Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 3, 695–706

Persistent URL: http://dml.cz/dmlcz/140510

Czechoslovak Mathematical Journal, 59 (134) (2009), 695–706

STRONG CONVERGENCE THEOREMS OF k-STRICT PSEUDO-CONTRACTIONS IN HILBERT SPACES

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(Received January 29, 2008)

Abstract. Let K be a nonempty closed convex subset of a real Hilbert space H such that $K \pm K \subset K$, $T\colon K \to H$ a k-strict pseudo-contraction for some $0 \leqslant k < 1$ such that $F(T) = \{x \in K\colon x = Tx\} \neq \emptyset$. Consider the following iterative algorithm given by

$$\forall x_1 \in K, \quad x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_K S x_n, \quad n \geqslant 1,$$

where $S\colon K \to H$ is defined by $Sx = kx + (1-k)Tx$, P_K is the metric projection of H onto K, A is a strongly positive linear bounded self-adjoint operator, f is a contraction. It is proved that the sequence $\{x_n\}$ generated by the above iterative algorithm converges strongly to a fixed point of T, which solves a variational inequality related to the linear operator A. Our results improve and extend the results announced by many others.

Keywords: Hilbert space, nonexpansive mapping, strict pseudo-contraction, iterative algorithm, fixed point

MSC 2010: 47H09, 4710

1. Introduction and preliminaries

Throughout this paper, we use $F(T)$ to denote the fixed point set of the mapping T and P_K to denote the metric projection of the Hilbert space H onto its closed convex subset K.

Recall that a self mapping $f\colon K \to K$ is a contraction on K, if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leqslant \alpha \|x - y\|, \quad \forall x, y \in K. \tag{1.1}$$

We use Π_K to denote the collection of all contractions on K. That is, $\Pi_K = \{f;\ f\colon K \to K \text{ a contraction}\}$. An operator A is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geqslant \overline{\gamma}\|x\|^2, \quad \forall x \in K. \tag{1.2}$$

Recall that a mapping $T\colon K \to H$ is said to be a k-strict pseudo-contraction if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \tag{1.3}$$

for all x, $y \in K$.

Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mappings T on K such that

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in K. \tag{1.4}$$

That is, T is a nonexpansive mapping if and only if T is a 0-strict pseudo-contraction. It is also said to be a pseudo-contraction if $k = 1$. T is said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0, 1)$ such that $T + \lambda I$ is pseudo-contractive. Clearly, the class of k-strict pseudo-contractions falls between the classes of nonexpansive mappings and pseudo-contractions. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of k-strict pseudo-contractions (see, e.g., [2]–[4]).

It is very clear that, in a real Hilbert space H, (1.3) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leqslant \|x - y\|^2 - \frac{1-k}{2}\|(I - T)x - (I - T)y\|^2 \tag{1.5}$$

for all $x, y \in K$. T is pseudo-contractive if and only if

$$\langle Tx - Ty, x - y\rangle \leqslant \|x - y\|^2. \tag{1.6}$$

T is strongly pseudo-contractive if and only if there exists a positive constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, x - y\rangle \leqslant (1 - \lambda)\|x - y\|^2. \tag{1.7}$$

for all $x, y \in K$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping (Browder [3]). More precisely, take $t \in (0, 1)$ and define a contraction $T_t\colon K \to K$ by

$$T_t x = tu + (1 - t)Tx, \quad x \in K, \tag{1.8}$$

where $u \in K$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in K. It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [3] proved the following well-known strong convergence theorem.

Theorem 1.1. *Let K be a bounded closed convex subset of a Hilbert space H, T a nonexpansive mapping on K. Fix $u \in K$ and define $z_t \in K$ as $z_t = tu + (1-t)Tz_t$ for $t \in (0,1)$. Then $\{z_t\}$ converges strongly to a element of $F(T)$ nearest to u.*

For a sequence $\{\alpha_n\}$ of real numbers in $[0,1]$ and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by

$$x_0 \in K, \quad x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geqslant 0. \tag{1.9}$$

The recursion formula (1.9) was first introduced in 1967 by Halpern [5] in the framework of Hilbert spaces. He proved the strong convergence of $\{x_n\}$ to a fixed point of T where $\alpha_n = n^{-\theta}$.

