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# A CHARACTERIZATION OF TOTALLY $\eta$-UMBILICAL REAL HYPERSURFACES AND RULED REAL HYPERSURFACES OF A COMPLEX SPACE FORM

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*Abstract.* We give a characterization of totally $\eta$-umbilical real hypersurfaces and ruled real hypersurfaces of a complex space form in terms of totally umbilical condition for the holomorphic distribution on real hypersurfaces. We prove that if the shape operator $A$ of a real hypersurface $M$ of a complex space form $M^n(c)$, $c \neq 0$, $n \geqslant 3$, satisfies $g(AX, Y) = ag(X, Y)$ for any $X, Y \in T_0(x)$, $a$ being a function, where $T_0$ is the holomorphic distribution on $M$, then $M$ is a totally $\eta$-umbilical real hypersurface or locally congruent to a ruled real hypersurface. This condition for the shape operator is a generalization of the notion of $\eta$-umbilical real hypersurfaces.

*Keywords*: real hypersurface, totally $\eta$-umbilical real hypersurface, ruled real hypersurface

*MSC 2010*: 53C40, 53C55, 53C25

## 1. Introduction

Let $M^n(c)$ be an $n$-dimensional complex space form with constant holomorphic sectional curvature $4c$, and let $M$ be a real hypersurface of $M^n(c)$. We denote by $J$ the complex structure of $M^n(c)$. Then $M$ has an almost contact metric structure $(\varphi, \xi, \eta, g)$ induced from $J$.

If the shape operator $A$ of a real hypersurface $M$ is of the form $A = aI$, where $I$ is the identity, then $M$ is said to be totally umbilical. In Tashiro-Tachibana [12], it was proved that no real hypersurface of $M^n(c)$, $c \neq 0$, is totally umbilical. So we need the notion of totally $\eta$-umbilical real hypersurfaces, that is, the shape operator $A$ is of the form $A = aI + b\eta \otimes \xi$. Totally $\eta$-umbilical real hypersurfaces of a complex projective space $CP^n$ and a complex hyperbolic space $CH^n$ are determined by Takagi [11] and Montiel [7].

If a real hypersurface $M$ of $M^n(c)$, $c \neq 0$, is totally $\eta$-umbilical, then the structure vector field $\xi$ is a principal vector field of the shape operator $A$ of $M$, that is, $A\xi = \alpha\xi$. On the other hand, for any ruled real hypersurface $M$ of $M^n(c)$, we see that the structure vector field $\xi$ is not principal vector field of $A$. But the shape operator $A$ of a ruled real hypersurface $M$ satisfies $g(AX, Y) = 0$ for any vectors $X, Y \in T_0(x) = \{X \in T_x(M)\colon\ \eta(X) = 0\}$, where $T_0$ is the holomorphic distribution on $M$ (see [4]).

It is an interesting and important problem to determine real hypersurfaces of complex space forms with respect to some conditions for the holomorphic distribution on real hypersurfaces. For instance, Kimura [3] classified real hypersurfaces of a complex projective space $CP^n$, $n \geqslant 3$, on which the sectional curvature of the holomorphic 2-plane spanned by a unit tangent vector orthogonal to the structure vector field $\xi$ is constant. When the ambient manifold is the complex hyperbolic space, the corresponding result is given by M. Ortega and J. D. Pérez [8], and D. J. Sohn and Y. J. Suh [10] (see also [9]).

So, we consider the condition for the holomorphic distribution on real hypersurfaces such that the shape operator $A$ of a real hypersurface $M$ satisfies $g(AX, Y) = ag(X, Y)$ for any $X, Y \in T_0$, $a$ being a function, which includes the notion of totally $\eta$-umbilical real hypersurfaces and is independent of the condition with respect to the structure vector field $\xi$.

Our main theorem states that if the shape operator $A$ of a real hypersurface $M$ of a complex space form $M^n(c)$, $c \neq 0$, $n \geqslant 3$, satisfies the condition above, then $M$ is a totally $\eta$-umbilical real hypersurface or locally congruent to a ruled real hypersurface.

## 2. Preliminaries

Let $M^n(c)$ denote the complex space form of complex dimension $n$ (real dimension $2n$) with constant holomorphic sectional curvature $4c$. We denote by $J$ the almost complex structure of $M^n(c)$. The Hermitian metric of $M^n(c)$ will be denoted by $G$.

