Czechoslovak Mathematical Journal

Jeffrey L. Stuart
The eavesdropping number of a graph

Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 3, 623–636

Persistent URL: http://dml.cz/dmlcz/140505

Czechoslovak Mathematical Journal, 59 (134) (2009), 623–636

THE EAVESDROPPING NUMBER OF A GRAPH

JEFFREY L. STUART, Tacoma

(Received December 31, 2007)

Abstract. Let G be a connected, undirected graph without loops and without multiple edges. For a pair of distinct vertices u and v, a minimum $\{u,v\}$-separating set is a smallest set of edges in G whose removal disconnects u and v. The edge connectivity of G, denoted $\lambda(G)$, is defined to be the minimum cardinality of a minimum $\{u,v\}$-separating set as u and v range over all pairs of distinct vertices in G. We introduce and investigate the eavesdropping number, denoted $\varepsilon(G)$, which is defined to be the maximum cardinality of a minimum $\{u,v\}$-separating set as u and v range over all pairs of distinct vertices in G. Results are presented for regular graphs and maximally locally connected graphs, as well as for a number of common families of graphs.

Keywords: eavesdropping number, edge connectivity, maximally locally connected, cartesian product, vertex disjoint paths

MSC 2010: 05C40

1. EAVESDROPPING

Suppose that a spy agency needs to maintain teams to eavesdrop on wireline communications between secured communications centers. If the agency will only have short notice as to which two centers will be in communication, what is the smallest number of eavesdropping teams that must be kept ready so that the agency is guaranteed to have an adequate number of teams to intercept the communications no matter which centers are involved and no matter which wirelines are employed? We investigate the solution of this problem via graph theory.

2. Notation, basic definitions and useful results

Let $G = G(V, E)$ be an undirected graph with vertex set V, with edge set E, without loops and without multiple edges. Unless otherwise specified, n denotes $|V|$. For each vertex $v \in V$, $d(v)$ will denote the degree of v. If the degree sequence for G is written in ascending order $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_{n-1} \leqslant d_n$, then we let $\delta(G) = d_1$, $\Delta = \Delta(G) = d_n$, and $\Delta'(G) = d_{n-1}$. For a vertex v, S_v denotes the set of all edges that meet v. For adjacent vertices u and v, uv will denote the edge between them. The vertex connectivity of G will be denoted by $k(G)$ and the edge connectivity of G will be denoted by $\lambda(G)$.

For $n \geqslant 1$, let K_n denote the complete graph on n vertices. For $n \geqslant 3$, let C_n denote the cycle on n vertices. For $n \geqslant 1$, let Q_n denote the n-dimensional hypercube on 2^n vertices.

If u and v are distinct vertices of a connected G, then a nonempty set S of edges in G is called a *$\{u, v\}$-separator* if the removal of S leaves u and v in distinct connected components of the resulting graph. The set S is called a *minimum* $\{u, v\}$-separator if it has the smallest cardinality among all $\{u, v\}$-separators. We will denote the size of a minimum $\{u, v\}$-separator by $\lambda(u, v)$.

The properties of $\lambda(u, v)$ have been extensively studied. The following result is well known (see [3, Theorem 5.8]).

Lemma 1. *Let G be a graph. If u and v are distinct vertices in a graph G, then $\lambda(u, v)$ equals the maximum number of edge disjoint paths between u and v.*

The following result due to Mader [8, Theorem 1] is perhaps less well known:

Theorem 2. *Let G be a connected graph. Then there exists a pair of adjacent vertices u and v in G that are joined by at least $\delta(G)$ edge disjoint paths. Consequently, $\lambda(u, v) \geqslant \delta(G)$ for some pair of adjacent vertices u and v in G.*

A connected graph G with at least two vertices is said to have *edge connectivity* $\lambda = \lambda(G)$ if the removal of some set of λ edges disconnects G, but there is no smaller set of edges whose removal disconnects G.

Observe that

$$\begin{aligned}\lambda(G) &= \min\{|S|\colon\ S \text{ is a } \{u, v\}\text{-separator},\ u, v \in V \text{ with } u \neq v\}\\ &= \min\{|S|\colon\ S \text{ is a minimum } \{u, v\}\text{-separator},\ u, v \in V \text{ with } u \neq v\}\\ &= \min\{\lambda(u, v)\colon\ u, v \in V \text{ with } u \neq v\}.\end{aligned}$$

Edge connectivity has been extensively studied, with important results dating to the well-known result of Whitney (1932) [9].

