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STATUSES AND BRANCH-WEIGHTS OF WEIGHTED TREES

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Abstract. In this paper we show that in a tree with vertex weights the vertices with the second smallest status and those with the second smallest branch-weight are the same.

Keywords: tree, status, branch-weight, median, centroid, second median, second centroid

MSC 2010: 05C12

1. Introduction

All graphs considered in this paper are finite, simple, and without loops. If G is a graph and there exists a weight function $w\colon V(G)\cup E(G)\to R^+$, then (G,w) is a *weighted graph*.

For a connected weighted graph (G,w), pertinent definitions and notation are given below.

For a path P in G, the *weight length* of P, denoted by $l_w(P)$, is defined by

$$l_w(P)=\sum_{e\in E(P)} w(e).$$

For vertices x, y in G, the *weight distance* between x and y, denoted by $d_w(x,y)$, is defined by

$$d_w(x,y)=\min l_w(P),$$

where the minimum is taken over all paths P joining x and y.

For any vertex x of G, the *status* of x, denoted by $s(x)$, is defined by

$$s(x)=\sum_{y\in V(G)} w(y)d_w(y,x).$$

The *median* of G, denoted by $M_1(G)$, is the set of vertices in G with the smallest status, i.e., $M_1(G) = \{t \in V(G)\colon\ s(t) \leqslant s(x) \text{ for all } x \in V(G)\}$.

The *second median* of G, denoted by $M_2(G)$, is the set of vertices in G with the second smallest status, i.e., $M_2(G) = \{t \in V(G) - M_1(G)\colon\ s(t) \leqslant s(x) \text{ for all } x \in V(G) - M_1(G)\}$.

The *weight* of G, denoted by $w(G)$, is defined by

$$w(G) = \sum_{x \in V(G)} w(x).$$

Note that by definition, the weight of a connected weighted graph is independent of the weights of its edges.

If T is a tree and (T, w) is a weighted graph, then we say that (T, w) is a *weighted tree*.

For a weighted tree (T, w), we give the following definitions and notation.

For any vertex x of T, the *branch-weight* of x, denoted by $bw(x)$, is the maximum weight of any component of $T - x$.

The *centroid* of T, denoted by $C_1(T)$, is the set of vertices in T with the smallest branch-weight, i.e., $C_1(T) = \{t \in V(T)\colon\ bw(t) \leqslant bw(x) \text{ for all } x \in V(T)\}$.

The *second centroid* of T, denoted by $C_2(T)$, is the set of vertices in T with the second smallest branch-weight, i.e., $C_2(T) = \{t \in V(T) - C_1(T)\colon\ bw(t) \leqslant bw(x) \text{ for all } x \in V(T) - C_1(T)\}$.

B. Zelinka [4] showed that any tree with constant weight function has its median equal to its centroid. A. Kang and D. Ault [2] extended the result to any weighted tree with constant vertex weight. O. Kariv and S. L. Hakimi [3] extended the result further to any weighted tree.

Proposition 1.1 [3, Lemma 3.1]**.** *Let (T, w) be a weighted tree and x a vertex in T. Then x is in the centroid of T if and only if $bw(x) \leqslant \frac{1}{2}w(T)$.*

Proposition 1.2 [3, Theorem 3.1]**.** *Any weighted tree has its median equal to its centroid.*

The purpose of this paper is to prove the following result.

Theorem 3.3. *Let (T, w) be a weighted tree with $w(e) = 1$ for each $e \in E(T)$. Then $M_2(T) = C_2(T)$.*

2. Some remarks

In this section, we give some remarks which we need for our discussions. Let us begin with those about statuses and medians. Though the main result deals with weighted trees, we state the remarks for connected weighted graphs if possible.

Remark 2.1. Let (G, w) be a connected weighted graph. Suppose that x, y are vertices in G such that xy is a cut edge. Let G_x, G_y be the components of $G - xy$ with $x \in V(G_x)$, $y \in V(G_y)$. Then we have $s(x) - s(y) = w(xy)(w(G_y) - w(G_x))$.

