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A GENERALIZATION OF BAER'S LEMMA

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Abstract. There is a classical result known as Baer's Lemma that states that an R-module E is injective if it is injective for R. This means that if a map from a submodule of R, that is, from a left ideal L of R to E can always be extended to R, then a map to E from a submodule A of any R-module B can be extended to B; in other words, E is injective. In this paper, we generalize this result to the category q_ω consisting of the representations of an infinite line quiver. This generalization of Baer's Lemma is useful in proving that torsion free covers exist for q_ω.

Keywords: Baer's Lemma, injective, representations of quivers, torsion free covers

MSC 2010: 13D30, 18G05

1. Introduction

One of the most fruitful concepts in the theory of modules and homological algebra is that of an injective object. Recall that a module is defined in the same way as an abstract vector space except that the scalars are permitted to be elements of a ring instead of a field. All rings considered here have a multiplicative identity. They are associative, but not necessarily commutative. Henceforth, the ring R is considered fixed, and modules are unital left R-modules.

By a map φ from one module to another we mean a linear homomorphism, that is, $\varphi(x+y)=\varphi(x)+\varphi(y)$ and $\varphi(cx)=c\varphi(x)$ when c is a scalar. The standard definition of an injective R-module is that an R-module E is injective if any map from an R-module A into E can be extended to a map from B into E whenever B is an R-module containing A. This condition can also be stated by saying that the following diagram commutes:

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle\varphi}\downarrow & \swarrow{\scriptstyle\phi} & \\ E & & \end{array}$$

By its very nature, the criterion in the definition of an injective module can be exhaustive to verify since it requires a verification for *all* modules B and submodules A. However, Reinhold Baer [1] succeeded in reducing the criterion to a special case that is much more manageable. The result is widely known as Baer's Lemma [1].

Baer's Lemma. An R-module E is injective if (and only if) every map from a left ideal L of R to E can be extended to R.

We sometimes refer to Baer's Lemma by saying that an R-module E is injective if it is injective for R. This should be interpreted to mean that E is injective if every map from any R-submodule (that is, any left ideal) of R into E can be extended to R itself.

2. The category q_ω

Define q_ω to be the category of representations of the quiver

$$\bullet \to \bullet \to \bullet \to \dots$$

Specifically, objects in q_ω have the form

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$$

where for all i, it is understood that A_i is an R-module and $f_i : A_i \to A_{i+1}$ is a map in R-Mod. A sequence $(\varphi_1, \varphi_2, \varphi_3, \dots)$ of maps in R-Mod is a map in the category q_ω from the object

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$$

to the object

$$B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} B_3 \xrightarrow{g_3} \dots$$

provided that $\varphi_i : A_i \to B_i$ is a map in R-Mod for which the equations $\varphi_{i+1} \circ f_i = g_i \circ \varphi_i$ are satisfied for each i. In other words, the following diagram commutes:

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots \\ \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_2} & & \downarrow{\scriptstyle\phi_3} & & \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots \end{array}$$

We say that the object

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$$

is a *subobject* of

$$B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} B_3 \xrightarrow{g_3} \dots$$

if A_i is a submodule of B_i and the following diagram commutes where $j : A_i \to B_i$ denotes the inclusion map:

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots \\ {\scriptstyle j}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle j}\downarrow & & \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots \end{array}$$

When the meaning is clear, we will denote the generic object

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$$

in q_ω simply by $\mathbf{A}$. Similarly,

$$B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} B_3 \xrightarrow{g_3} \dots$$

is denoted by $\mathbf{B}$.

By definition, an object

$$\mathbf{E} = E_1 \xrightarrow{\delta_1} E_2 \xrightarrow{\delta_2} E_3 \xrightarrow{\delta_3} \dots$$

is injective in the category q_ω if every map from $\mathbf{A}$ into $\mathbf{E}$ can be extended to a map from $\mathbf{B}$ to $\mathbf{E}$ whenever $\mathbf{A}$ is a subobject of $\mathbf{B}$.