Theorem 3. *Let G be a connected graph. Then $k(G) \leqslant \lambda(G) \leqslant \delta(G)$.*

3. The eavesdropping number $\varepsilon(G)$

For a connected graph G with at least two vertices, the *eavesdropping number* of G, denoted $\varepsilon = \varepsilon(G)$, is defined by

$$\begin{aligned}\varepsilon(G) &= \max\{|S|\colon\ S \text{ is a minimum } \{u,v\}\text{-separator},\ u,v \in V \text{ with } u \neq v\}\\ &= \max\{\lambda(u,v)\colon\ u,v \in V \text{ with } u \neq v\}.\end{aligned}$$

A minimum $\{u,v\}$-separator of maximum cardinality (as u and v range over V with u and v distinct) is called an *eavesdropping set* for G. A vertex v is called a *critical* vertex if there is another vertex u in G such that a minimum $\{u,v\}$-separator is an eavesdropping set. The pair of vertices $\{u,v\}$ is called a *critical pair* if $u \neq v$ and $\lambda(u,v) = \varepsilon(G)$.

In view of Lemma 1, the following two results are immediate.

Lemma 4. *Let G be a connected graph with at least two vertices. Then $\varepsilon(G)$ is the maximum number of edge disjoint paths between a pair of vertices in G where the maximum is taken over all distinct pairs of vertices. Further, if $\lambda(u,v) = \varepsilon(G)$ for some pair of vertices $u,v \in V$, then both u and v are critical vertices for G.*

Lemma 5. *Let G be a connected graph with at least two vertices.Then $\lambda(G) = \varepsilon(G)$ if and only if every minimum $\{u,v\}$-separator (as u and v range over all distinct pairs of vertices in V) has the same cardinality.*

Corollary 6. *Let G be a tree with at least two vertices. Then $k(G) = \lambda(G) = \varepsilon(G) = \delta(G) = 1$.*

Lemma 7. *Let u and v be distinct vertices of a connected graph G. Then each of S_u and S_v is a $\{u,v\}$-separator, and thus $\lambda(u,v) \leqslant \min\{d(u), d(v)\}$.*

Proof. Removing all edges in the set S_u disconnects u from every other vertex in G, hence from v; thus S_v is a $\{u,v\}$-separator with $|S_u| = d(u)$. Similarly, S_v is a $\{u,v\}$-separator with $|S_v| = d(v)$. If S is a $\{u,v\}$-separator of minimal cardinality, the inequality follows. $\square$

Theorem 8. *Let G be a connected graph with at least two vertices. Then*

$$\delta(G) \leqslant \varepsilon(G) \leqslant \Delta'(G).$$

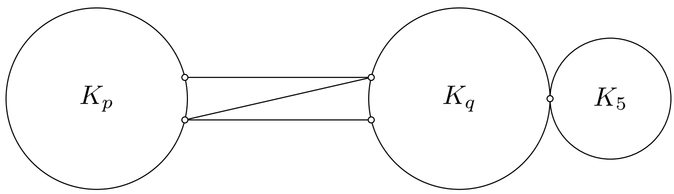
P r o o f. ¿From Theorem 2, $\delta(G) \leqslant \lambda(u, v)$ for some pair of vertices u and v. By definition, $\lambda(u, v) \leqslant \varepsilon(G)$. For the remaining inequality apply Lemma 7 and note that $\min\{d(u), d(v)\} \leqslant \min\{\Delta'(G), \Delta(G)\} = \Delta'(G)$. □

Note that for a tree T on n vertices, $\Delta'(T)$ and $\Delta(T)$ can be large since $\Delta'(T) \leqslant \frac{1}{2}n - 1$, and this bound can be attained by the central vertices of a symmetric double star. Consequently, $\varepsilon(G)$ can be much smaller than $\Delta'(G)$.

The following example shows that the parameters discussed in this paper can all have distinct values.

Example 9. Let p and q be positive integers with $5 \leqslant p < q$. Let G be constructed from K_p, K_q and K_5 by the addition of three edges as indicated below. Then $\delta(G) = 4$, $\Delta'(G) = q + 1$, and $\Delta(G) = q + 3$. Further, $k(G) = 1$, $\lambda(G) = 3$ and $\varepsilon(G) = q$. Thus

$$k(G) < \lambda(G) < \delta(G) < \varepsilon(G) < \Delta'(G) < \Delta(G).$$



Algorithms exist to find the eavesdropping number for a graph and an eavesdropping set in polynomial time.