Proof.

$$\begin{aligned} s(x) - s(y) &= \sum_{t \in V(G)} w(t)(d_w(t,x) - d_w(t,y)) \\ &= \sum_{t \in V(G_x)} w(t)(d_w(t,x) - d_w(t,y)) + \sum_{t \in V(G_y)} w(t)(d_w(t,x) - d_w(t,y)) \\ &= \sum_{t \in V(G_x)} w(t)(-w(xy)) + \sum_{t \in V(G_y)} w(t)(w(xy)) \\ &= w(xy)(w(G_y) - w(G_x)). \end{aligned}$$

□

Remark 2.2. Let (G, w) be a connected weighted graph. Suppose that $x_1, x_2, \ldots, x_k$ $(k \geqslant 2)$ are vertices in G such that $x_1 x_2 \ldots x_k$ is a path and, for $i = 1, 2, \ldots, k-1$, each $x_i x_{i+1}$ is a cut edge of G with $w(x_i x_{i+1}) = 1$. Let $G_1, G_2, \ldots, G_k$ be the components of $G - \{x_1x_2,\ x_2x_3, \ldots, x_{k-1}x_k\}$ with $x_i \in V(G_i)$ for $i = 1, 2, \ldots, k$. Then

$$s(x_1) - s(x_k) = \sum_{i=1}^{k} (-k - 1 + 2i) w(G_i).$$

Proof. We prove the result by induction on k. By Remark 2.1 this is true for $k = 2$. Suppose the result holds for $k \geqslant 2$. Now $x_1 x_2 \ldots x_k x_{k+1}$ is a path in G such that each $x_i x_{i+1}$ is a cut edge of G with $w(x_i x_{i+1}) = 1$ for $i = 1, 2, \ldots, k$, and $G_1, G_2, \ldots, G_k, G_{k+1}$ are components of $G - \{x_1x_2, x_2x_3, \ldots,\ x_{k-1}x_k, x_kx_{k+1}\}$ with $x_i \in V(G_i)$, $i = 1, 2, \ldots, k, k+1$. Applying the induction hypothesis to the path $x_1 x_2 \ldots x_k$ we have $s(x_1) - s(x_k) = \sum_{i=1}^{k-1} (-k-1+2i) w(G_i) + (k-1)(w(G_k) + w(G_{k+1}))$. Considering the path $x_k x_{k+1}$ we have $s(x_k) - s(x_{k+1}) = -(w(G_1) + w(G_2) + \ldots +$

$w(G_k)) + w(G_{k+1})$. Thus

$$\begin{aligned} s(x_1) - s(x_{k+1}) &= (s(x_1) - s(x_k)) + (s(x_k) - s(x_{k+1})) \\ &= \sum_{i=1}^{k-1} (-k - 1 + 2i) w(G_i) + (k-1)(w(G_k) + w(G_{k+1})) \\ &\quad - (w(G_1) + w(G_2) + \ldots + w(G_k)) + w(G_{k+1}) \\ &= \sum_{i=1}^{k-1} (-k - 2 + 2i) w(G_i) + (k-2) w(G_k) + k \cdot w(G_{k+1}) \\ &= \sum_{i=1}^{k+1} (-k - 2 + 2i) w(G_i) \\ &= \sum_{i=1}^{k+1} (-(k+1) - 1 + 2i) w(G_i). \end{aligned}$$

This completes the proof. □

Remark 2.3. Let (G, w) be a connected weighted graph. Suppose that $x_1, x_2, \ldots, x_k$ ($k \geqslant 3$) are vertices in G such that $x_1 x_2 \ldots x_k$ is a path, and for $i = 1, 2, \ldots, k-1$, $x_i x_{i+1}$ is a cut edge of G. If $s(x_1) \leqslant s(x_2)$, then $s(x_2) < s(x_3) < s(x_4) < \ldots < s(x_k)$.

P r o o f. It suffices to show that $s(x_2) < s(x_3)$. Let G_1, G_2, G_3 be the components of $G - \{x_1x_2, x_2x_3\}$ such that $x_i \in V(G_i)$, $i = 1, 2, 3$. By Remark 2.1,

$$\begin{aligned} s(x_1) - s(x_2) &= w(x_1x_2)((w(G_2) + w(G_3)) - w(G_1)), \\ s(x_2) - s(x_3) &= w(x_2x_3)(w(G_3) - (w(G_1) + w(G_2))). \end{aligned}$$

Since $s(x_1) \leqslant s(x_2)$, we have $w(G_2) + w(G_3) - w(G_1) \leqslant 0$, which implies that $w(G_3) - w(G_1) - w(G_2) < 0$ for $w(G_2) > 0$. Thus $s(x_2) < s(x_3)$. □

The above remark for trees with constant weight functions appeared in [1, Theorem 3.3].

Remark 2.4. The median of a weighted tree consists either of one single vertex or two vertices which are adjacent.

P r o o f. This follows immediately from Remark 2.3. □

The following remarks concern branch-weights of weighted trees.