3. The generalization of Baer's lemma

The object

$$R \xrightarrow{j} R \xrightarrow{j} R \xrightarrow{j} \dots$$

in q_ω, where j is the identity map, is denoted by $\mathbf{R}$. As in the case of R-Mod, we say that an object $\mathbf{E}$ in q_ω is injective for $\mathbf{R}$ if each mapping from a subobject $\mathbf{S}$ of $\mathbf{R}$ to $\mathbf{E}$ can be extended to $\mathbf{R}$.

Theorem 3.1. *Let*

$$\mathbf{E} = E_1 \xrightarrow{\delta_1} E_2 \xrightarrow{\delta_2} E_3 \xrightarrow{\delta_3} \dots$$

be an object in the category q_ω*. Then* $\mathbf{E}$ *is injective in the category* q_ω *if and only if* $\mathbf{E}$ *is injective for* $\mathbf{R}$.

P r o o f. Clearly the condition is necessary since it is a special case of the criterion stated in the definition of an injective object in q_ω. Conversely, assume now that $\mathbf{E}$ is injective for $\mathbf{R}$. That is, assume that whenever

$$\mathbf{L} = L_1 \subseteq L_2 \subseteq L_3 \subseteq \dots$$

is an ascending sequence of left ideals of the ring R, any map in q_ω from $\mathbf{L}$ into $\mathbf{E}$ can be extended to $\mathbf{R}$.

To verify that $\mathbf{E}$ is injective, let

$$\mathbf{B} = B_1 \xrightarrow{f_1} B_2 \xrightarrow{f_2} B_3 \xrightarrow{f_3} \dots$$

be an arbitrary object in q_ω and let let

$$\mathbf{A} = A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$$

be a subobject of $\mathbf{B}$. Let π be a map in q_ω from $\mathbf{A}$ to $\mathbf{E}$. We want to show that π can be extended to a map in q_ω from $\mathbf{B}$ to $\mathbf{E}$. Toward this end, suppose that π has been extended to a maximal subobject

$$\mathbf{C} = C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xrightarrow{f_3} \dots$$

of

$$\mathbf{B} = B_1 \xrightarrow{f_1} B_2 \xrightarrow{f_2} B_3 \xrightarrow{f_3} \dots$$

that contains

$$\mathbf{A} = A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots.$$

It suffices to prove that $\mathbf{C} = \mathbf{B}$.

Assume, by way of contradiction, that $\mathbf{C} \neq \mathbf{B}$. Then there must be a $k > 0$ such that C_k is a proper submodule of B_k. Choose an element $b_k \in B_k$ not in C_k. We will use this element b_k to construct another subobject of

$$\mathbf{B} = B_1 \xrightarrow{f_1} B_2 \xrightarrow{f_2} B_3 \xrightarrow{f_3} \dots.$$

Specifically, the object is

$$\mathbf{D} = 0 \to \ldots \to 0 \to Rb_k \to Rf_k(b_k) \to Rf_{k+1}(b_{k+1}) \to \ldots$$

where $b_{n+1} = f_n(b_n)$ for all $n > k$. For each n, define a submodule S_n of R by $S_n = Rb_n + C_n$ and consider the subobject

$$\mathbf{S} = S_1 \xrightarrow{f_1} S_2 \xrightarrow{f_2} S_3 \xrightarrow{f_3} \ldots.$$

The proof will be completed if we can show that π can be extended to $\mathbf{S}$ since $\mathbf{S}$ is a subobject of $\mathbf{B}$ properly containing $\mathbf{C}$, which was chosen maximal. We will show that indeed π can be extended to $\mathbf{S}$ by finding a mapping $\gamma = \{\gamma_n\}$ in q_ω from the subobject $\mathbf{D}$ of $\mathbf{B}$ to $\mathbf{E}$ that agrees with π on $\mathbf{C}$. Let $L_n = \{r \in R : rb_n \in C_n\}$. Observe that if $rb_n = c_n \in C_n$, then $rb_{n+1} = rf_n(b_n) = f_n(rb_n) = f_n(c_n) \in C_{n+1}$. So $L_n \subseteq L_{n+1}$. Define $\phi_n : L_n \to E_n$ by $\phi_n(x) = \pi_n(xb_n)$ if $x \in L_n$. Then we have a map $\phi = \{\phi_n\}$ in q_ω from

