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THE EQUALITY CASES FOR THE INEQUALITIES OF OPPENHEIM AND SCHUR FOR POSITIVE SEMI-DEFINITE MATRICES

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Abstract. In this paper, necessary and sufficient conditions for equality in the inequalities of Oppenheim and Schur for positive semidefinite matrices are investigated.

Keywords: Oppenheim's inequality, Schur's inequality, positive semidefinite matrix, Hadamard product

MSC 2010: 15A45, 15A57

1. INTRODUCTION

The following Hadamard inequality for positive semidefinite matrices is well-known (see e.g. [1]).

Theorem 1.1. *Let $A = (a_{ij})$ be an $n \times n$ positive semidefinite matrix. Then*

$$\det A \leqslant a_{11}a_{22}\ldots a_{nn}$$

with equality if and only if A is diagonal or has a row of zeros.

The *Hadamard product* of two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined by $A \circ B \equiv (a_{ij}b_{ij})$. The Hadamard product arises in a wide variety of ways. It was perhaps the first significant result published about the Hadamard product that the class of positive semidefinite matrices of a given size is closed under the Hadamard

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product. For more information, the authors may refer to [1], [3] and [4]. Oppenheim in [7] proved the following:

Theorem 1.2. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ positive semidefinite matrices. Then*

$$\det(A \circ B) \geqslant \det A \prod_{i=1}^{n} b_{ii}.$$

Moreover, Schur (e.g. see [6] or [7]) proved the following

Theorem 1.3. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ positive semidefinite matrices. Then*

$$\det(A \circ B) \geqslant \det A \prod_{i=1}^{n} b_{ii} + \det B \prod_{i=1}^{n} a_{ii} - \det A \det B.$$

It is of interest to know when equalities in Oppenheim's and Schur's inequalities occur. Oppenheim in [7] gave partial answer to this question. He showed that

Theorem 1.4. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ positive definite matrices. Then*

(1) *Equality in Oppenheim's inequality occurs if and only if A is diagonal.*
(2) *Equality in Schur's inequality occurs if and only if there exists a permutation matrix P such that*

$$P^T AP = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad P^T BP = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix},$$

where A_{11} and B_{11} are 2×2 matrices; A_{22} and B_{22} are diagonal.

The main topic of this paper is to characterize when equalities in Oppenheim's and Schur's inequalities hold. The main results are the following:

Theorem 1.5. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ positive semidefinite matrices. Then equality in Oppenheim's inequality occurs if and only if one of the following holds*

(1) *$A \circ B$ is singular;*
(2) *There exists an $n \times n$ matrix $T = \operatorname{diag}(t_{11}, \ldots, t_{nn})$ with $|t_{ii}| = \sqrt{b_{ii}}$ such that $A \circ B = TAT$.*

Theorem 1.6. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ positive semidefinite matrices. Then equality in Schur's inequality occurs if and only if one of the following holds.*

(1) *A and B are nonsingular and there exists a permutation matrix P such that*

$$P^T AP = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \qquad P^T BP = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix},$$

where A_{11} and B_{11} are 2×2 matrices; A_{22} and B_{22} are diagonal.

(2) *$A \circ B$ is singular;*

(3) *B is singular and there exists an $n \times n$ matrix $T = \mathrm{diag}(t_{11}, \ldots, t_{nn})$ with $|t_{ii}| = \sqrt{b_{ii}}$ such that $A \circ B = TAT$.*

(4) *A is singular and there exists an $n \times n$ matrix $T = \mathrm{diag}(t_{11}, \ldots, t_{nn})$ with $|t_{ii}| = \sqrt{a_{ii}}$ such that $A \circ B = TBT$.*

The rest of this paper is organized as follows: In Section 2, we present some preliminary results. In Section 3, we investigate some conditions for the Hadamard product of two positive semidefinite matrices to be singular. These results, in Section 4, are applied to provide proofs of Theorems 1.5 and 1.6.