In 1977, Lions [6] improved the result of Halpern [5], still in Hilbert spaces, by proving the strong convergence of $\{x_n\}$ to a fixed point of T where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$(C1)\colon \lim_{n\to\infty} \alpha_n = 0,\ (C2)\colon \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad (C3)\colon \lim_{n\to\infty} \frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+1}^2} = 0.$$

It was observed that both Halperns and Lions conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\{\alpha_n\} = (n+1)^{-1}$. This was overcome in 1992 by Wittmann [11], who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

$$(C1)\colon \lim_{n\to\infty} \alpha_n = 0,\ (C2)\colon \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad (C4)\colon \sum_{n=1}^{\infty} |\alpha_{n+1}-\alpha_n| < \infty.$$

In 2002, Xu [14] (see also [13]) improved the result of Lions. To be more precise, he weakened the condition $(C3)$ by removing the square in the denominator so that the canonical choice of $\{\alpha_n\} = (n+1)^{-1}$ is possible.

More recently, Xu [15] studied the following iterative process by so-called viscosity approximation which was first introduced by Moudafi [9].

$$x_0 = x \in K, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geqslant 0. \tag{1.10}$$

Xu [15] proved the following theorem in Hilbert spaces.

Theorem 1.2. *Let H be a Hilbert space, K a closed convex subset of H, $T\colon K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f\colon K \to K$ a contraction. Let $\{x_n\}$ be generated by* (1.10). *Then under the hypotheses*

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(C2) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(C5) *either* $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim\limits_{n\to\infty} (\alpha_{n+1}/\alpha_n) = 1$,

$\{x_n\}$ converges strongly to a fixed point of T, which is the unique solution of some variational inequality.

Very Recently, Marino and Xu [14] improved the result of Xu [15] by introducing the following iterative algorithm

$$x_0 \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geqslant 0. \tag{1.11}$$

To be more precise, Marino and Xu [8] obtained the following theorem.

Theorem 1.3. *Let H be a Hilbert space, K a closed convex subset of H, $T\colon H \to H$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let A be a strong positive bounded linear operator with coefficient $\overline{\gamma}$ and $f\colon H \to H$ a contraction with the contractive coefficient* $(0 < \alpha_n < 1)$ *such that* $0 < \gamma < \overline{\gamma}/\alpha$. *Let $\{x_n\}$ be generated by* (1.11). *Then under the hypotheses* (C1), (C2) *and* (C5), *$\{x_n\}$ converges strongly to a fixed point of T, which is the unique solution of some variational inequality related to the linear operator A.*

In this paper, motivated by Browder [3], Halpern [5], Witmann [11], Moudafi [9], Xu [12]–[15], Marino and Xu [7], [8] and Zhou [16], we introduce a general iterative algorithm and prove strong convergence theorems for a k-strict pseudo-contraction. Our results improve and extend the corresponding ones announced by many others.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 ([13], [14]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leqslant (1 - \gamma_n)\alpha_n + \delta_n,$$

where γ_n is a sequence in $(0,1)$ *and $\{\delta_n\}$ is a sequence such that*

(i) $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup\limits_{n\to\infty} \delta_n/\gamma_n \leqslant 0$ *or* $\sum\limits_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim\limits_{n\to\infty} \alpha_n = 0$.

Lemma 1.2 ([8]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with the coefficient $\overline{\gamma} > 0$ and $0 < \varrho \leqslant \|A\|^{-1}$. Then $\|I - \varrho A\| \leqslant 1 - \varrho\overline{\gamma}$.*

Lemma 1.3 ([8]). *Let H be a Hilbert space. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let $T\colon H \to H$ be a nonexpansive mapping with a fixed point $x_t \in H$ of the contraction $x \mapsto t\gamma f(x) + (1 - tA)Tx$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\bar{x}$ of T, which solves the variational inequality*

$$\langle (A - \gamma f)\bar{x}, z - \bar{x}\rangle \leqslant 0, \quad \forall z \in F(T).$$

Lemma 1.4. *In a Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, (x + y)\rangle, \quad x, y \in H.$$

Lemma 1.5 ([16]). *If T is a k-strict pseudo-contraction on a closed convex subset K of a real Hilbert space H, then the fixed point set $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 1.6 ([16]). *Let $T\colon K \to H$ be a k-strict pseudo-contraction with $F(T) \neq \emptyset$. Then $F(P_K T) = F(T)$. Define $S\colon K \to H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for each $x \in K$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

Lemma 1.7 ([10]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let β_n be a sequence in $[0, 1]$ with $0 < \liminf\limits_{n\to\infty} \beta_n \leqslant \limsup\limits_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geqslant 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0.$$

Then $\lim\limits_{n\to\infty} \|y_n - x_n\| = 0$.