Remark 2.5. Let x be a vertex of a weighted tree (T, w) and B a component of $T - x$ such that $bw(x) = w(B)$. Then
(1) $bw(y) > bw(x)$ for each $y \in V(T) - (V(B) \cup \{x\})$,
(2) if $y \in V(T) - \{x\}$ and $bw(y) \leqslant bw(x)$ then $y \in V(B)$.

P r o o f. (1) Let B' be the component of $T - y$ such that $x \in V(B')$. Then $V(B') \supset \{x\} \cup V(B)$. Hence $bw(y) \geqslant w(B') > w(B) = bw(x)$.

(2) This follows from (1). □

Remark 2.6. Let $x_1 x_2 \ldots x_k$ ($k \geqslant 3$) be a path in a weighted tree (T, w) where $bw(x_1) \leqslant bw(x_2)$. Then $bw(x_2) < bw(x_3) < bw(x_4) < \ldots < bw(x_k)$.

P r o o f. It suffices to show that $bw(x_2) < bw(x_3)$. Let B be a component of $T - x_2$ such that $bw(x_2) = w(B)$. By Remark 2.5(2), $x_1 \in V(B)$. From $x_1 \in V(B)$ and $x_2 \notin V(B)$, we see that $x_3 \notin V(B)$. Since B is a component of $T - x_2$ such that $bw(x_2) = w(B)$, by Remark 2.5(1), we conclude that $bw(x_3) > bw(x_2)$. □

3. Main result

For a graph G and $A \subset V(G)$, we use $N(A)$ to denote the set

$$\{x \in V(G) - A \colon \ x \text{ is adjacent to some vertex in } A\}.$$

Lemma 3.1. *Let (T, w) be a weighted tree with $w(e) = 1$ for each $e \in E(T)$. For each $x \in N(M_1(T))$, let T_x denote the component of $T - M_1(T)$ with $x \in V(T_x)$. Then we have*

(1) $s(x) - s(y) = 2w(T_y) - 2w(T_x)$ *if* $x, y \in N(M_1(T))$ *and* $x \neq y$,

(2) $M_2(T) = \{x \in N(M_1(T)) \colon \ w(T_x) \geqslant w(T_y) \text{ for all } y \in N(M_1(T))\}$.

P r o o f. (1) By Remark 2.4, $M_1(T)$ consists either of a single vertex or of two adjacent vertices. Let $x, y \in N(M_1(T))$ with $x \neq y$. We distinguish two cases.

Case 1. x, y are adjacent to the same vertex in $M_1(T)$, say x, y are adjacent to $m \in M_1(T)$.

We see that T_x, T_y are the components of $T - \{xm, my\}$ such that $x \in V(T_x)$, $y \in V(T_y)$. Applying Remark 2.2 to the path xmy, we obtain $s(x) - s(y) = -2w(T_x) + 2w(T_y)$ since $w(e) = 1$ for all $e \in E(T)$.

Case 2. x, y are adjacent to distinct vertices in $M_1(T)$, say, x is adjacent to m_1, y is adjacent to m_2 where $m_1, m_2 \in M_1(T)$ and $m_1 \neq m_2$.

By Remark 2.4 m_1, m_2 are adjacent. Let $T' = T - \{xm_1, m_1m_2, m_2y\}$. We see that T_x, T_y are components of T' such that $x \in V(T_x), y \in V(T_y)$. Let T_1, T_2 be the components of T' such that $m_1 \in V(T_1)$, $m_2 \in V(T_2)$. Applying Remark 2.2 to the path xm_1m_2y, we obtain $s(x) - s(y) = -3w(T_x) - w(T_1) + w(T_2) + 3w(T_y)$ again since $w(e) = 1$ for all $e \in E(T)$. Since m_1, m_2 are in the median of T, we have, by Remark 2.1, $w(T_x) + w(T_1) = w(T_2) + w(T_y)$. Thus $s(x) - s(y) = -2w(T_x) + 2w(T_y)$.