$$\mathbf{L} = L_1 \subseteq L_2 \subseteq L_3 \subseteq \ldots$$

to

$$\mathbf{E} = E_1 \xrightarrow{\delta_1} E_2 \xrightarrow{\delta_2} E_3 \xrightarrow{\delta_3} \ldots$$

represented by the following commutative diagram with j denoting the inclusion map:

$$\begin{array}{ccccccc}
L_1 & \xrightarrow{j} & L_2 & \xrightarrow{j} & L_3 & \xrightarrow{j} & \cdots \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \\
E_1 & \xrightarrow{\delta_1} & E_2 & \xrightarrow{\delta_2} & E_3 & \xrightarrow{\delta_3} & \cdots
\end{array}$$

To see that the diagram is commutative and that ϕ is actually a map in q_ω, observe that for every $x \in L_n$ we have

$$\delta_n \phi_n(x) = \delta_n \pi_n(xb_n) = \pi_{n+1} f_n(xb_n) = \pi_{n+1}(xb_{n+1}) = \phi_{n+1}(x) = \phi_{n+1}(j(x))$$

because $\pi = \{\pi_n\}$ is a map in q_ω from $\mathbf{A}$ to $\mathbf{E}$.

By hypothesis, the map $\phi = \{\phi_n\}$ in q_ω from

$$\mathbf{L} = L_1 \subseteq L_2 \subseteq L_3 \subseteq \ldots$$

to $\mathbf{E}$ can be extended to a map from $\mathbf{R}$ to $\mathbf{E}$ since we are assuming that $\mathbf{E}$ is injective for $\mathbf{R}$. Therefore, we can now define a map $\gamma = \{\gamma_n\}$ where $\gamma_n : Rb_n \to E_n$ is defined by $\gamma_n(rb_n) = \phi_n(r)$ for every $r \in R$.

It is crucial to our argument that γ_n agrees with π_n on $Rb_n \cap C_n$. Suppose $x \in Rb_n \cap C_n$ and let $rb_n = x = c_n$, where $r \in R$ and $c_n \in C_n$. Then

$$\gamma_n(x) = \gamma_n(rb_n) = \phi_n(r) = \pi(c_n).$$

Because of the agreement of γ_n and π_n, it follows that the mapping ϱ_n from S_n to E_n defined by

$$\varrho_n(c_n + rb_n) = \pi_n(c_n) + \gamma_n(rb_n)$$

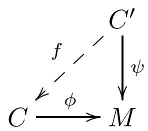
is well defined. Therefore, the mapping $\varrho = \{\varrho_n\}$ from

$$\mathbf{S} = S_1 \xrightarrow{f_1} S_2 \xrightarrow{f_2} S_3 \xrightarrow{f_3} \ldots$$

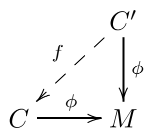
extends $\pi = \{\pi_n\}$ in q_ω from $\mathbf{C}$ to $\mathbf{E}$ to a mapping in q_ω from $\mathbf{S}$ to $\mathbf{E}$. Since $\mathbf{C}$ was chosen as a maximal extension, we conclude that $\mathbf{S} = \mathbf{B}$, and we have shown that $\mathbf{E}$ is an injective object in q_ω. □

4. An application

For an R-module M, a morphism $\phi : C \to M$ where C is torsion free is called a torsion free precover of M if for any $\psi : C' \to M$ where C' is torsion free, there is a map $f : C' \to C$ such that $\phi \circ f = \psi$. That is, the following diagram commutes:



If $\phi : C \to M$ is a torsion free precover and if every $f : C \to C$ such that $\phi \circ f = \phi$ is an automorphism, then ϕ is a torsion free cover of M:



In [2], E. Enochs proved that torsion free covers exist for integral domains. That is, he showed that any module over an integral domain has a torsion free cover. Enochs' proof uses injectives and their properties in R-Mod in a fundamental way, and therefore Baer's Lemma for R-Mod comes into play. For example, Enochs uses the well-known fact that every torsion-free module over an integral domain can be imbedded in a torsion free injective module.

In [3], the question was raised whether objects in the category q_ω have torsion free covers. By using the above generalization of Baer's Lemma, we will show in a forthcoming paper that torsion free covers exist for the category q_ω.

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