2. Preliminary results

First of all, recall the notion of majorization. Given a real vector $x = (x_1, \ldots, x_n)^T$, we rearrange its components as $x_{[1]} \geqslant x_{[2]} \geqslant \ldots \geqslant x_{[n]}$. Let $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T$. We say x is *majorized* by y and denote by $x \preceq y$, if $\sum\limits_{i=1}^{k} x_{[i]} \leqslant \sum\limits_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum\limits_{i=1}^{n} x_{[i]} = \sum\limits_{i=1}^{n} y_{[i]}$. Moreover, if there exists $1 \leqslant t \leqslant n$ such that $\sum\limits_{i=1}^{t} x_{[i]} < \sum\limits_{i=1}^{t} y_{[i]}$, we say that x is *strictly majorized* by y and denote by $x \prec y$. We first need the following Lemma (e.g., see [5]).

Lemma 2.1 ([5]). *Let $f(x)$ be a convex function on an open interval I. If $x = (x_1, \ldots, x_n)^T \prec y = (y_1, \ldots, y_n)^T$ with $x_i, y_i \in I$, then $\sum\limits_{i=1}^{n} f(x_i) < \sum\limits_{i=1}^{n} f(y_i)$.*

We also need the following notions. A positive semidefinite matrix whose all diagonal entries are 1 is called *correlation matrix.* Let A be an $n \times n$ positive semidefinite matrix with eigenvalues $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$, where $\lambda_1(A) \geqslant \ldots \geqslant \lambda_n(A)$. From [2], we have

Lemma 2.2 ([2]). *Let A be an $n\times n$ positive semidefinite matrix and C be an $n\times n$ correlation matrix. Then $\lambda(A\circ C)\preceq\lambda(A)$.*

Lemma 2.3. *Let $C=(c_{ij})$ be an $n\times n$ correlation matrix with $n\geqslant 3$. If $|c_{i_1,i_2}|=|c_{i_2,i_3}|=1$ with $i_1\neq i_2\neq i_3$, then $c_{i_1,i_3}=c_{i_1,i_2}c_{i_2,i_3}$ and $|c_{i_1,i_3}|=1$.*

Proof. Without loss of generality, we assume that $i_1=1, i_2=2, i_3=3$. Then the 3×3 leading principal minor of C is equal to

$$\det\begin{pmatrix} 1 & c_{12} & c_{13}\\ c_{21} & 1 & c_{23}\\ c_{31} & c_{32} & 1\end{pmatrix} = 1+2c_{12}c_{23}c_{13}-c_{12}^2-c_{23}^2-c_{13}^2=-(c_{13}-c_{12}c_{23})^2\geqslant 0.$$

Hence $c_{13}=c_{12}c_{23}$ and $|c_{13}|=1$. □

We also need the following notions. Let A be an $n\times n$ real symmetric matrix. We define a simple graph $G(A)=(V,E)$ associated with A, where $V=\{v_1,\ldots,v_n\}$ and $\{v_i,v_j\}\in E$ if and only if $a_{ij}\neq 0$ for $i\neq j$. Let $X=(x_{ij})$ be an $n\times n$ matrix. Denote by $|X|=(|x_{ij}|)$ the nonnegative matrix whose entries are given by $|x_{ij}|$.

Lemma 2.4. *Let G be a connected graph of order n. Then there exists a vertex u such that $G-u$ is still connected.*

Proof. Let T be a spanning tree of G and u be a pendent vertex of T. Then $T-u$ is still connected. Since $T-u$ is a spanning subgraph of $G-u$, $G-u$ is connected. □

Lemma 2.5. *Let $A=(a_{ij})$ be an $n\times n$ symmetric matrix and $C=(c_{ij})$ be an $n\times n$ correlation matrix. If $|A\circ C|=|A|$ and $G(A)$ is connected, then there exists an n column vector α such that $|\alpha|=e$ and $C=\alpha\alpha^T$, where e is a column vector of all ones.*