(2) From Remark 2.3, we see that $M_2(T) \subset N(M_1(T))$. Thus $M_2(T) = \{x \in N(M_1(T))\colon\ s(x) \leqslant s(y)$ for all $y \in N(M_1(T))\}$. By(1), for $x, y \in N(M_1(T))$ with $x \neq y$, we have $s(x) \leqslant s(y)$ if and only if $w(T_x) \geqslant w(T_y)$. Thus $M_2(T) = \{x \in N(M_1(T))\colon\ w(T_x) \geqslant w(T_y)$ for all $y \in N(M_1(T))\}$. This completes the proof. □

Lemma 3.2. *Let (T, w) be a weighted tree. For each $x \in N(C_1(T))$, let T_x be the component of $T - C_1(T)$ with $x \in V(T_x)$. Then we have*
(1) *$bw(x) = w(T) - w(T_x)$ if $x \in N(C_1(T))$,*
(2) *$C_2(T) = \{x \in N(C_1(T))\colon\ w(T_x) \geqslant w(T_y)$ for all $y \in N(C_1(T))\}$.*

Proof. (1) Let $x \in N(C_1(T))$. Suppose that x is adjacent to c where $c \in C_1(T)$. We see that T_x is the component of $T - c$ with $x \in V(T_x)$. By Proposition 1.1, $bw(c) \leqslant \frac{1}{2}w(T)$. Thus $w(T_x) \leqslant bw(c) \leqslant \frac{1}{2}w(T)$. Let $A_0, A_1, \ldots, A_k$ be the components of $T - x$ where $c \in V(A_0)$. Then $V(T_x) = \{x\} \cup V(A_1) \cup V(A_2) \cup \ldots \cup V(A_k)$, which implies that for $i = 1, 2, \ldots, k$ we have $w(A_i) < w(T_x) \leqslant \frac{1}{2}w(T)$. Note also that $w(A_0) = w(T) - w(T_x) \geqslant \frac{1}{2}w(T)$. Hence $bw(x) = \max\limits_{0 \leqslant i \leqslant k} w(A_i) = w(A_0) = w(T) - w(T_x)$.

(2) From Remark 2.6, we see that $C_2(T) \subset N(C_1(T))$. Thus $C_2(T) = \{x \in N(C_1(T))\colon\ bw(x) \leqslant bw(y)$ for all $y \in N(C_1(T))\}$. By(1), for $x, y \in N(C_1(T))$ with $x \neq y$, we have $bw(x) \leqslant bw(y)$ if and only if $w(T_x) \geqslant w(T_y)$. Thus $C_2(T) = \{x \in N(C_1(T))\colon\ w(T_x) \geqslant w(T_y)$ for all $y \in N(C_1(T))\}$. This completes the proof. □

Since by Proposition 1.2 $M_1(T) = C_1(T)$ for any weighted tree T, the main result of this paper now follows from Lemmas 3.1(2) and 3.2(2).

Theorem 3.3. *Let (T, w) be a weighted tree with $w(e) = 1$ for each $e \in E(T)$. Then $M_2(T) = C_2(T)$.*

The above theorem cannot be extended to trees the edge weights of which are not constant. Consider the following example. Let T be the tree in Fig. 1 with $w(a) = w(b) = w(c) = w(d) = w(e) = 1$, $w(ab) = w(bc) = w(de) = 1, w(bd) = 4$. Then $s(a) = s(c) = 14, s(b) = 11, s(d) = 15, s(e) = 18$; thus $M_2(T) = \{a, c\}$. Further, $bw(a) = bw(c) = bw(e) = 4, bw(d) = 3, bw(b) = 2$; thus $C_2(T) = \{d\}$. We have $M_2(T) \neq C_2(T)$.

Also in a weighted tree, the vertices with the third smallest status need not be the same as those with the third smallest branch-weight, even if the weight function of the tree is a constant. Consider the following example. Let T be the tree in Fig. 2 with constant vertex weight 1 and constant edge weight 1. Then $s(a) = 26$, $s(b) = 19$, $s(c) = 14$, $s(d) = 11$, $s(e) = s(f) = s(g) = s(h) = s(i) = 18$, and $bw(a) = bw(e) = bw(f) = bw(g) = bw(h) = bw(i) = 8$, $bw(b) = 7$, $bw(c) = 6$,

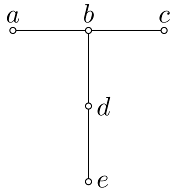

Fig. 1

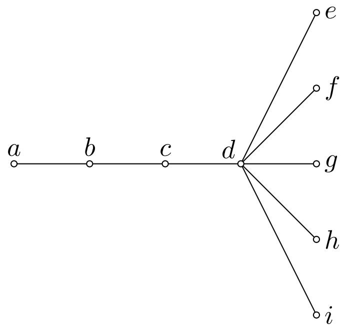

Fig. 2

$bw(d) = 3$. Thus the vertices e, f, g, h, i are those with the third smallest status, and the vertex b is the one with the third smallest branch-weight.

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