2,3,\ldots,p$, then this graph is the friendship graph F_{p+1}), the graph obtained by $C_3, P_{i_1}, P_{i_2}, \ldots, P_{i_p}$ intersecting in a single vertex ($i_1 \geqslant 1$), the graph obtained by $C_3, P_{i_1}, C_{i_2}, \ldots, C_{i_p}$ ($i_j \neq 4$, $j = 2,3,\ldots,p$, $i_1 \geqslant 1$) intersecting in a single vertex, etc.)

Lai and Yan [40] proved that

Theorem 15. *If $n \geqslant 5r+18$, $r+1 \geqslant k \geqslant 7$, and $j \geqslant 6$, then*

$$\sigma(K_{r+1}-U,n)=\begin{cases}(r-1)(2n-r)-3(n-r)-1 & \text{if } n-r \text{ is odd,}\\ (r-1)(2n-r)-3(n-r) & \text{if } n-r \text{ is even}\end{cases}$$

where U is a graph on k vertices and j edges which contains a graph $(K_3 \cup P_3)$ but does not contain a cycle on 4 vertices and not contains Z_4.

There are a number of graphs on k vertices and j edges which contains a graph $(K_3 \cup P_3)$ but do not contain a cycle on 4 vertices and do not contain Z_4. (For example, $C_3 \cup C_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ ($i_j \neq 4, j = 2,3,\ldots,p, i_1 \geqslant 5$), $C_3 \cup P_{i_1} \cup P_{i_2} \cup \ldots \cup P_{i_p}$ ($i_1 \geqslant 3$), $C_3 \cup P_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ ($i_j \neq 4$, $j = 2,3,\ldots,p$, $i_1 \geqslant 3$), etc.)

A harder question is to characterize the potentially H-graphic sequences without zero terms. Luo [53] characterized the potentially C_k-graphic sequences for each $k = 3,4,5$.

Theorem 16. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 3$. Then π is potentially C_3-graphic if and only if $d_3 \geqslant 2$ except for 2 case: $\pi = (2^4)$ and $\pi = (2^5)$.*

Theorem 17. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 4$. Then π is potentially C_4-graphic if and only if the following conditions hold:*

(1) $d_4 \geqslant 2$.

(2) $d_1 = n-1$ *implies* $d_2 \geqslant 3$.

(3) *If* $n = 5,6$, *then* $\pi \neq (2^n)$.

Theorem 18. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially C_5-graphic if and only if the following conditions hold:*
(1) $d_5 \geqslant 2$.
(2) *For $i = 1, 2$, $d_1 = n - i$ implies $d_{4-i} \geqslant 3$.*
(3) *If $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$, then $d_1 + d_2 \leqslant n + k - 2$.*

Chen [2] characterized the potentially C_k-graphic sequences for $k = 6$.

Theorem 19. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 6$. Then π is potentially C_6-graphic if and only if the following conditions hold:*
(1) $d_6 \geqslant 2$.
(2) *If $n = 7, 8$, then $\pi \neq (2^n)$.*
(3) *For $i = 1, 2, 3$, $d_1 = n - i$ implies $d_{5-i} \geqslant 3$.*
(4) *If $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$, then $d_1 + d_2 \leqslant n + k - 2$; if $\pi = (d_1, d_2, 3, 2^k, 1^{n-k-3})$, then $d_1 + d_2 \leqslant n + k$; if $\pi = (d_1, d_2, 3, 3, 2^k, 1^{n-k-4})$, then $d_1 + d_2 \leqslant n + k + 2$.*

Yin, Chen and Chen [60] characterized the potentially ${}_kC_l$-graphic sequences for each $k = 3, 4 \leqslant l \leqslant 5$ and $k = 4$, $l = 5$.