2. Main results

Theorem 2.1. *Let K be a nonempty closed convex subset of a real Hilbert space H such that $K \pm K \subset K$ and $T\colon K \to H$ a k-strict pseudo-contraction for some $0 \leqslant k < 1$ with a fixed point. Let A be a strongly positive linear bounded self-adjoint* *operator on K with the coefficient $\overline{\gamma}$ and $f \in \Pi_K$ a contraction with the contractive coefficient $(0 < \alpha < 1)$ such that $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\forall x_1 \in K, \quad x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_K S x_n, \quad n \geqslant 1,$$

where $S\colon K \to H$ is defined by $Sx = kx + (1-k)Tx$. If the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ converges strongly to a fixed point q of T, which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leqslant 0, \quad \forall p \in F(T).$$

Proof. We divide the proof into three parts.

Step 1. First, we show the sequence $\{x_n\}$ is bounded.

From Lemma 1.6, we see that $S\colon K \to H$ is a nonexpansive mapping and $F(S) = F(T)$. By our assumptions on T, we know $F(T) \neq \emptyset$ and hence $F(S) \neq \emptyset$. By Lemma 1.6, we see that$F(P_K S) = F(S) \neq \emptyset$. Since $P_K\colon H \to K$ is a nonexpansive mapping, we conclude that $P_K S\colon K \to K$ is nonexpansive. From the condition (i), we may assume, without loss of generality, that $\alpha_n \leqslant (1-\beta_n)\|A\|^{-1}$ for all $n \geqslant 1$. Since A is a strongly positive bounded linear operator on K, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle|\colon\ x \in K, \|x\| = 1\}.$$

Observe that

$$\langle ((1-\beta_n)I - \alpha_n A)x, x\rangle = 1 - \beta_n - \alpha_n \langle Ax, x\rangle \geqslant 1 - \beta_n - \alpha_n \|A\| \geqslant 0,$$

that is, $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned} \|(1-\beta_n)I - \alpha_n A\| &= \sup\{\langle ((1-\beta_n)I - \alpha_n A)x, x\rangle\colon\ x \in K, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n \langle Ax, x\rangle\colon\ x \in K, \|x\| = 1\} \\ &\leqslant 1 - \beta_n - \alpha_n \overline{\gamma}. \end{aligned}$$

Therefore, taking a point $p \in F(T)$, we obtain

$$\begin{aligned} &\|x_{n+1} - p\| \\ &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(P_K S x_n - p)\| \\ &\leqslant (1 - \beta_n - \alpha_n \overline{\gamma})\|P_K S x_n - p\| + \beta_n \|x_n - p\| + \alpha_n \|\gamma f(x_n) - Ap\| \end{aligned}$$

$$\leqslant (1-\beta_n-\alpha_n\overline{\gamma})\|x_n-p\|+\beta_n\|x_n-p\|+\alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|$$
$$=[1-\alpha_n(\overline{\gamma}-\gamma\alpha)]\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|.$$

By simple inductions, we have

$$\|x_n-p\|\leqslant \max\Big\{\|x_0-p\|,\frac{\|Ap-\gamma f(p)\|}{\overline{\gamma}-\gamma\alpha}\Big\},\quad n\geqslant 1,$$

which gives that the sequence $\{x_n\}$ is bounded.

Step 2. In this part, we show that $\lim_{n\to\infty}\|P_KSx_n-x_n\|=0$.