Proof. We prove the assertion by induction on n. It is easy to see that the assertion holds for $n=1,2$. Assume that the assertion holds for all positive integers less than n. We proceed to show that the assertion holds for n. Since $G(A)$ is connected, by Lemma 2.4, we may assume that $G-v_1$ is connected. Hence A can be partitioned into the form

$$A=\begin{pmatrix} a_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\qquad C=\begin{pmatrix} c_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}.$$

with $G(A_{22})=G(A)-v_1$. Thus $|A_{22}\circ C_{22}|=|A_{22}|$ and $G(A_{22})$ is connected. By the induction hypothesis, there exists an $n-1$ column vector $\alpha_2=(x_2,\ldots,x_n)^T$

such that $|\alpha_2| = e$ and $C_{22} = \alpha_2\alpha_2^T$. Hence $c_{ij} = x_i x_j$ and $|c_{ij}| = 1$ for $2 \leqslant i, j \leqslant n$. On the other hand, since $G(A)$ is connected, without loss of generality, we assume that $a_{12} \neq 0$. By $|A \circ C| = |A|$, we have $|a_{12}c_{12}| = |a_{12}| \neq 0$ and $|c_{12}| = 1$. Let $x_1 = c_{12}/x_2$. Thus $|x_1| = 1$ and $c_{12} = x_1 x_2$. For any $3 \leqslant p \leqslant n$, by Lemma 2.3, we have $c_{1p} = c_{12}c_{2p} = x_1x_2x_2x_p = x_1x_p$ and $|c_{1p}| = 1$. Let $\alpha = (x_1, \alpha_2^T)^T$. Then $|\alpha| = e$ and $C = \alpha\alpha^T$. $\square$

Corollary 2.6. *Let $A = (a_{ij})$ be an $n \times n$ positive semidefinite matrix and $C = (c_{ij})$ be an $n \times n$ correlation matrix. If $|A \circ C| = |A|$, then there exists a diagonal matrix T with $|T| = I$ (the identity matrix) such that $A \circ C = TAT$.*

P r o o f. Let $G_1, \ldots, G_k$ be the connected components of $G(A)$. Then there exists a permutation matrix P such that $PAP^T = \operatorname{diag}(A_{11}, \ldots, A_{kk})$, where $G_i = G(A_{ii})$. Let $PCP^T = (C_{ij})$. Thus $P(A \circ C)P^T = \operatorname{diag}(A_{11} \circ C_{11}, \ldots, A_{kk} \circ C_{kk})$. Hence $|A_{ii} \circ C_{ii}| = |A_{ii}|$ for $i = 1, \ldots, k$. It follows from Lemma 2.5 that there exists a diagonal matrix T_i with $|T_i| = I$ such that $A_{ii} \circ C_{ii} = T_i A_{ii} T_i$. Let $T = P^T \operatorname{diag}(T_1, \ldots, T_k)P$. Thus $|T| = I$ and

$$\begin{aligned} P(A \circ C)P^T &= \operatorname{diag}(A_{11} \circ C_{11}, \ldots, A_{kk} \circ C_{kk}) \\ &= \operatorname{diag}(T_1A_{11}T_1, \ldots, T_kA_{kk}T_k) = PTATP^T. \end{aligned}$$

So $A \circ C = TAT$. $\square$

Now we can present the main result of this section.

Theorem 2.7. *Let A be an $n \times n$ positive semifinite matrix and C be an $n \times n$ correlation matrix. If $A \circ C$ is nonsingular, then the following statements are equivalent:*
(1) $\det(A \circ C) = \det A$;
(2) $\lambda(A \circ C) = \lambda(A)$;
(3) $|A \circ C| = |A|$;
(4) *There exists a diagonal matrix T with $|T| = I$ such that $A \circ C = TAT$.*

P r o o f. We prove that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2). By Lemma 2.2, $\lambda(A \circ C) \preceq \lambda(A)$. Suppose that $\lambda(A \circ C) \neq \lambda(A)$. Thus $\lambda(A \circ C) \prec \lambda(A)$. Clearly $f(x) = -\ln x$ is a convex function on an open interval $(0, \infty)$. Moreover, $\lambda_i(A \circ C) > 0$ and $\lambda_i(A) > 0$ for $i = 1, \ldots, n$, since $A \circ C$ is nonsingular. By Lemma 2.1, we have $\sum\limits_{i=1}^n f(\lambda_i(A \circ C)) < \sum\limits_{i=1}^n f(\lambda_i(A))$. Thus $\prod\limits_{i=1}^n \lambda_i(A) < \prod\limits_{i=1}^n \lambda_i(A \circ C)$. So $\det A < \det(A \circ C)$. This is a contradiction and (2) holds.