Theorem 20. *Let $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ be a potentially C_4-graphic sequence. Then π is potentially ${}_3C_4$-graphic if and only if π satisfies one of the following conditions:*
(1) *$d_2 \geqslant 3$ and $\pi \neq (3^2, 2^4)$;*
(2) *$\pi = (d_1, 2^k, 1^{n-k-1})$ with $2 \leqslant d_1 \leqslant 3$ and $k \geqslant 6$, and $\pi \neq (2^8)$ and (2^9);*
(3) *$\pi = (d_1, 2^k, 1^{n-k-1})$ with $4 \leqslant d_1 \leqslant n-2$ and $k \geqslant 5$, and $\pi \neq (4, 2^6)$ and $(4, 2^7)$.*

Theorem 21. *Let $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ be a potentially C_5-graphic sequence. Then π is potentially ${}_3C_5$-graphic if and only if π satisfies one of the following conditions:*
(1) *$d_2 \geqslant 3$ and $\pi \neq (3^2, 2^4)$ and $(3^2, 2^5)$;*
(2) *$\pi = (d_1, 2^k, 1^{n-k-1})$ with $2 \leqslant d_1 \leqslant 3$ and $k \geqslant 11$, and $\pi \neq (2^{13})$ and (2^{14});*
(3) *$\pi = (d_1, 2^k, 1^{n-k-1})$ with $4 \leqslant d_1 \leqslant 5$ and $k \geqslant 10$, and $\pi \neq (4, 2^{11})$ and $(4, 2^{12})$;*
(4) *$\pi = (d_1, 2^k, 1^{n-k-1})$ with $6 \leqslant d_1 \leqslant n-4$ and $k \geqslant 9$, and $\pi \neq (4, 2^{10})$ and $(4, 2^{11})$.*

Theorem 22. *Let $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ be a potentially C_5-graphic sequence. Then π is potentially ${}_4C_5$-graphic if and only if π satisfies one of the following conditions:*
(1) *$d_2 \geqslant 3$;*
(2) *$\pi = (d_1, 2^k, 1^{n-k-1})$ with $2 \leqslant d_1 \leqslant 3$ and $k \geqslant 8$, and $\pi \neq (2^{10})$ and (2^{11});*

(3) $\pi = (d_1, 2^k, 1^{n-k-1})$ *with* $4 \leqslant d_1 \leqslant n-4$ *and* $k \geqslant 7$, *and* $\pi \neq (4, 2^8)$ *and* $(4, 2^9)$.

Chen, Yin and Fan [10] characterized the potentially ${}_kC_l$-graphic sequences for each $3 \leqslant k \leqslant 5, l = 6$.

Theorem 23. *Let* $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$, $n \geqslant 6$, *and* $\pi \neq (3^2, 2^{10})$, (2^{19}), (2^{20}), $(4, 2^{17})$, $(4, 2^{18})$, $(6, 2^{16})$, $(6, 2^{17})$, $(8, 2^{15})$, $(8, 2^{16})$. *Then* π *is potentially* ${}_3C_6$-*graphic if and only if* π *be a potentially* C_6-*graphic sequence, and* π *satisfies one of the following conditions:*

(1) $d_3 \geqslant 3$, *and if* $d_1 = d_3 = 3$, $d_4 = 2$, *then* $d_{10} = 2$;

(2) $d_2 \geqslant 4$, $d_3 = 2$, $d_7 = 2$;

(3) $d_2 = 3$, $d_3 = 2$, *and if* $4 \geqslant d_1 \geqslant 3$, *then* $d_{10} = 2$, *and if* $n-4 \geqslant d_1 \geqslant 5$, *then* $d_9 = 2$;

(4) $d_2 = 2$, *and if* $3 \geqslant d_1 \geqslant 2$, *then* $d_{18} = 2$, *and if* $5 \geqslant d_1 \geqslant 4$, *then* $d_{17} = 2$, *and if* $7 \geqslant d_1 \geqslant 6$, *then* $d_{16} = 2$, *and if* $n-7 \geqslant d_1 \geqslant 8$, *then* $d_{15} = 2$.