Put $l_n=(x_{n+1}-\beta_nx_n)/(1-\beta_n)$. That is,

$$x_{n+1}=(1-\beta_n)l_n+\beta_nx_n,\quad n\geqslant 1. \tag{2.1}$$

Now, we compute $l_{n+1}-l_n$. Observing that

$$\begin{aligned}l_{n+1}-l_n&=\frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)P_KSx_{n+1}}{1-\beta_{n+1}}\\&\quad-\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)P_KSx_n}{1-\beta_n}\\&=\frac{\alpha_{n+1}(\gamma f(x_{n+1})-AP_KSx_{n+1})}{1-\beta_{n+1}}-\frac{\alpha_n(\gamma f(x_n)-AP_KSx_n)}{1-\beta_n}\\&\quad+P_KSx_{n+1}-P_KSx_n,\end{aligned}$$

we have

$$\begin{aligned}\|l_{n+1}-l_n\|&\leqslant\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-AP_KSx_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|AP_KSx_n-\gamma f(x_n)\|\\&\quad+\|P_KSx_{n+1}-P_KSx_n\|\\&\leqslant\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-AP_KSx_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|AP_KSx_n-\gamma f(x_n)\|\\&\quad+\|x_{n+1}-x_n\|.\end{aligned}$$

It follows from the conditions (i) and (iii) that

$$\limsup_{n\to\infty}\{\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\|\}\leqslant 0.$$

From Lemma 1.7, we have

$$\lim_{n\to\infty}\|x_n-l_n\|=0. \tag{2.2}$$

Observing (2.1) again, we have

$$\|x_{n+1} - x_n\| = (1 - \beta_n)\|x_n - l_n\|.$$

From the condition (iii) and (2.2), we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2.3}$$

Notice that

$$\begin{aligned}\|x_n - P_K S_n\| &\leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - P_K S x_n\| \\ &\leqslant \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - AP_K S x_n\| + \beta_n\|x_n - P_K S x_n\|,\end{aligned}$$

which yields that

$$(1 - \beta_n)\|x_n - P_K S_n\| \leqslant \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - AP_K S x_n\|.$$

It follows from the conditions (i), (iii) and (2.3) that

$$\lim_{n\to\infty} \|x_n - P_K S x_n\| = 0. \tag{2.4}$$

Step 3. Finally, we show that $x_n \to q$, as $n \to \infty$.

First, we claim that

$$\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle \leqslant 0, \tag{2.5}$$

where $q = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto t\gamma f(x) + (I - tA)P_K S x.$$

Then x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)P_K S x_t$, where $t \in (0, \min\{1, \|A\|^{-1}\})$. Thus we have

$$\|x_t - x_n\| = \|(I - tA)(P_K S x_t - x_n) + t(\gamma f(x_t) - Ax_n)\|.$$

It follows from Lemma 1.4 that

$$\begin{aligned}\|x_t - x_n\|^2 &= \|(I - tA)(P_K S x_t - x_n) + t(\gamma f(x_t) - Ax_n)\|^2 \\ &\leqslant (1 - \overline{\gamma} t)^2\|P_K S x_t - x_n\|^2 + 2t\langle \gamma f(x_t) - Ax_n, x_t - x_n\rangle \\ &\leqslant (1 - 2\overline{\gamma} t + (\overline{\gamma} t)^2)\|x_t - x_n\|^2 + f_n(t) \\ &\quad + 2t\langle \gamma f(x_t) - Ax_t, x_t - x_n\rangle + 2t\langle Ax_t - Ax_n, x_t - x_n\rangle,\end{aligned} \tag{2.6}$$

where

$$f_n(t) = (2\|x_t - x_n\| + \|x_n - P_K S x_n\|)\|x_n - P_K S x_n\| \to 0, \quad \text{as } n \to 0. \tag{2.7}$$

Observing A is linear and strongly positive and using (1.2), we have

$$\langle Ax_t - Ax_n, x_t - x_n \rangle = \langle A(x_t - x_n), x_t - x_n \rangle \geqslant \overline{\gamma}\|x_t - x_n\|^2. \tag{2.8}$$

Combining (2.6) and (2.8), we obtain

$$\begin{aligned}
&2t\langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \\
&\leqslant (\overline{\gamma}^2 t^2 - 2\overline{\gamma} t)\|x_t - x_n\|^2 + f_n(t) + 2t\langle Ax_t - Ax_n, x_t - x_n \rangle \\
&\leqslant (\overline{\gamma} t^2 - 2t)\langle A(x_t - x_n), x_t - x_n \rangle + f_n(t) + 2t\langle Ax_t - Ax_n, x_t - x_n \rangle \\
&\leqslant \overline{\gamma} t^2 \langle A(x_t - x_n), x_t - x_n \rangle + f_n(t).
\end{aligned}$$