Theorem 10. *Let G be a connected graph on n vertices. Then $\varepsilon(G)$ can be computed and an eavesdropping set can be found in $O(n^6)$ operations. If G has m edges, then $\varepsilon(G)$ can be computed and an eavesdropping set can be found in $O(n^2m^2)$ operations.*

P r o o f. For each vertex v in G, the Max-Flow Min-Cut Algorithm [3, Algorithm 5.1] can be applied to G to find $\lambda(u, v)$ for every vertex u in G in $O(nm^2)$ operations. Since there are n choices for v, it follows that $\varepsilon(G)$ can be computed and an eavesdropping set can be found in $O(n^2m^2)$ operations. Since $m \leqslant \binom{n}{2}$, m is bounded by $O(n^2)$, and thus, $O(n^2m^2) \leqslant O(n^6)$. □

4. Maximally locally connected graphs

Let G be a connected graph with at least two vertices. The *distance between two vertices in* G is the length of the shortest path connecting them. The *diameter of* G, denoted $\operatorname{diam}(G)$, is the maximum distance between distinct vertices in G. When $G = K_1$, $\operatorname{diam}(G) = 0$. When G is not connected, $\operatorname{diam}(G) = \infty$.

The graph G is called *maximally locally connected* if $\lambda(u, v) = \min\{d(u), d(v)\}$ for all distinct $u, v \in V(G)$. In [4], Fricke, Oellermann and Swart showed the following result:

Theorem 11. *If G is a graph with $\operatorname{diam}(G) \leqslant 2$, then $\lambda(u, v) = \min\{d(u), d(v)\}$ for all pairs of distinct vertices u and v.*

In [7], Hellwig and Volkmann generalized a result on p-partite graphs from [4] to obtain

Theorem 12. *Let p be a positive integer with $p \geqslant 2$. Let G be a graph that does not contain a complete subgraph of order $p + 1$. If*

$$|V(G)| \leqslant 2\left\lfloor \frac{p\delta(G)}{p-1} \right\rfloor - 1,$$

then G is maximally locally connected.

In the same paper, Hellwig and Volkmann proved

Theorem 13. *Let G be a bipartite graph with bipartition $V = V' \cup V''$ with $V' \cap V'' = \emptyset$. Let $n = |V|$, and suppose that $\delta(G) \geqslant 2$. If $d(x) + d(y) \geqslant \frac{1}{2}(n + 1)$ whenever $x, y \in V'$ with $x \neq y$ and whenever $x, y \in V''$ with $x \neq y$, then G is maximally locally connected.*

The following result connects maximally locally connectedness to the eavesdropping number.

Theorem 14. *Let G be a maximally locally connected graph. Then $\varepsilon(G) = \Delta'(G)$.*

P r o o f. If G is maximally locally connected, then

$$\begin{aligned}\varepsilon(G) &= \max\{\lambda(u, v)\colon\ u, v \in V(G) \text{ with } u \neq v\}\\ &= \max\{\min\{d(u), d(v)\}\colon\ u, v \in V(G) \text{ with } u \neq v\}.\end{aligned}$$

Choose u and v with $u \neq v$ and $\{d(u), d(v)\} = \{\Delta'(G), \Delta(G)\}$. This is clearly the maximizer. □

The next result is a consequence of Theorem 8.

Theorem 15. *Let G be r-regular for some positive integer r. Then $\varepsilon(G) = r$.*

P r o o f. Since each connected component of G is r-regular, apply the preceding theorem to any connected component, and note that $\delta(G) = r = \Delta'(G)$. □

The following three results are consequences of the preceding theorem since in each case, it is known that $\lambda(G) = \Delta'(G)$.

Example 16. Let $n \geqslant 2$. Then

$$k(K_n) = \lambda(K_n) = \varepsilon(K_n) = \delta(K_n) = \Delta'(K_n) = \Delta(K_n) = n - 1.$$

Example 17. Let $n \geqslant 3$. Then

$$k(C_n) = \lambda(C_n) = \varepsilon(C_n) = \delta(C_n) = \Delta'(C_n) = \Delta(C_n) = 2.$$

Example 18. Let $n \geqslant 2$. Then

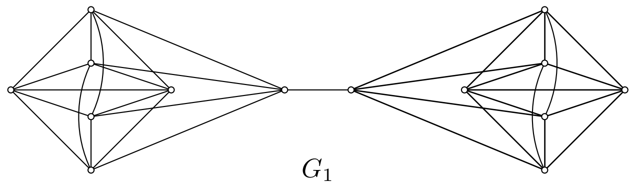
$$k(Q_n) = \lambda(Q_n) = \varepsilon(Q_n) = \delta(Q_n) = \Delta'(Q_n) = \Delta(Q_n) = n.$$

More generally, there exist families of regular graphs G for which $k(G)$ and $\lambda(G)$ are small, but for which $\varepsilon(G)$ is arbitrarily large.

Example 19. Let m be a positive integer. The following construction produces an r-regular graph G_m on $8m + 6$ vertices with $\varepsilon(G) = r = 4m + 1$ but $k(G_m) = \lambda(G_m) = 1$.