Theorem 24. *Let* $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$, $n \geqslant 6$, *and* $\pi \neq (2^{16})$, (2^{17}), $(4, 2^{14})$, $(4, 2^{15})$, $(6, 2^{13})$, $(6, 2^{14})$, *Then* π *is potentially* ${}_4C_6$-*graphic if and only if* π *is a potentially* C_6-*graphic sequence, and* π *satisfies one of the following conditions:*

(1) $d_3 \geqslant 3$, *and if* $d_1 = d_3 = 3$, $d_4 = 2$, *then* $d_{10} = 2$;

(2) $d_2 \geqslant 4$, $d_3 = 2$, $d_7 = 2$;

(3) $d_2 = 3$, $d_3 = 2$, *and if* $4 \geqslant d_1 \geqslant 3$, *then* $d_{10} = 2$, *and if* $n-4 \geqslant d_1 \geqslant 5$, *then* $d_9 = 2$;

(4) $d_2 = 2$, *and if* $3 \geqslant d_1 \geqslant 2$, *then* $d_{15} = 2$, *and if* $5 \geqslant d_1 \geqslant 4$, *then* $d_{14} = 2$, *and if* $n-7 \geqslant d_1 \geqslant 6$, *then* $d_{13} = 2$.

Theorem 25. *Let* $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$, $n \geqslant 6$, *and* $\pi \neq (2^{12})$, (2^{13}), $(4, 2^{10})$, $(4, 2^{11})$, *Then* π *is potentially* ${}_5C_6$-*graphic if and only if* π *is a potentially* C_6-*graphic sequence, and* π *satisfies one of the following conditions:*

(1) $d_2 \geqslant 3$;

(2) $3 \geqslant d_1 \geqslant 2$, $d_2 = 2$, $d_{11} = 2$;

(3) $n-6 \geqslant d_1 \geqslant 4$, $d_2 = 2$, $d_{10} = 2$.

Luo and Warner [54] characterized the potentially K_4-graphic sequences.

Theorem 26. *Let* $\pi = (d_1, d_2, \ldots, d_n)$ *be a graphic sequence without zero terms and with* $d_4 \geqslant 3$ *and* $n \geqslant 4$. *Then* π *is potentially* K_4-*graphic if and only if* $d_4 \geqslant 3$ *and* $\pi \neq (n-1, 3^s, 1^{n-s-1})$ *for each* $s = 4, 5$ *except the following sequences:*

$n = 5$: $(4, 3^4)$, $(3^4, 2)$;

$n = 6$: (4^6), $(4^2, 3^4)$, $(4, 3^4, 2)$, (3^6), $(3^5, 1)$, $(3^4, 2^2)$;

$n=7$: (4^7), $(4^3,3^4)$, $(4,3^6)$, $(4,3^5,1)$, $(3^6,2)$, $3^5,2,1)$;
$n=8$: $(3^7,1)$, $(3^6,1^2)$.

Eschen and Niu [14] and Lai [31] independently characterized the potentially K_4-e-graphic sequences.

Theorem 27. *Let $\pi=(d_1,d_2,\ldots,d_n)$ be a graphic sequence with $n\geqslant 4$. Then π is potentially (K_4-e)-graphic if and only if the following conditions hold:*
(1) $d_2\geqslant 3$.
(2) $d_4\geqslant 2$.
(3) *If $n=5,6$, then $\pi\neq(3^2,2^{n-2})$ and $\pi\neq(3^6)$.*

Yin and Yin [73] characterized the potentially K_5-e and K_6-graphic sequences.

Theorem 28. *Let $n\geqslant 5$ and $\pi=(d_1,d_2,\ldots,d_n)\in\mathrm{NS}_n$ be a positive graphic sequence with $d_3\geqslant 4$ and $d_5\geqslant 3$. Then π is potentially K_5-e-graphic if and only if π is not one of the following sequences:* $(n-1,4^6,1^{n-7})$, $(n-1,4^2,3^4,1^{n-7})$, $(n-1,4^2,3^3,1^{n-6})$;
$n=6$: $(4^6),(4^4,3^2),(4^3,3^2,2)$;
$n=7$: $(4^3,3^4)$, $(5^2,4,3^4)$, (4^7), $(4^5,3^2)$, $(5,4^3,3^3)$, $(5^2,4^5)$, $(5,4^5,3)$, $(4^3,3^2,2^2)$, $(4^4,3^2,2)$, $(5,4^2,3^3,2)$, $(4^6,2)$, $(4^3,3^3,1)$;
$n=8$: (5^8), (4^8), $(5^2,4^6)$, $(6,4^7)$, $(4^4,3^4)$, $(5,4^2,3^5)$, $(4^6,3^2)$, $(5,4^6,3)$, $(4^3,3^4,2)$, $(4^7,2)$, $(4^4,3^3,1)$, $(5,4^2,3^4,1)$, $(4^3,3^3,2,1)$, $(4^6,3,1)$, $(5,4^6,1)$;
$n=9$: (4^9), $(4^3,3^5,1)$, $(4^8,2)$, $(4^7,3,1)$, $(5,4^7,1)$, $(4^3,3^4,1^2)$, $(4^7,1^2)$;
$n=10$: $(4^8,1^2)$.

