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# ON THE DISTRIBUTIVE RADICAL OF AN ARCHIMEDEAN LATTICE-ORDERED GROUP

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*Abstract.* Let $G$ be an Archimedean $\ell$-group. We denote by $G^d$ and $R_D(G)$ the divisible hull of $G$ and the distributive radical of $G$, respectively. In the present note we prove the relation $(R_D(G))^d = R_D(G^d)$. As an application, we show that if $G$ is Archimedean, then it is completely distributive if and only if it can be regularly embedded into a completely distributive vector lattice.

*Keywords*: Archimedean $\ell$-group, divisible hull, distributive radical, complete distributivity

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Throughout the paper, $\ell$-group will be used as a shorthand for lattice-ordered group.

The distributive radical $R_D(G)$ of an $\ell$-group $G$ was investigated by Byrd and Lloyd [2]; cf. also Darnel [3], Section 2.2.

Assume that $G$ is an Archimedean $\ell$-group. The symbol $G^d$ denotes the divisible hull of $G$. In this paper we prove that the relation

$$(R_D(G))^d = R_D(G^d)$$

is valid.

In other words, we prove that the operators $d$ and $R_D$ commute on the class $\mathscr{A}$ of all Archimedean $\ell$-groups defined by

$$d\colon\ G \to G^d,$$
$$R_D\colon\ G \to R_D(G).$$

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As an application, we show that if $G$ is Archimedean, then it is completely distributive if and only if it can be regularly embedded into a completely distributive vector lattice.

## 1. $\ell$-ideals in $G$

The group operation in an $\ell$-group will be written additively; for the terminology, cf. [1] and [2].

It is well-known that each Abelian $\ell$-group can be embedded into a divisible $\ell$-group. Each Archimedean $\ell$-group is Abelian. If $G$ is an Archimedean $\ell$-group, then according to the results of [4], the divisible hull $G^d$ is an Archimedean $\ell$-group and is characterized by the following properties:

(i) for each element $y$ of $G^d$ there exist a positive integer $n$ and an element $x \in G$ such that $ny = x$; in such case we write

$$y = \frac{x}{n}; \tag{1}$$

(ii) for each $z \in G^d$ and each positive integer $m$ there is $t \in G^d$ with $mt = z$;

(iii) $G$ is regularly embedded into $G^\wedge$ (that is, all suprema and infima that exist in $G$ are preserved by the embedding into $G^\wedge$).

This terminology is in accordance with that applied by Sikorski [8] for Boolean algebras.

In what follows we assume that $G$ is an Archimedean $\ell$-group.

If $H$ is an $\ell$-group, then we denote by $J(H)$ the system of all $\ell$-ideals of $H$; this system is partially ordered by the set-theoretic inclusion. In fact, $J(H)$ is a complete lattice. For an Abelian $\ell$-group $H$, the notion of $\ell$-ideal coincides with the notion of convex $\ell$-subgroup of $H$.

The following three lemmas are easy to verify; the proofs will be omitted.

**Lemma 1.1.** *Let $H_1$ be a subset of an $\ell$-group $H$ such that*

(i) *$H_1$ is a subgroup of the group $H$;*

(ii) *if $0 < x \in H_1$, $0 < y \in H$ and $y \leqslant x$, then $y \in H_1$.*

*Then $H_1$ is a convex $\ell$-subgroup of $H$.*

Let $A \in J(G)$. We denote by $f(A)$ the set of all elements $y$ of $G^d$ which can be expressed in the form (1), where $x \in A$ and $n \in \mathbb{N}$.

**Lemma 1.2.** *Let $A \in J(G)$. Then $f(A) \in J(G^d)$.*

Let $B \in J(G^d)$. We put $g(B) = B \cap G$.

**Lemma 1.3.** *Under the notation as above, $g(B)$ belongs to $J(G)$.*

**Lemma 1.4.** *Let $A \in J(G)$ and $B = f(A)$. Then $g(B) = A$.*

P r o o f. a) Let $x \in A$. Then $x \in B$, whence $x \in g(B)$. Thus $A \subseteq g(B)$.

b) Let $z \in (g(B))^+$. Hence there are $n \in \mathbb{N}$ and $x \in A$ such that $nz = x$. In view of $z \geqslant 0$ we have $x \geqslant 0$. Moreover, $0 \leqslant z \leqslant x$. Since $z \in G$, we infer that $z \in A$. Thus $(g(B))^+ \subseteq A$.

c) If $x \in A$, then $-x$ also belongs to $A$. Hence b) yields $(g(B))^- \subseteq A$. The group $g(B)$ is generated by its subset

$$(g(B))^+ \cup (g(B))^-$$

and thus $g(B) \subseteq A$. □

As a consequence of 1.3 and 1.4 we get that $f$ is a one-to-one mapping of the set $J(G)$ onto the set $J(G^d)$.