To build H_m proceed as follows. Start with two distinct copies of K_{2m+2}, call them L_1 and L_2, and a singleton vertex w. Select two distinct vertices, u_1 and v_1 in L_1, and two distinct vertices u_2 and v_2 in L_2. Label the vertices in $V(L_1) \setminus \{u_1, v_1\}$ as $a_1, \ldots, a_{2m}$. Label the vertices in $V(L_2) \setminus \{u_2, v_2\}$ as $b_1, \ldots, b_{2m}$. Identify u_1 with u_2 (call the vertex u), v_1 with v_2 (call the vertex v), and the edge u_1v_1 with the edge u_2v_2 (call the edge uv). Create an edge between each pair of vertices a_i and b_j exactly when $i \neq j$. Join every vertex a_i and every vertex b_j to w. It is easy to verify that H_m contains $4m + 3$ vertices, that every vertex except w has degree $4m + 1$, and that w has degree $4m$.

Now G_m is constructed from two copies of H_m by connecting w in each copy of H_m with an edge. Deleting either copy of w or the edge that connects them will disconnect G_m. Then G_m has $8m + 6$ vertices and is r-regular with $r = 4m + 1$.



G_1

Note that if there were an r-regular graph G on $8m+6$ vertices with $r \geqslant 4m+3$, then by a result due to Chartrand and Harary [2], $k(G) \geqslant 2$. Thus the graph G_m has almost the highest degree that a regular graph with $k(G) = 1$ can have.

6. Cartesian products

If G_1 and G_2 are graphs, then the *Cartesian product* of G_1 and G_2, denoted $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$, and edge set $E = \{(u_1, v_1)(u_2, v_2)$: either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$, or $v_1 = v_2$ and $u_1u_2 \in E(G_1)\}$. For $i = 1, 2$ and for $v \in G_i$, let $d^{(i)}(v)$ denote the degree of the vertex v in G_i. Cartesian products of graphs have been extensively studied. For example, the edge-connectivity of Cartesian products has been studied in [10]. We state without proof a simple result that will be useful in the rest of this section.

Lemma 20. *Let G_1 and G_2 be graphs. For each $u \in V(G_1)$ and each $v \in V(G_2)$, the degree of the vertex (u, v) in $G_1 \times G_2$ is $d^{(1)}(u) + d^{(2)}(v)$. Thus, $\Delta(G_1 \times G_2) = \Delta(G_1) + \Delta(G_2)$, and*

$$\Delta'(G_1 \times G_2) = \max\{\Delta'(G_1) + \Delta(G_2), \Delta(G_1) + \Delta'(G_2)\}.$$

Theorem 21. *Let G_1 and G_2 be graphs each of which contains at least one edge. Then $\varepsilon(G_1 \times G_2)$ satisfies:*

$$\max\{\varepsilon(G_1) + \Delta(G_2), \varepsilon(G_2) + \Delta(G_1)\} \leqslant \varepsilon(G_1 \times G_2)$$

and

$$\varepsilon(G_1 \times G_2) \leqslant \max\{\Delta'(G_1) + \Delta(G_2), \Delta(G_1) + \Delta'(G_2)\}.$$

P r o o f. The upper bound follows from Theorem 8 and the preceding lemma. Suppose that a and b are a critical pair of vertices for the graph G_1. Since G_1 contains an edge, there must be a path from a to b in G_1. Suppose that w is a vertex of degree $\Delta(G_2)$ in G_2, and that its neighbors in G_2 are $v_1, v_2, \ldots, v_{\Delta_2}$. Observe that for $1 \leqslant j \leqslant \Delta(G_2)$, $(a,w)(a,v_j)$ and $(b,w)(b,v_j)$ are distinct edges in $G_1 \times G_2$. For each $v \in V(G_2)$, let $G_1 \times v$ denote the subgraph of $G_1 \times G_2$ induced by $V(G_1) \times \{v\}$. Then for $1 \leqslant j \leqslant \Delta(G_2)$ there is a path P_j in $G_1 \times v_j$ from a to b. Thus there are $\Delta(G_2)$ paths in $G_1 \times G_2$ from (a,w) to (b,w) of the form $(a,w)(a,v_j), P_j, (b,w)(b,v_j)$; and further, these paths are edge disjoint and contain no edges from $G_1 \times w$. Finally, since a and b are critical for G_1, there exist $\varepsilon(G_1)$ paths from (a,w) to (b,w) that lie entirely inside $G_1 \times w$. Thus there are at least $\varepsilon(G_1) + \Delta(G_2)$ edge disjoint paths between two vertices in $G_1 \times G_2$. Interchanging the roles of G_1 and G_2, the lower bound inequality follows. □

The next result, which is a corollary of Theorem 15 as well as of the previous theorem, shows that both inequalities in the previous result are sharp.