Theorem 29. *Let $n\geqslant 18$ and $\pi=(d_1,d_2,\ldots,d_n)\in\mathrm{NS}_n$ be a positive graphic sequence with $d_6\geqslant 5$. Then π is potentially A_6-graphic if and only if $\pi_6\notin\{(2),(2^2),(3,1),(3^2),(3,2,1),(3^2,2),(3^3,1),(3^2,1^2)\}$.*

Yin and Chen [61] characterized the potentially $K_{r,s}$-graphic sequences for $r=2$, $s=3$ and $r=2$, $s=4$.

Theorem 30. *Let $n\geqslant 5$ and $\pi=(d_1,d_2,\ldots,d_n)\in\mathrm{GS}_n$. Then π is potentially $K_{2,3}$-graphic if and only if π satisfies the following conditions:*
(1) *$d_2\geqslant 3$ and $d_5\geqslant 2$;*
(2) *if $d_1=n-1$ and $d_2=3$, then $d_5=3$;*
(3) *$\pi\neq(3^2,2^4)$, $(3^2,2^5)$, $(4^3,2^3)$, $(n-1,3^5,1^{n-6})$ and $(n-1,3^6,1^{n-7})$.*

Theorem 31. *Let $n \geqslant 6$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$. Then π is potentially $K_{2,4}$-graphic if and only if π satisfies the following conditions:*

(1) $d_2 \geqslant 4$ *and* $d_6 \geqslant 2$;

(2) *if* $d_1 = n-1$ *and* $d_2 = 4$, *then* $d_3 = 4$ *and* $d_6 \geqslant 3$;

(3) $\pi \neq (4^3, 2^4)$, $(4^2, 2^5)$, $(4^2, 2^6)$, $(5^2, 4, 2^4)$, $(5^3, 3, 2^3)$, $(6, 5^2, 2^5)$, $(5^3, 2^4, 1)$, $(6^3, 2^6)$, $(n-1, 4^2, 3^4, 1^{n-7})$, $(n-1, 4^2, 3^5, 1^{n-8})$, $(n-2, 4^2, 2^3, 1^{n-6})$, *and* $(n-2, 4^3, 2^2, 1^{n-6})$.

Chen [3] characterized the potentially $K_5 - 2K_2$-graphic sequences for $5 \leqslant n \leqslant 8$. Hu and Lai [23] characterized the potentially $K_5 - P_3$, $K_5 - A_3$, $K_5 - K_3$, $K_5 - K_{1,3}$ and $K_5 - 2K_2$-graphic sequences where A_3 is $P_2 \cup K_2$.

Theorem 32. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially $K_5 - P_3$-graphic if and only if the following conditions hold:*

(1) $d_1 \geqslant 4$, $d_3 \geqslant 3$ *and* $d_5 \geqslant 2$.

(2) $\pi \neq (4, 3^2, 2^3)$, $(4, 3^2, 2^4)$ *and* $(4, 3^6)$.

Theorem 33. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially $K_5 - A_3$-graphic if and only if the following conditions hold:*

(1) $d_4 \geqslant 3$ *and* $d_5 \geqslant 2$.