(2) $\Longrightarrow$ (3). Since $\lambda(A \circ C) = \lambda(A)$, the sum of minors of all 2×2 principal submatrices of $A \circ C$ is equal to the sum of minors of all 2×2 principal submatrices of A. On the other hand, the sum of minors of all 2×2 principal submatrices of $A \circ C$ is equal to

$$\sum_{1 \leqslant i < j \leqslant n} (a_{ii}a_{jj}c_{ii}c_{jj} - |a_{ij}|^2|c_{ij}|^2) = \sum_{1 \leqslant i < j \leqslant n} (a_{ii}a_{jj} - |a_{ij}|^2|c_{ij}|^2),$$

and the sum of minors of all 2×2 principal submatrices of A is equal to

$$\sum_{1 \leqslant i < j \leqslant n} (a_{ii}a_{jj} - |a_{ij}|^2).$$

Hence

$$\begin{aligned}\sum_{1 \leqslant i < j \leqslant n} (a_{ii}a_{jj}c_{ii}c_{jj} - |a_{ij}|^2|c_{ij}|^2) &= \sum_{1 \leqslant i < j \leqslant n} (a_{ii}a_{jj} - |a_{ij}|^2|c_{ij}|^2) \\ &= \sum_{1 \leqslant i < j \leqslant n} (a_{ii}a_{jj} - |a_{ij}|^2).\end{aligned}$$

Thus

$$\sum_{1 \leqslant i < j \leqslant n} |a_{ij}|^2(1 - |c_{ij}|^2) = 0.$$

Because $|c_{ij}| \leqslant 1$, $|a_{ij}|^2 = |a_{ij}|^2|c_{ij}|^2$ for $1 \leqslant i < j \leqslant n$. Hence $|a_{ij}c_{ij}| = |a_{ij}|$ for $1 \leqslant i, j \leqslant n$.

(3) $\Longrightarrow$ (4). This follows from Corollary 2.6.

(4) $\Longrightarrow$ (1). This is obvious. □

3. Singularity of Hadamard product of two positive semidefinite matrices

In this section, we present some equivalent conditions for Hadamard product of two positive semidefinite matrices being singular.

Lemma 3.1. *Let A be an $n \times n$ positive semidefinite matrix and x be an n vector. Then $Ax = 0$ if and only if $x^H Ax = 0$, where H stands for transpose conjugate.*

Proof. Since A is positive semidefinite, there exists a positive semidefinite matrix B such that $A = B^H B$. Thus $x^H Ax = (Bx)^H(Bx)$. Hence $Ax = 0$ if and only if $x^H Ax = 0$. □

Lemma 3.2. *Let $\{x_1, \dots, x_p\}$ and $\{y_1, \dots, y_q\}$ be orthogonal sets of eigenvectors of $n \times n$ positive semidefinite matrices A and B corresponding to all nonzero eigenvalues $\lambda_1 \dots \lambda_p$ of A and $\mu_1, \dots, \mu_q$ of B respectively (including multiplicities). Then $A \circ B$ is singular if and only if the dimension of the subspace spanned by $\{x_i \circ y_j, i = 1, \dots, p, j = 1, \dots, q\}$ is less than n.*

Proof. Clearly, $A = \sum\limits_{i=1}^{p} \lambda_i x_i x_i^H$ and $B = \sum\limits_{j=1}^{q} \mu_j y_j y_j^H$. Thus

$$A \circ B = \left(\sum_{i=1}^{p} \lambda_i x_i x_i^H \right) \circ \left(\sum_{j=1}^{q} \mu_j y_j y_j^H \right) = \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i \mu_j (x_i \circ y_j)(x_i \circ y_j)^H.$$