Corollary 22. *Let r_1 and r_2 be positive integers. For $i = 1, 2$, let G_i be an r_i-regular graph. Then $G_1 \times G_2$ is an $(r_1 + r_2)$-regular graph, and further,*

$$\varepsilon(G_1 \times G_2) = \varepsilon(G_1) + \varepsilon(G_2).$$

7. Edge cutsets, vertex cutsets and the eavesdropping number

Theorem 23. *Let G be a connected graph with at least two vertices. Suppose that F is an edge cutset for G with $|F| = h$, and suppose that the deletion of the edges in F results in disjoint, nonempty graphs G_1 and G_2. Then*

$$\varepsilon(G) \leqslant \max\Big\{h, \varepsilon(G_1) + \Big\lfloor \frac{h}{2} \Big\rfloor, \varepsilon(G_2) + \Big\lfloor \frac{h}{2} \Big\rfloor\Big\}.$$

Proof. Suppose that a and b are any pair of distinct vertices in the graph G_1. Then there are at most $\varepsilon(G_1)$ edge disjoint paths between a and b in G_1. Since there are h edges connecting G_1 to G_2, there are at most an additional $\lfloor \frac{1}{2}h \rfloor$ edge disjoint paths between a and b in G. In particular, if a and b are a critical pair of vertices for G, then it follows that $\varepsilon(G) \leqslant \varepsilon(G_1) + \lfloor \frac{1}{2}h \rfloor$. Similarly, every pair of distinct vertices contained in G_2 is joined by at most $\varepsilon(G_2) + \lfloor \frac{1}{2}h \rfloor$ edges, and hence if G_2 contains a critical pair of vertices for G, then $\varepsilon(G) \leqslant \varepsilon(G_2) + \lfloor \frac{1}{2}h \rfloor$. Further, if a lies in G_1 and b lies in G_2, then there are at most h edge disjoint paths between a and b, and if a and b are a critical pair for G, then $\varepsilon(G) \leqslant h$. Finally, at least one of the following holds: G_1 contains a critical pair for G, G_2 contains a critical pair for G, or there is a critical pair for G with one vertex in G_1 and one vertex in G_2. □

Corollary 24. *Let G be a connected graph with at least three vertices. Suppose that G has a pendant vertex v. Then $\varepsilon(G) = \varepsilon(G - v)$.*

Theorem 25. *Let G be a connected graph with at least two vertices. Suppose that $W = \{w_1, w_2, \ldots, w_k\}$ is a vertex cutset for G. Let p be*

$$p = \sum_{i=1}^{k} \left\lfloor \frac{d(w_i)}{2} \right\rfloor.$$

Suppose that the deletion of the vertices in W results in disjoint, nonempty graphs G_1 and G_2. Then

$$\varepsilon(G) \leqslant \max\Big\{p, \varepsilon(G_1) + \left\lfloor \frac{p}{2} \right\rfloor, \varepsilon(G_2) + \left\lfloor \frac{p}{2} \right\rfloor\Big\}.$$

Proof. Observe that the maximum number of edge disjoint paths passing through a vertex v is $\left\lfloor \frac{1}{2} d(v) \right\rfloor$. Consequently, the maximum possible number of edge disjoint paths in G between vertices in G_1 and vertices in G_2 is p, the maximum possible number of edge disjoint paths passing through vertices in W.

Suppose that a and b are any pair of distinct vertices in the graph G_1. Then there are at most $\varepsilon(G_1)$ edge disjoint paths between a and b in G_1. Since there are at most p edge disjoint paths connecting G_1 to G_2 in G, there are at most an additional $\left\lfloor \frac{1}{2} p \right\rfloor$ edge disjoint paths between a and b in G. In particular, if a and b are a critical pair of vertices for G, then it follows that $\varepsilon(G) \leqslant \varepsilon(G_1) + \left\lfloor \frac{1}{2} p \right\rfloor$. Similarly, every pair of distinct vertices contained in G_2 is joined by at most $\varepsilon(G_2) + \left\lfloor \frac{1}{2} p \right\rfloor$ edges, and hence if G_2 contains a critical pair of vertices for G, then $\varepsilon(G) \leqslant \varepsilon(G_2) + \left\lfloor \frac{1}{2} p \right\rfloor$. Further, if a lies in G_1 and b lies in G_2, then there are at most p edge disjoint paths between a and b, and if a and b are a critical pair for G, then $\varepsilon(G) \leqslant p$. Finally, at least one of the following holds: G_1 contains a critical pair for G, G_2 contains a critical pair for G, or there is a critical pair for G with one vertex in G_1 and one vertex in G_2. □

8. Further results

Proposition 26. *Let $n_1, n_2, \ldots, n_k$ be positive integers for some positive integer $k \geqslant 2$. Let $K(n_1, n_2, \ldots, n_k)$ denote the complete, k-partite graph with partition sets of size $n_1, n_2, \ldots, n_k$.*

(i) *If $1 = n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$, then*

$$\varepsilon(K(n_1, n_2, \ldots, n_k)) = 1 + \sum_{i=3}^{k} n_i,$$

where the summation is zero when $k = 2$.