From the definitions of the mappings $f$ and $g$ we obtain that if $A_1, A_2 \in J(G)$, then

$$A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2).$$

Similarly, if $B_1, B_2 \in J(G^d)$, then

$$B_1 \subseteq B_2 \Rightarrow g(B_1) \subseteq g(B_2).$$

Hence we conclude

**Lemma 1.5.** *The mapping $f$ is an isomorphism of the lattice $J(G)$ onto the lattice $J(G^d)$.*

For each $\ell$-group $H$ and each subset $X$ of $H$ the polar $X^{\delta(H)}$ is defined by

$$X^{\delta(H)} = \{y \in H\colon\ |y| \wedge |x| = 0 \quad \text{for all } x \in X\}.$$

It is well-known that $X^{\delta(H)}$ is a convex $\ell$-subgroup of $H$.

From the definition of polar we easily obtain

**Lemma 1.6.** *Let $H$ be an Abelian $\ell$-group and let $A$ be an $\ell$-ideal of $H$. Then $A^{\delta(H)}$ is the largest element of the set*

$$\{A_1 \in J(H)\colon\ A_1 \wedge A = \{0\}\},$$

*where $A_1 \wedge A$ denotes the infimum of $\{A_1, A\}$ in the lattice $J(H)$.*

**Lemma 1.7.** *Let $A \in J(G)$. Then*

$$f(A^{\delta(G)}) = f(A)^{\delta(G^d)}.$$

Proof. This is a consequence of 1.5 and 1.6. □

**Lemma 1.8.** *Let $A \in J(G)$. The following conditions are equivalent:*

(i) *$A$ is a prime ideal in $G$.*

(ii) *$f(A)$ is a prime ideal in $G^d$.*

Proof. a) Let (i) be valid and let $y_1, y_2 \in G^d$ be such that $y_1 \wedge y_2 = 0$. Hence there are $x_1, x_2 \in G$ and $n_1, n_2 \in \mathbb{N}$ such that

$$y_i = \frac{x_i}{n_i} \quad (i = 1, 2).$$

Then we have $x_i \geqslant 0$ $(i = 1, 2)$. Moreover, from the relation $y_1 \wedge y_2 = 0$ we obtain

$$(n_1 y_1) \wedge (n_2 y_2) = 0,$$

whence $x_1 \wedge x_2 = 0$. The condition (i) yields that either $x_1 \in A$ or $x_2 \in A$. Thus either $y_1 \in f(A)$ or $y_2 \in f(A)$. Therefore (ii) is valid.

b) Conversely, assume that (ii) holds. Let $x_1, x_2 \in G$, $x_1 \wedge x_2 = 0$. Then $x_1, x_2 \in G^d$ and the relation $x_1 \wedge x_2 = 0$ is valid in $G^d$. In view of (ii), either $x_1 \in f(A)$ or $x_2 \in f(A)$. Hence according to 1.4, either $x_1 \in A$ or $x_2 \in A$. Therefore (i) holds. □

## 2. Distributive radical and complete distributivity

We recall the notions concerning higher degrees of distributivity.

Let $\alpha$ and $\beta$ be nonzero cardinals. Further, let $T$ and $S$ be nonempty sets with $\operatorname{card} T \leqslant \alpha$ and $\operatorname{card} S \leqslant \beta$. A lattice $L$ is $(\alpha,\beta)$-*distributive* if the following identities hold in $L$

(d$_1$) $\bigwedge_{t\in T}\bigvee_{s\in S} x_{t,s} = \bigvee_{\varphi\in S^T}\bigwedge_{t\in T} x_{t,\varphi(t)}$,

(d$_2$) $\bigvee_{t\in T}\bigwedge_{s\in S} x_{t,s} = \bigwedge_{\varphi\in S^T}\bigvee_{t\in T} x_{t,\varphi(t)}$

whenever all joins and meets appearing in (d$_1$) or (d$_2$) exist in $L$.

If $L$ is $(\alpha,\beta)$-distributive for any nonzero cardinals $\alpha$ and $\beta$, then $L$ is said to be completely distributive.