(2) $\pi \neq (n-1, 3^3, 2^{n-k}, 1^{k-4})$ *where* $n \geqslant 6$ *and* $k = 4, 5, \ldots, n-2$, n *and* k *have the same parity.*

(3) $\pi \neq (3^4, 2^2)$, (3^6), $(3^4, 2^3)$, $(3^6, 2)$, $(4, 3^6)$, $(3^7, 1)$, (3^8), $(n-1, 3^5, 1^{n-6})$ *and* $(n-1, 3^6, 1^{n-7})$.

Theorem 34. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially $K_5 - K_3$-graphic if and only if the following conditions hold:*

(1) $d_2 \geqslant 4$ *and* $d_5 \geqslant 2$.

(2) $\pi \neq (4^2, 2^4)$, $(4^2, 2^5)$, $(4^3, 2^3)$ *and* (4^6).

Theorem 35. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially $K_5 - K_{1,3}$-graphic if and only if the following conditions hold:*

(1) $d_1 \geqslant 4$ *and* $d_4 \geqslant 3$.

(2) $\pi \neq (4, 3^4, 2)$, (4^6), $(4^2, 3^4)$, $(4, 3^6)$, (4^7), $(4, 3^5, 1)$, $(n-1, 3^4, 1^{n-5})$ *and* $(n-1, 3^5, 1^{n-6})$.

Theorem 36. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially $K_5 - 2K_2$-graphic if and only if the following conditions hold:*

(1) $d_1 \geqslant 4$ *and* $d_5 \geqslant 3$;

(2)

$$\pi \neq \begin{cases} (n-i, n-j, 3^{n-i-j-2k}, 2^{2k}, 1^{i+j-2}), & n-i-j \text{ is even;} \\ (n-i, n-j, 3^{n-i-j-2k-1}, 2^{2k+1}, 1^{i+j-2}), & n-i-j \text{ is odd.} \end{cases}$$

where $1 \leqslant j \leqslant n-5$ *and* $0 \leqslant k \leqslant [\frac{1}{2}(n-j-i-4)]$.

(3) $\pi \neq (4^2, 3^4)$, $(4, 3^4, 2)$, $(5, 4, 3^5)$, $(5, 3^5, 2)$, (4^7), $(4^3, 3^4)$, $(4^2, 3^4, 2)$, $(4, 3^6)$, $(4, 3^5, 1)$,$(4, 3^4, 2^2)$, $(5, 3^7)$, $(5, 3^6, 1)$, (4^8), $(4^2, 3^6)$, $(4^2, 3^5, 1)$, $(4, 3^6, 2)$, $(4, 3^5, 2, 1)$, $(4, 3^7, 1)$, $(4, 3^6, 1^2)$, $(n-1, 3^5, 1^{n-6})$ *and* $(n-1, 3^6, 1^{n-7})$.

Hu and Lai [21] characterized the potentially $K_5 - C_4$-graphic sequences.

Theorem 37. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially $(K_5 - C_4)$-graphic if and only if the following conditions hold:*

(1) $d_1 \geqslant 4$.

(2) $d_5 \geqslant 2$.

(3) $\pi \neq ((n-2)^2, 2^{n-2})$ *for* $n \geqslant 6$, *where the symbol x^y stands for y consecutive terms x.*

(4) $\pi \neq (n-k, k+i, 2^i, 1^{n-i-2})$ *where* $i = 3, 4, \ldots, n-2k$ *and* $k = 1, 2, \ldots, [\frac{1}{2}(n-1)] - 1$.

(5) *If* $n = 6$, *then* $\pi \neq (4, 2^5)$.

(6) *If* $n = 7$, *then* $\pi \neq (4, 2^6)$.

Hu and Lai [22] characterized the potentially $K_5 - Z_4$-graphic sequences.

Theorem 38. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially $(K_5 - Z_4)$-graphic if and only if the following conditions hold:*

(1) $d_1 \geqslant 4$, $d_2 \geqslant 3$ *and* $d_4 \geqslant 2$.

Hu, Lai and Wang [25] characterized the potentially $K_5 - P_4$ and $K_5 - Y_4$-graphic sequences where Y_4 is a tree on 5 vertices and 3 leaves.