By Lemma 3.1, $A \circ B$ is singular if and only if there exists a nonzero column vector z such that $z^H(A \circ B)z = 0$, i.e.,

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i \mu_j z^H (x_i \circ y_j)(x_i \circ y_j)^H z = 0.$$

This happens if and only if each term of the above equation is zero, since $\lambda_i > 0, \mu_j > 0$ for $i = 1, \dots, p$, $j = 1, \dots, q$, therefore, if and only if $(x_i \circ y_j)^H z = 0$ for $i = 1, \dots, p$, $j = 1, \dots, q$. Thus

$$\left((x_1 \circ y_1, \dots, x_p \circ y_q)\right)^H z = 0.$$

Hence, if and only if the rank of the matrix $\left((x_1 \circ y_1, \dots, x_p \circ y_q)\right)^H$ is less than n and the dimension of the subspace spanned by $\{x_i \circ y_j, i = 1, \dots, p, j = 1, \dots, q\}$ is less than n. $\square$

Now we present some equivalent conditions for the Hadamard product of two positive semidefinite matrices to be singular, which is, in essence, attributable to [4].

Theorem 3.3. *Let A and B be two $n \times n$ positive semidefinite matrices. Thus the following are equivalent:*

(1) *$A \circ B$ is singular, i.e., there exists an n nonzero vector $z = (z_1, \dots, z_n)^T$ such that $(A \circ B)z = 0$;*
(2) *$z^H(A \circ B)z = 0$;*
(3) *$ADB = 0$, where $D = \operatorname{diag}(z_1, \dots, z_n)$;*
(4) *$DADB = 0$;*
(5) *$\operatorname{tr}(DADB) = 0$.*

Proof. (1) $\Longleftrightarrow$ (2) follows from Lemma 3.1.

(2) $\Longrightarrow$ (3). Since B is positive semidefinite matrix, we write $B = \sum_{i=1}^{k} v_i v_i^H$, where $\operatorname{rank} B = k$ and $v_i = (y_{i1}, \ldots, y_{in})^H$ for $i = 1, \ldots, k$. Then

$$0 = z^H (A \circ B) z = \sum_{i=1}^{k} (z_1 y_{i1}, \ldots, z_1 y_{in}) A (z_1 y_{i1}, \ldots, z_1 y_{in})^H.$$

Since A is semidefinite, $(z_1 y_{i1}, \ldots, z_1 y_{in}) A (z_1 y_{i1}, \ldots, z_1 y_{in})^H = 0$ for $i = 1, \ldots, k$. By Lemma 3.1, $A(z_1 y_{i1}, \ldots, z_n y_{in})^H = 0$, which implies $A \operatorname{diag}(z_1, \ldots, z_n) v_i = 0$ for $i = 1, \ldots, k$. Therefore

$$ADB = \sum_{i=1}^{k} A \operatorname{diag}(z_1, \ldots, z_n) v_i v_i^H = 0.$$

(3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) is obvious.

(5) $\Longrightarrow$ (2). Clearly, the (i,i) entry of $DADB$ is equal to $\sum_{k=1}^{n} a_{ik} z_i z_k b_{ki}$. Hence

$$0 = tr(DADB) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ik} z_i z_k = z^H (A \circ B) z.$$

□

Remark. Since Theorem 2.7 holds for $A \circ B$ being nonsingular, it is natural to ask whether the assertion is still true for $A \circ B$ being singular. The answer is negative. For example, let

$$A = \begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix}.$$

Clearly, A is positive semidefinite and B is a correlation matrix. Moreover, $\det(A \circ B) = \det A = 0$, but $|A \circ B| \neq |A|$. However we still have the following result.

Theorem 3.4. *Let A and B be two $n \times n$ positive semidefinite matrices. If $\operatorname{rank}(A) = n - 1$, then $\det(A \circ B) = \det A = 0$ if and only if B has a row of zeros or there exists a permutation P with*

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad PBP^T = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

such that A_{11} is singular and $A_{11} \circ B_{11} = T_1 A_{11} T_1$.