**Definition 2.1** (Cf. [2]). Let $H$ be an $\ell$-group. The distributive radical $R_D(H)$ of $H$ is defined to be the set

$$\bigcap A_i^{\delta(H)} \quad (i \in I),$$

where $\{A_i\}_{i\in I}$ is the system of all minimal prime ideals of $H$.

**Lemma 2.2.** $R_D(G^d) = f(R_D(G))$.

Proof. Let $A \in J(G)$. In view of 1.7, 1.8 and 1.5 the following conditions are equivalent:

(i) $A$ is a minimal prime ideal of $G$.
(ii) $f(A)$ is a minimal prime ideal of $G^d$.

By applying 1.5 again we conclude that the assertion of the lemma is valid. □

**Corollary 2.2.1.** $R_D(G) = \{0\}$ *if and only if* $R_D(G^d) = \{0\}$.

Proof. This is a consequence of 2.2 and 1.5. □

**Theorem 2.3.** *Let $G$ be an Archimedean $\ell$-group. Then the relation*

$$(R_D(G))^d = R_D(G^d)$$

*is valid.*

Proof. Denote

$$P = R_D(G), \quad Q = R_D(G^d).$$

We have to verify that $Q = P^d$.

It is obvious that $P$ is an $\ell$-subgroup of $Q$ and that $Q$ is Archimedean. Consider the conditions (i), (ii) and (iii) from the definition of the divisible hull of an $\ell$-group (cf. Section 1). Let us apply these conditions for $P$ and $Q$.

Let $q \in Q$. In view of 2.2, there exist $p \in P$ and $n \in \mathbb{N}$ such that $nq = p$. Hence (i) is valid.

In view of the definition of the divisible hull, $Q$ is divisible (since it is an $\ell$-ideal of a divisible $\ell$-group); thus the condition (ii) is satisfied.

Suppose that $X \subseteq P$, $x \in P$ and that the relation $x = \sup X$ is valid in $P$. Since $P$ is an $\ell$-ideal in $G$, the relation $x = \sup X$ holds in $G$. Then, according to the definition of the divisible hull, $x = \sup X$ is valid in $G^d$ as well. Since $Q$ is an $\ell$-ideal of $G^d$ and $X \subseteq Q$, $x \in Q$, we conclude that the relation $x = \sup X$ holds in $Q$. Analogously we verify the corresponding dual condition. Therefore the condition (iii) is satisfied. □

The following theorem is due to Byrd and Lloyd [2].

**Theorem 2.4.** *Let $H$ be an $\ell$-group. The following conditions are equivalent:*

(i) $R_D(H) = \{0\}$.
(ii) *$H$ is completely distributive.*

**Theorem 2.5.** *The following conditions are equivalent:*

(i) *$G$ is completely distributive.*
(ii) *$G^d$ is completely distributive.*

Proof. This is a consequence of 2.2.1 and 2.4. □

The Dedekind completion of an Archimedean $\ell$-group $H$ will be denoted by $H^\wedge$.

We remark that the implication (i)⇒(ii) is a consequence of the following more general result which is due to Darnel (private communication):

(∗) Let $H$ be an Abelian $\ell$-group. Suppose that $H$ is completely distributive. Then the divisible hull $H^d$ of $H$ is completely distributive as well.

Let $\alpha$ and $\beta$ be cardinals. The question whether the complete distributivity in (∗) can be replaced by $(\alpha, \beta)$-distributivity remains open.

**Lemma 2.6.** (Cf. [4], [6].) *$G^{d\wedge}$ is a complete vector lattice.*

The following result was proved independently in [6] and [7].

**Lemma 2.7.** (Cf. [6], [7].) *$G$ is regularly embedded into $G^{d\wedge}$.*

**Lemma 2.8.** *The following conditions are equivalent:*

(i) $G^d$ *is completely distributive.*

(ii) $G^{d\wedge}$ *is completely distributive.*

P r o o f. This follows from Theorem 2.2 in [5]. □

**Theorem 2.9.** *Let $G$ be an Archimedean $\ell$-group. The following conditions are equivalent:*

(i) $G$ *is completely distributive.*

(ii) $G$ *can be regularly embedded into a completely distributive complete vector lattice.*

P r o o f. The implication (ii) $\Rightarrow$ (i) is obvious. The converse implication is a consequence of 2.5, 2.6, 2.7, 2.3 and 2.8. (Thus the only new ingredient to prove Theorem 2.9 is Theorem 2.3.) □

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