Theorem 39. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially $K_5 - P_4$-graphic if and only if the following conditions hold:*

(1) $d_2 \geqslant 3$.

(2) $d_5 \geqslant 2$.

(3) $\pi \neq (n-1, k, 2^t, 1^{n-2-t})$ *where* $n \geqslant 5$, $k, t = 3, 4, \ldots, n-2$, *and*, k *and* t *have different parities.*

(4) *For* $n \geqslant 5$, $\pi \neq (n-k, k+i, 2^i, 1^{n-i-2})$ *where* $i = 3, 4, \ldots, n-2k$ *and* $k = 1, 2, \ldots, [\frac{1}{2}(n-1)] - 1$.

(5) *If* $n = 6, 7$, *then* $\pi \neq (3^2, 2^{n-2})$.

Theorem 40. *Let* $\pi = (d_1, d_2, \ldots, d_n)$ *be a graphic sequence with* $n \geqslant 5$. *Then* π *is potentially* $K_5 - Y_4$*-graphic if and only if the following conditions hold:*

(1) $d_3 \geqslant 3$.

(2) $d_4 \geqslant 2$.

(3) $\pi \neq (3^6)$.

Hu and Lai [24] characterized the potentially $K_{3,3}$ and $K_6 - C_6$-graphic sequences.

Theorem 41. *Let* $\pi = (d_1, d_2, \ldots, d_n)$ *be a graphic sequence with* $n \geqslant 6$. *Then* π *is potentially* $K_{3,3}$*-graphic if and only if the following conditions hold:*

(1) $d_6 \geqslant 3$;

(2) *for* $i = 1, 2$, $d_1 = n - i$ *implies* $d_{4-i} \geqslant 4$;

(3) $d_2 = n - 1$ *implies* $d_3 \geqslant 5$ *or* $d_6 \geqslant 4$;

(4) $d_1 + d_2 = 2n - i$ *and* $d_{n-i+3} = 1 (3 \leqslant i \leqslant n-4)$ *implies* $d_3 \geqslant 5$ *or* $d_6 \geqslant 4$;

(5) $d_1 + d_2 = 2n - i$ *and* $d_{n-i+4} = 1 (4 \leqslant i \leqslant n-3)$ *implies* $d_3 \geqslant 4$;

(6) $\pi = (d_1, d_2, 3^4, 2^t, 1^{n-6-t})$ or $(d_1, d_2, 4^2, 3^2, 2^t, 1^{n-6-t})$ *implies* $d_1 + d_2 \leqslant n + t + 2$;

(7) $\pi = (d_1, d_2, 4, 3^4, 2^t, 1^{n-7-t})$ *implies* $d_1 + d_2 \leqslant n + t + 3$;

(8) *for* $t = 5, 6$, $\pi \neq (n-i, k+i, 4^t, 2^{k-t}, 1^{n-2-k})$ *where* $i = 1, \ldots, [\frac{1}{2}(n-k)]$ *and* $k = t, \ldots, n-2i$;

(9) $\pi \neq (5^4, 3^2, 2)$, (4^6), $(3^6, 2)$, $(6^4, 3^4)$, $(4^2, 3^6)$, $(4, 3^6, 2)$, $(3^6, 2^2)$, (3^8), $(3^7, 1)$, $(4, 3^8)$, $(4, 3^7, 1)$, $(3^8, 2)$, $(3^7, 2, 1)$, $(3^9, 1)$, $(3^8, 1^2)$, $(n-1, 4^2, 3^4, 1^{n-7})$, $(n-1, 4^2, 3^5, 1^{n-8})$, $(n-1, 5^3, 3^3, 1^{n-7})$, $(n-2, 4, 3^5, 1^{n-7})$, $(n-2, 4, 3^6, 1^{n-8})$, $(n-3, 3^6, 1^{n-7})$, $(n-3, 3^7, 1^{n-8})$.