P r o o f. It is clear for sufficiency. We only consider necessity. We assume that $b_{ii} \neq 0$ for $i = 1, \ldots, n$. By Theorem 3.3, there exists a diagonal matrix $X \neq 0$ such that $AXB = 0$. Then there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad PBP^T = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

and $PXP^T = \operatorname{diag}(X_1, 0)$, where X_1 is an $m \times m$ nonsingular diagonal matrix with $1 \leqslant m \leqslant n$. Hence $A_{11}X_1B_{11} = 0$. So $\operatorname{rank}(A_{11}) + \operatorname{rank}(X_1B_{11}) \leqslant m$. On the other hand, since $\operatorname{rank}(A) = n-1$, $\operatorname{rank}(A_{11}) \geqslant m-1$. Therefore, $\operatorname{rank}(X_1B_{11}) = 1$, which implies that $\operatorname{rank}(B_{11}) = 1$. Then let $B_{11} = (\sqrt{b_{11}}, \ldots, \sqrt{b_{mm}})^T(\sqrt{b_{11}}, \ldots, \sqrt{b_{mm}})$. Let $T_1 = \operatorname{diag}(\sqrt{b_{11}}, \ldots, \sqrt{b_{mm}})$. We conclude that $A_{11} \circ B_{11} = T_1AT_1$. Moreover, $\operatorname{rank}(A_{11}) = m - 1$. □

4. Proofs of theorems 1.5 and 1.6

Lemma 4.1. *Let $A = (a_{ij})$ be an $n \times n$ positive definite matrix and $B = (b_{ij})$ be an $n \times n$ positive semidefinite matrix. Then $\det(A \circ B) = \det A \prod_{i=1}^{n} b_{ii}$ if and only if B has a row of zeros or there exists a diagonal matrix $T = \operatorname{diag}(t_1, \ldots, t_n)$ with $|t_i| = \sqrt{b_{ii}}$ for $i = 1, \ldots, n$ such that $A \circ B = TAT$.*

P r o o f. Sufficiency. If B has a row of zeros, then $A \circ B$ has a row of zeros and $\det(A \circ B) = 0 = \det A \prod_{i=1}^{n} b_{ii}$. If $A \circ B = TAT$, then $\det(A \circ B) = \det(TAT) = \det A \prod_{i=1}^{n} b_{ii}$.

Necessity. Without loss of generality, we assume that $b_{ii} > 0$ for $i = 1, \ldots, n$. Let $D = \operatorname{diag}(b_{11}, \ldots, b_{nn})$ and $C = D^{-1/2}BD^{-1/2}$. Thus C is a correlation matrix and $\det(A \circ C) = \det A$. By Theorem 2.7, there exists a diagonal matrix T_1 with $|T_1| = I$ such that $A \circ C = T_1AT_1$. Let $T = D^{1/2}T_1$. Then $A \circ B = TAT$ with $|t_i| = \sqrt{b_{ii}}$ for $i = 1, \ldots, n$. □

Corollary 4.2. *Let $A = (a_{ij})$ be an $n \times n$ positive definite matrix and $B = (b_{ij})$ be an $n \times n$ positive semidefinite matrix. Suppose that $G(A)$ is connected. Then $\det(A \circ B) = \det A \prod_{i=1}^{n} b_{ii}$ if and only if B has a row of zeros or $B = \alpha\alpha^H$, where $|\alpha| = (\sqrt{b_{11}}, \ldots, \sqrt{b_{nn}})^H$.*

P r o o f. It follows from Lemmas 4.1 and 2.5. □

P r o o f o f T h e o r e m 1.5. Sufficiency. Clearly, it is obvious.

Necessity. If A is singular, then $\det(A \circ B) = 0$, which implies that $A \circ B$ is singular. If A is nonsingular, it follows from Lemma 4.1 that the assertion holds. □

Proof of Theorem 1.6. Sufficiency. Clearly it is obvious.

Necessity. If both A and B are nonsingular, it follows from Theorem 1.4 that (1) holds. If either A or B is singular, it follows from Theorem 1.5 that either (3) or (4) holds. If A and B are singular, then $\det(A \circ B) = 0$ and $A \circ B$ is singular. $\square$

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