It follows that

$$\langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leqslant \frac{\overline{\gamma} t}{2}\langle Ax_t - Ax_n, x_t - x_n \rangle + \frac{1}{2t} f_n(t). \tag{2.9}$$

Let $n \to \infty$ in (2.9) and note that (2.7) yields

$$\limsup_{n\to\infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leqslant \frac{t}{2} M_1, \tag{2.10}$$

where $M_1 > 0$ is an appropriate constant such that $M_1 \geqslant \overline{\gamma}\langle Ax_t - Ax_n, x_t - x_n \rangle$ for all $t \in (0,1)$ and $n \geqslant 1$. Taking $t \to 0$ in (2.10), we have

$$\limsup_{t\to 0} \limsup_{n\to\infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leqslant 0. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}
\langle \gamma f(q) - Aq, x_n - q \rangle = {} & \langle \gamma f(q) - Aq, x_n - q \rangle - \langle \gamma f(q) - Aq, x_n - x_t \rangle \\
& + \langle \gamma f(q) - Aq, x_n - x_t \rangle - \langle \gamma f(q) - Ax_t, x_n - x_t \rangle \\
& + \langle \gamma f(q) - Ax_t, x_n - x_t \rangle - \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\
& + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q \rangle \\
&\qquad \leqslant \|\gamma f(q) - Aq\|\|x_t - q\| + \|A\|\|x_t - q\| \lim_{n\to\infty} \|x_n - x_t\| \\
&\qquad\quad + \gamma\alpha\|q - x_t\| \lim_{n\to\infty} \|x_n - x_t\| + \limsup_{n\to\infty} \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle.
\end{aligned}$$

Therefore, from (2.11), we have

$$
\begin{aligned}
&\limsup_{n\to\infty}\langle \gamma f(q)-Aq, x_n-q\rangle \\
&=\limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(q)-Aq, x_n-q\rangle \\
&\leqslant \limsup_{t\to 0}\|\gamma f(q)-Aq\|\|x_t-q\|+\limsup_{t\to 0}\|A\|\|x_t-q\|\lim_{n\to\infty}\|x_n-x_t\| \\
&\quad +\limsup_{t\to 0}\gamma\alpha\|q-x_t\|\lim_{n\to\infty}\|x_n-x_t\|+\limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(x_t)-Ax_t, x_n-x_t\rangle \\
&\leqslant 0.
\end{aligned}
$$

Hence, (2.5) holds. Now from Lemma 1.4, we have

$$
\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&=\|((1-\beta_n)I-\alpha_nA)(P_KSx_n-q)+\beta_n(x_n-p)+\alpha_n(\gamma f(x_n)-Aq)\|^2 \\
&\leqslant \|((1-\beta_n)I-\alpha_nA)(P_KSx_n-q)+\beta_n(x_n-p)\|^2 \\
&\quad +2\alpha_n\langle \gamma f(x_n)-Aq, x_{n+1}-q\rangle \\
&\leqslant (1-\alpha_n\overline{\gamma})^2\|x_n-q\|^2+\alpha_n\gamma\alpha(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\
&\quad +2\alpha_n\langle \gamma f(q)-Aq, x_{n+1}-q\rangle,
\end{aligned} \tag{2.12}
$$

which implies that

$$
\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&\leqslant \frac{(1-\alpha_n\overline{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle \gamma f(q)-Aq, x_{n+1}-q\rangle \\
&\leqslant \Big[1-\frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\Big]\|x_n-q\|^2 \\
&\quad +\frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\Big[\frac{1}{\overline{\gamma}-\alpha\gamma}\langle \gamma f(q)-Aq, x_{n+1}-q\rangle+\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\alpha\gamma)}M_2\Big],
\end{aligned} \tag{2.13}
$$

where M_2 is an appropriate constant such that $M_2 \geqslant \sup_{n\geqslant 1}\{\|x_n-q\|^2\}$. Put $j_n = 2\alpha_n(\overline{\gamma}-\alpha\gamma)/(1-\alpha_n\alpha\gamma)$ and

$$
t_n=\frac{1}{\overline{\gamma}-\alpha\gamma}\langle \gamma f(q)-Aq, x_{n+1}-q\rangle+\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\alpha\gamma)}M_2.
$$