Theorem 42. *Let* $\pi = (d_1, d_2, \ldots, d_n)$ *be a graphic sequence with* $n \geqslant 6$. *Then* π *is potentially* $K_6 - C_6$*-graphic if and only if the following conditions hold:*

(1) $d_6 \geqslant 3$;

(2) *for* $i = 1, 2$, $d_1 = n - i$ *implies* $d_{4-i} \geqslant 4$;

(3) $d_2 = n - 1$ *implies* $d_4 \geqslant 4$;

(4) $d_1 + d_2 = 2n - i$ *and* $d_{n-i+3} = 1 (3 \leqslant i \leqslant n-4)$ *implies* $d_4 \geqslant 4$;

(5) $d_1 + d_2 = 2n - i$ *and* $d_{n-i+4} = 1 (4 \leqslant i \leqslant n-3)$ *implies* $d_3 \geqslant 4$;

(6) $\pi = (d_1, d_2, d_3, 3^k, 2^t, 1^{n-3-k-t})$ *implies* $d_1 + d_2 + d_3 \leqslant n + 2k + t + 1$;

(7) $\pi = (d_1, d_2, 3^4, 2^t, 1^{n-6-t})$ *implies* $d_1 + d_2 \leqslant n + t + 2$;

(8) $\pi \neq (n-i, k, t, 3^t, 2^{k-i-t-1}, 1^{n-2-k+i})$ *where* $i = 1, \ldots, [\frac{1}{2}(n-t-1)]$ *and* $k = i+t+1, \ldots, n-i$ *and* $t = 4, 5, \ldots, k-i-1$;

(9) $\pi \neq (3^6,2)$, $(4^2,3^6)$, $(4,3^6,2)$, $(3^6,2^2)$, (3^8), $(3^7,1)$, $(4,3^8)$, $(4,3^7,1)$, $(3^8,2)$, $(3^7,2,1)$, $(3^9,1)$, $(3^8,1^2)$, $(n-1,4^2,3^4,1^{n-7})$, $(n-1,4^2,3^5,1^{n-8})$, $(n-2,4,3^5,1^{n-7})$, $(n-2,4,3^6,1^{n-8})$, $(n-3,3^6,1^{n-7})$, $(n-3,3^7,1^{n-8})$.

Xu and Hu [57] characterized the potentially $K_{1,4}+e$-graphic sequences. Chen and Li [8] characterized the potentially $K_{1,t}+e$-graphic sequences.

Theorem 43. *Let $\pi=(d_1,d_2,\ldots,d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially $K_{1,4}+e$-graphic if and only if $d_1 \geqslant 4$, $d_3 \geqslant 2$.*

Theorem 44. *Let $t \geqslant 3$, $\pi=(d_1,d_2,\ldots,d_n)$ is a graphic sequence with $n \geqslant t+1$. Then π is potentially $K_{1,t}+e$-graphic if and only if $d_1 \geqslant t$, $d_3 \geqslant 2$.*

Open problems

Problem 1. Determine $\sigma(K_{r+1}-G,n)$ for G is a graph on k vertices and j edges which contains a graph $K_3 \cup K_{1,3}$ but does not contain a cycle on 4 vertices and does not contain Z_4 and P_3.

Problem 2. Determine $\sigma(K_{r+1}-G,n)$ for $G=K_3 \cup iK_2 \cup jP_2 \cup tK_3$.

Problem 3. Determine $\sigma(K_{r+1}-G,n)$ for graph G which contains $C_3, C_4, \ldots, C_l$ but does not contain a cycle on $l+1$ vertices ($4 \leqslant l \leqslant r$).

Problem 4. Determine $\sigma(K_{r+1}-G,n)$ for a graph G which contains C_3, $C_4, \ldots, C_{r+1}$.

Problem 5. Determine $\sigma(K_{r+1}-G,n)$ for small n.

Problem 6. Characterize potentially $K_{r+1}-G$-graphic sequences for the remaining G.

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Authors' addresses: Chunhui Lai (correspondent author), Department of Mathematics, Zhangzhou Teachers College, Zhangzhou, Fujian 363000, P. R. China, e-mail: zjlaichu@public.zzptt.fj.cn; Lili Hu, Department of Mathematics, Zhangzhou Teachers College, Zhangzhou, Fujian 363000, P. R. China, e-mail: jackey2591924@163.com.