Czechoslovak Mathematical Journal

R. C. Smith

Proper uniform algebras are flat

Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 2, 285–286

Persistent URL: http://dml.cz/dmlcz/140479

Czechoslovak Mathematical Journal, 59 (134) (2009), 285–286

PROPER UNIFORM ALGEBRAS ARE FLAT

R. C. Smith, Mississippi State

(Received March 2, 2005)

Abstract. In this brief note, we see that if A is a proper uniform algebra on a compact Hausdorff space X, then A is flat.

Keywords: proper uniform algebra, Hausdorff space

MSC 2010: 46J10, 46B20, 46E15

A Banach space E is *flat* if there is a curve of length two in the unit sphere of E with antipodal endpoints; i.e., E is flat if there is a continuous function $\gamma\colon [0,1] \to E$ such that $\|\gamma(t)\| = 1$ for each $t \in [0,1]$, $\sup\Big\{\sum\limits_{n=1}^{m} \|\gamma(t_n) - \gamma(t_{n-1})\|\colon 0 = t_0 < t_1 < \ldots < t_m = 1\Big\} = 2$ and $\gamma(0) = -\gamma(1)$. Schäffer's monograph [4] contains a wealth of information about flat spaces and related topics. In [4], the scalar field is always the real numbers, but in this paper we are more interested the complex case. It's clear that if E is a complex Banach space with a flat real-linear subspace, then E itself is also flat and so this will not cause any problems. For a compact Hausdorff space X, let $C(X)$ denote the complex-valued continuous functions on X and let $C(X,\mathbb{R})$ denote the real-valued continuous functions on X. Equip both with the supremum norm $\|f\|_\infty = \sup\{|f(x)|\colon x \in X\}$. The compact Hausdorff space X is *scattered* if every nonempty subset of X contains a relatively isolated point. From work of Niykos and Schäffer [2] (also see [4]) we have $C(X,\mathbb{R})$ (and thus $C(X)$) is flat whenever X is not scattered. A subalgebra A of $C(X)$ is a *uniform algebra* on X if A is closed, contains the constant functions, and separates the points of X. If A is a uniform algebra on X and $A \neq C(X)$, A is said to be a *proper* uniform algebra on X. An old result of Rudin [5] asserts that if X is a compact Hausdorff space and there exists a proper unifrom algebra on X then X is not scattered. In view of these results, it seems natural to determine whether every proper uniform algebra is flat. As a final preliminary, for $K \subset \mathbb{C}$ compact, let $P(K)$ denote the closure of the polynomials (in one complex variable) in $C(K)$.

Theorem. *Every proper uniform algebra is flat.*

P r o o f. Let A be a proper uniform algebra on a compact Hausdorff space X. Let $f \in A$. For any polynomial p, $p \circ f \in A$ and $\|p \circ f\|_\infty = \sup\{|p(z)|\colon\ z \in f(X)\}$. Since A is closed in $C(X)$, $P(f(X))$ is isometric to a subalgebra of A. If $f(X)$ is countable, then $f(X)$ has no interior and the complement of $f(X)$ is connected. By Mergelyan's Theorem, $P(f(X)) = C(f(X))$ and so, the complex conjugate of f, $\overline{f} \in A$. Since $A \neq C(X)$, it follows from the Stone-Weierstrass Theorem that there is some $g \in A$ such that $\overline{g} \notin A$. Thus $K = g(X)$ is an uncountable compact metric space. By a theorem of Pełczyński [3], $P(K)$ contains a subspace isometric to $C([0,1])$. As mentioned above, $C([0,1])$ is flat hence $P(K)$ is flat. Since A contains a subspace isometric to $P(K)$, A is flat as well. □

Remarks. In the proof above, the argument that $g(X)$ is uncountable for some $g \in A$ is essentially Rudin's argument in [5]. The full generality of Pełczyński's result from [3] is not needed, we only need this result for $P(K)$, where $K \subset \mathbb{C}$ is compact and uncountable. One can show that the outer boundary of K contains an uncountable compact set S of harmonic measure zero so S is a peak interpolation set for $P(K)$. Applying the linear extention theorem of Michael and Pełczyński [1] yields a subpace of $P(K)$ isometric to $C(S)$. Since this argument does not seem to lead to anything beyond what we have above, the details are omitted.

References

[1] *E. Michael and A. Pełczyński*: A linear extension theorem. Illinois J. Math. *11* (1967), 563–579.

[2] *P. Nyikos and J. J. Schäffer*: Flat spaces of continuous functions. Studia Math. *42* (1972), 221–229.

[3] *A. Pełczyński*: Some linear topological properties of separable function algebras. Proc. Amer. Math. Soc. *18* (1967), 652–660.

[4] *J. J. Schäffer*: Geometry of spheres in normed spaces, Lecture Notes in Pure and Applied Mathematics, No. 20. Marcel-Dekker, Inc., New York-Basel, 1976.

[5] *W. Rudin*: Continuous functions on compact spaces without perfect subsets. Proc. Amer. Math. Soc. *8* (1957), 39–42.

Author's address: R. C. Smith, P.O. Drawer MA, Mississippi State, Mississippi 39762 USA, e-mail: `smith@math.msstate.edu`.