(ii) *If* $2 \leqslant n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$, *then*

$$\varepsilon(K(n_1, n_2, \ldots, n_k)) = \sum_{i=2}^{k} n_i.$$

P r o o f. Part (ii). For each j, let V_j denote the vertex partition subset of size n_j.

C a s e 1. Choose $j \in \{1, 2, \ldots, k\}$. Choose $x, y \in V_j$ with $x \neq y$. Then for each $v \in V - V_j$, there is a path $P_v = \{(x, v), (v, y)\}$, and if $w \in V - V_j$ with $w \neq v$, then P_v and P_w are edge disjoint. Thus $\lambda(x, y) \geqslant |V - V_j| = \sum\limits_{i \neq j} n_i = d(x) = d(y)$. Since $\lambda(x, y) \leqslant d(y)$ by Lemma 7, $\lambda(x, y) = d(y)$.

C a s e 2. Choose $j, l \in \{1, 2, \ldots, k\}$ with $j < l$. Then choose $x \in V_j$ and $y \in V_l$. Note that $n_j \leqslant n_l$ implies $d_x \geqslant d_y$, and hence, by Lemma 7, $\lambda(x, y) \leqslant d_y$. Since $\varepsilon(G)$ is the maximum of $\lambda(x, y)$ over all pairs of distinct vertices, it suffices to maximize d_y over all choices of $y \in V$. This maximum occurs when n_j is minimized.

Part (i). The proof is similar to that of Part (ii). If $x \in V_1$ and $y \in V_2$, then $\lambda(x, y) = |V - (V_1 \cup V_2)| + 1 = |V| - n_2$. The proof that this is the maximum is similar to the proof of Part (i). □

For a disconnected graph G, $\varepsilon(G)$ and the several of the other parameters used in this paper are determined from the components. Consequently, the stipulation that G is connected can be removed when discussing the eavesdropping number.

Proposition 27. *Let G be a disconnected graph with connected components $G_1, G_2, \ldots, G_p$ for some $p \geqslant 2$. Then $k(G) = \lambda(G) = 0$, $\delta(G) = \min\limits_j \delta(G_j)$, $\Delta(G) = \max\limits_j \Delta(G_j)$, $\varepsilon(G) = \max\limits_j \varepsilon(G_j)$, and $\Delta'(G) \geqslant \max\limits_j \Delta'(G_j)$.*

Note that the inequality in Proposition 27 can be strict. For example, if G is the disjoint union of the complete bipartite graphs $K_{1,5}$ and $K_{1,6}$, then $\Delta'(G) = 5$ but $\max\limits_j \Delta'(G_j) = 1$. Also note that $\varepsilon(G) = 0$ exactly when G contains no edges.

Employing Theorem 8 yields

Theorem 28. *Let G be a disconnected graph with connected components $G_1, G_2, \ldots, G_p$ for some $p \geqslant 2$. Then*

$$\max_j \delta(G_j) \leqslant \varepsilon(G) \leqslant \max_j \Delta'(G_j).$$

It is well-known that a connected graph G contains a cycle if and only if $\lambda(G) \geqslant 2$. Consequently, we have

Lemma 29. *Let G be a graph with at least one edge. Then G is either a tree or a forest if and only if $\varepsilon(G) = 1$. Also, G contains a cycle if and only if $\varepsilon(G) \geqslant 2$.*

Next, we examine the impact of vertex and edge deletions.

Example 30. Let $m \geqslant 2$ be a positive integer. If G is obtained by joining a single vertex in each of two copies K_m to a singleton vertex w, then $\varepsilon(G) = \varepsilon(G - \{w\}) = m - 1$. Let P be the path on 5 vertices, and label the vertices consecutively from one end to the other as $v_1, v_2, \ldots, v_5$. Let H be obtained by taking m copies of P, and identifying all copies of v_1 (call the common vertex u), identifying all copies of v_3 (call the common vertex w), and identifying all copies of v_5 (call the common vertex v). Then there are m edge disjoint paths from u to v, so $\varepsilon(H) = m$ but $\varepsilon(H - \{w\}) = 1$.

The next result follows from the definition of $\varepsilon(G)$, Lemma 4, and the observations in the previous example.