That is,

$$
\|x_{n+1}-q\|^2 \leqslant (1-j_n)\|x_n-q\|+j_nt_n. \tag{2.14}
$$

It follows from the conditions (i), (ii) and (2.5) that $\lim_{n\to\infty} j_n=0$, $\sum_{n=1}^{\infty} j_n=\infty$ and $\limsup_{n\to\infty} t_n\leqslant 0$. Apply Lemma 1.1 to (2.14) to conclude that $x_n\to q$, as $n\to\infty$. This completes the proof. □

3. Applications

As applications of Theorem 2.1, we have the following results immediately.

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H such that $K \pm K \subset K$ and $T\colon K \to H$ a nonexpansive mapping with a fixed point. Let A be a strongly positive linear bounded self-adjoint operator with the coefficient $\overline{\gamma}$ and $f \in \Pi_K$ a contraction with the contractive coefficient $(0 < \alpha < 1)$ such that $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\forall x_1 \in K, \quad x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_K T x_n, \quad n \geqslant 1.$$

If the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf\limits_{n\to\infty} \beta_n \leqslant \limsup\limits_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ converges strongly to a fixed point q of T, which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leqslant 0, \quad \forall p \in F(T).$$

Taking $A = I$, the identity mapping and $\gamma = 1$ in Theorem 3.1, we have the following.

Theorem 3.2. *Let K be a nonempty closed convex subset of a real Hilbert space H and $T\colon K \to H$ a nonexpansive mapping with a fixed point. Let $f\colon K \to K$ be a contraction with the contractive coefficient $(0 < \alpha < 1)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\forall x_1 \in K, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\beta_n-\alpha_n)P_K T x_n, \quad n \geqslant 1.$$

If the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf\limits_{n\to\infty} \beta_n \leqslant \limsup\limits_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ converges strongly to a fixed point q of T, which solves the following variational inequality

$$\langle f(q) - q, p - q \rangle \leqslant 0, \quad \forall p \in F(T).$$

References

[1] *G. L. Acedo and H. K. Xu*: Iterative methods for strict pseudo-contractions in Hilbert spaces. Nonlinear Anal. *67* (2007), 2258–2271.

[2] *F. E. Browder*: Fixed point theorems for noncompact mappings in Hilbert spaces. Proc. Natl. Acad. Sci. USA *53* (1965), 1272–1276.

[3] *F. E. Browder*: Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces. Arch. Ration. Mech. Anal. *24* (1967), 82–90.

[4] *F. E. Browder and W. V. Petryshyn*: Construction of fixed points of nonlinear mappings in Hilbert space. J. Math. Anal. Appl. *20* (1967), 197–228.

[5] *B. Halpern*: Fixed points of nonexpansive maps. Bull. Amer. Math. Soc. *73* (1967), 957–961.

[6] *P. L. Lions*: Approximation de points fixes de contractions. C.R. Acad. Sci. Paris Ser. A–B *284* (1977), A1357–A1359.

[7] *G. Marino and H. K. Xu*: Weak and strong convergence theorems for k-strict pseudo-contractions in Hilbert spaces. J. Math. Anal. Appl. *329* (2007), 336–349.

[8] *G. Marino and H. K. Xu*: A general iterative method for nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. *318* (2006), 43–52.

[9] *A. Moudafi*: Viscosity approximation methods for fixed points problems. J. Math. Anal. Appl. *241* (2000), 46–55.

[10] *T. Suzuki*: Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces. Fixed Point Theory Appl. (2005), 103–123.

[11] *R. Wittmann*: Approximation of fixed points of nonexpansive mappings. Arch. Math. *58* (1992), 486–491.

[12] *H. K. Xu*: An iterative approach to quadratic optimization. J. Optim. Theory Appl. *116* (2003), 659–678.

[13] *H. K. Xu*: Iterative algorithms for nonlinear operators. J. London Math. Soc. *66* (2002), 240–256.

[14] *H. K. Xu*: Another control condition in an iterative method for nonexpansive mappings. Bull. Austral. Math. Soc. *65* (2002), 109–113.

[15] *H. K. Xu*: Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. *298* (2004), 279–291.

[16] *H. Zhou*: Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert space. Nonlinear Analysis *69* (2008), 456–462.

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