Lemma 31. *Let G be a graph with at least two vertices and at least one edge. Then for each vertex v, $\varepsilon(G - \{v\}) \leqslant \varepsilon(G)$, and for each edge e, $\varepsilon(G) - 1 \leqslant \varepsilon(G - e) \leqslant \varepsilon(G)$.*

Turning to subgraphs, we have

Lemma 32. *Let G be a graph with at least two vertices. Let H be a subgraph of G with at least two vertices. Let u and v be distinct vertices in H. Then $\lambda_H(u, v) \leqslant \lambda_G(u, v)$.*

In the preceding result, it seems reasonable that every minimum $\{u, v\}$-separator in G should contain a minimum $\{u, v\}$-separator in H, and that every minimum $\{u, v\}$-separator in H should be contained in a minimum $\{u, v\}$-separator in G, but neither of these claims is known to be true. Nonetheless, the preceding lemma does yield

Proposition 33. *Let G be a graph, and let H be a subgraph of G. Then $\varepsilon(G) \geqslant \varepsilon(H)$.*

Corollary 34. *Let G be a graph. If G contains K_m as a subgraph for some $m \geqslant 2$, then $\varepsilon(G) \geqslant m - 1$.*

The following result says that when G contains a cycle, any pendent trees can be pruned from G without changing the eavesdropping number.

Theorem 35. *Suppose that G is a graph that contains a cycle. Suppose that the subgraph T of G is a pendent tree that meets the rest of G at a vertex u. Let the subgraph H of G be induced by the vertex set $(V(G) - V(T)) \cup \{u\}$. Then $\varepsilon(G) = \varepsilon(H)$.*

Proof. Since G contains a cycle, $\lambda(G) \geqslant 2$, and hence no vertex in $V(T) - \{u\}$ can be critical. If $\{v, w\}$ is a critical pair of vertices for G, then none of the vertices on any path from v to w can include edges in T. Thus $\lambda_G(v, w) = \lambda_H(v, w)$. □

The final two results concern contractions. The first addresses vertex contraction, the second, edge contraction.

Theorem 36. *Suppose that G has a degree two vertex a with nonadjacent neighbors b and c. Let H be the graph obtained from G be deleting the vertex a and replacing the edges ab and ac with a new edge bc. Then $\varepsilon(G) = \varepsilon(H)$.*

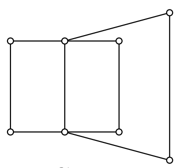
Proof. Since a, b and c do not lie on a triangle, H does not contain multiple edges. For any $w \in V - \{a, b, c\}$, $\lambda(w, a) \leqslant \lambda(w, b)$ and $\lambda(w, a) \leqslant \lambda(w, c)$. Thus there is a critical pair $\{u, v\}$ of vertices for G that does not contain a. Let S be a minimum $\{u, v\}$-separator in G. Then S contains at most one of the edges ab and ac. If S contains neither edge, then S is a $\{u, v\}$-separator in H, and it must be minimal for H. If S contains one of the edges ab and ac, without loss ab, then $(S - \{ab\}) \cup \{bc\}$ is a minimal $\{u, v\}$-separator in H. □

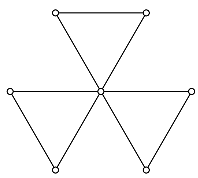
Theorem 37. *Suppose that G contains two adjacent vertices a and b that do not lie on a triangle. Let $e = ab$. Let H be the graph obtained by contracting the edge e (that is, by deleting e and identifying a and b). Then $\varepsilon(H) \leqslant \varepsilon(G)$. Further, $\{a, b\}$ is the unique critical pair for G if and only if $\varepsilon(H) < \varepsilon(G)$.*

Proof. Since e is not an edge of a triangle, contracting e does not produce multiple edges. Suppose that $\{u, v\}$ is a pair of distinct vertices in G. Then there are $\lambda_G(u, v)$ edge disjoint paths from u to v in G. There cannot be more paths from u to v in H than there are in G. Thus $\lambda_G(u, v) \geqslant \lambda_H(u, v)$, and hence, $\varepsilon(G) \geqslant \varepsilon(H)$. If none of the $\lambda_G(u, v)$ edge disjoint paths in G from u to v use e, then $\lambda_G(u, v) = \lambda_H(u, v)$. If from every set of $\lambda_G(u, v)$ edge disjoint paths in G from u to v, some path uses e, and if $\{u, v\} \neq \{a, b\}$, identifying a and b and deleting e does not eliminate that path in H, and thus $\lambda_G(u, v) = \lambda_H(u, v)$. If $\{u, v\} \neq \{a, b\}$ and $\{u, v\}$ is a critical pair for G, then it is also a critical pair for H, and hence, $\varepsilon(G) = \varepsilon(H)$. Suppose that $\{a, b\}$ is the unique critical pair for G. If $\{u, v\}$ is a critical pair for H, then $\{u, v\} \neq \{a, b\}$, and hence $\varepsilon(H) = \lambda_H(u, v) = \lambda_G(u, v) < \lambda_G(a, b) = \varepsilon(G)$. □

If $\{a, b\}$ in the preceding theorem is the unique critical pair for G, then e is in every eavesdropping set for G. Consequently, we would expect that H has a lower eavesdropping number. Alternatively, if $\{\Delta(G), \Delta'(G)\} = \{d_a, d_b\}$, contracting on e means $\Delta(H) = \Delta(G) + \Delta'(G) - 2$, which is large, but $\Delta'(H)$ could be much smaller than $\Delta'(G)$. Since Δ' is an upper bound for ε, this suggests that $\varepsilon(H)$ can be much smaller than $\varepsilon(G)$. The following example confirms this.

Example 38. Let m be a positive integer with $m \geqslant 2$. Let G_m be constructed from m copies of the four-cycle C_4 as follows. For each copy of C_4, label a pair of adjacent vertices as a and b, and label the edge between a and b as $e = ab$. Join all m four-cycles by identifying the vertices a, the vertices b, and the edges e. Then $\varepsilon(G_m) = m + 1$, and $\{a, b\}$ is the unique critical pair for G_m since every other vertex has degree 2. If the edge e is contracted to obtain H_m, then H_m consists of m triangles joined at a common vertex, and hence $\varepsilon(H) = 2$. Note that $\Delta(G_m) = \Delta'(G_m) = m+1$; $\Delta(H_m) = 2m$, which is large, but $\Delta'(H_m) = 2$, which can be much smaller than $m + 1$.


G_3


H_3

Corollary 39. *Suppose that G is a connected graph with more than two vertices. Suppose that $e = ab$ is a cutedge for G. Let the graph H be obtained by contracting the edge e. Then $\varepsilon(G) = \varepsilon(H)$.*

In closing, we mention several natural questions:

1. For a fixed value of $|V|$ or of $|V|$ and $|E|$, what values of $\varepsilon(G)$ can occur?
2. For a fixed values of $|V|$ and $\Delta(G)$ (or $\Delta'(G)$), what values of $\varepsilon(G)$ can occur?
3. What conditions on G imply that $\varepsilon(G) = \delta(G)$? That $\varepsilon(G) = \Delta'(G)$?
4. Is there a relationship between $\varepsilon(G)$ and the diameter of G?
5. Is restricting G to be bipartite useful?
6. What conditions on G imply that some eavesdropping set is actually the set of all edges incident at a critical vertex? That all eavesdropping sets are of this type?
7. What if multiple edges or loops are allowed in G?
8. What are the analogous results for directed graphs?

Each of these questions is analogous to questions about edge connectivity that have already been investigated. For example, how large a (minimally) n-edge connected graph can be has been studied in [1], [6], [8]. An extensive survey of results relating edge-connectivity, super edge-connectivity, minimum degree, clique number, and maximally locally connectedness can be found in [5].

References

[1] *C. Cai*: The maximal size of graphs with at most k edge-disjoint paths connecting any two adjacent vertices. Discrete Math. *85* (1990), 43–52.

[2] *G. Chartrand, F. Harary*: Graphs with prescribed connectivities. Theory of Graphs, Proc. Colloq., Tihany, Hungary, 1966. Académiai Kiadói, Budapest, 1968, pp. 61–63.

[3] *G. Chartrand, O. R. Oellermann*: Applied and Algorithmic Graph Theory. McGraw-Hill, New York, 1993.

[4] *G. Fricke, O. R. Oellermann, H. C. Swart*: The edge-connectivity, average edge-connectivity and degree conditions. Manuscript. 2000.

[5] *A. Hellwig*: Maximally connected graphs and digraphs. Doctoral dissertation. Rheinisch-Westfälishchen Technische Hochschule, Aachen, 2005.

[6] *A. Hellwig, L. Volkmann*: Maximally local-edge-connected graphs and digraphs. Ars. Comb. *72* (2004), 295–306.

[7] *A. Huck, H. Okamura*: Counterexamples to a conjecture of Mader about cycles through specified vertices in n-edge-connected graphs. Graphs Comb. *8* (1992), 253–258.

[8] *W. Mader*: Existenz gewisser Konfigurationen in n-gesättigten Graphen und in Graphen genügend grosser Kantendichte. Math. Ann. *194* (1971), 295–312. (In German.)

[9] *H. Whitney*: Congruent graphs and the connectivity of graphs. Am. J. Math. *54* (1932), 150–168.

[10] *J.-M. Xu, C. Yang*: Connectivity of Cartesian product graphs. Discrete Math. *306* (2006), 159–165.

Author's address: J. L. Stuart, Pacific Lutheran University, Tacoma, WA 98447, U.S.A., e-mail: jeffrey.stuart@plu.edu.