Let $M$ be a real $(2n-1)$-dimensional hypersurface immersed in $M^n(c)$. We denote by $g$ the Riemannian metric induced on $M$ from $G$. We take the unit normal vector field $N$ of $M$ in $M^n(c)$. For any vector field $X$ tangent to $M$, we define $\varphi$, $\eta$ and $\xi$ by

$$JX = \varphi X + \eta(X)N, \quad JN = -\xi,$$

where $\varphi X$ is the tangential part of $JX$, $\varphi$ is a tensor field of type (1,1), $\eta$ is a 1-form, and $\xi$ is the unit vector field on $M$. Then they satisfy

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0$$

for any vector field $X$ tangent to $M$. Moreover, we have

$$g(\varphi X, Y) + g(X, \varphi Y) = 0, \quad \eta(X) = g(X, \xi),$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Thus $(\varphi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M^n(c)$, and by $\nabla$ the one in $M$ determined by the induced metric. Then the *Gauss* and *Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX,$$

for any vector fields $X$ and $Y$ tangent to $M$. We call $A$ the *shape operator* of $M$.

For the contact metric structure on $M$ we have

$$\nabla_X \xi = \varphi AX, \quad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

We denote by $R$ the Riemannian curvature tensor field of $M$. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X, Y)Z = c\{&g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X \\ &- g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

and the *equation of Codazzi* by

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi\}.$$

From the equation of Gauss, the Ricci tensor $S$ of $M$ is given by

$$\begin{aligned} S(X, Y) = &(2n + 1)cg(X, Y) - 3c\eta(X)\eta(Y) \\ &+ \mathrm{Tr}A g(AX, Y) - g(AX, AY), \end{aligned}$$

where $\mathrm{Tr}\, A$ is the trace of $A$.

If the shape operator $A$ of $M$ is of the form $AX = aX + b\eta(X)\xi$ for some functions $a$ and $b$, then $M$ is said to be *totally $\eta$-umbilical* (see Tashiro-Tachibana [12]). It is well known that if $M$ is a totally $\eta$-umbilical real hypersurface of a complex space form $M^n(c)$, $c \neq 0$, $n \geqslant 2$, then $M$ has two constant principal curvatures (see Takagi [11]).

**Example 1.** Let $\mathbb{C}^n$ be the space of $(n+1)$-tuples of complex numbers $(z_1, \dots, z_{n+1})$. Put $S^{2n+1} = \Big\{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \colon \sum_{j=1}^{n+1} |z_j|^2 = 1\Big\}$. For a positive number $r$ we denote by $M'(2n, r)$ a hypersurface of $S^{2n+1}$ defined by

$$\sum_{j=1}^{n} |z_j|^2 = r|z_{n+1}|^2, \quad \sum_{j=1}^{n+1} |z_j|^2 = 1.$$

Let $\pi\colon S^{2n+1} \longrightarrow CP^n$ be the natural projection. Then $M(2n-1, r) = \pi(M'(2n, r))$ is a connected compact real hypersurface of $CP^n$ with two constant principal curvatures and totally $\eta$-umbilical. We call $M(2n-1, r)$ a *geodesic hypersphere* of $CP^n$. We have (see [1] and [11])

**Theorem A.** *Let $M$ be a totally $\eta$-umbilical real hypersurface of $CP^n$, $n \geqslant 2$, then $M$ is locally congruent to a geodesic hypersphere.*

Moreover, any totally $\eta$-umbilical real hypersurface of $M^n(c)$ is a pseudo-Einstein real hypersurface, that is, the Ricci tensor $S$ of $M$ satisfies $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$ for some functions $a$ and $b$ (cf. [13]).

**Example 2** ([7])**.** Let $H_1^{2n+1}$ be a $(2n+1)$-dimensional anti-de Sitter space in $\mathbb{C}^{n+1}$, which is a Lorentz manifold of constant sectional curvature $-1$. $H_1^{2n+1}$ is a principal $S^1$-bundle over the complex hyperbolic space $CH^n$ with projection map $\pi\colon H_1^{2n+1} \longrightarrow CH^n$. $CH^n$ is of constant holomorphic sectional curvature $-4$.

For integers $p$ and $q$ with $p + q = n - 1$ and $t \in \mathbb{R}$, $0 < t < 1$, we consider the Lorentz hypersurface $M'_{p,q}(t)$ of $H_1^{2n+1}$ defined by

$$-|z_0|^2 + \sum_{j=1}^{n} |z_j|^2 = -1, \quad t\Big(-|z_0|^2 + \sum_{j=1}^{p} |z_j|^2\Big) = -\sum_{k=p+1}^{n} |z_k|^2,$$

which is isometric to the product

$$H_1^{2p+1}(1/(t-1)) \times S^{2q+1}(t/(1-t)),$$

where $1/(t-1)$ and $t/(1-t)$ are the respective squares of the radii. We put $M_{p,q}(t) = \pi(M'_{p,q}(t))$. $M_{p,q}(t)$ is a real hypersurface of $CH^n$ with constant three principal curvatures $\tanh\theta$, $\cosh\theta$ and $2\coth 2\theta$ with multiplicities $2p$, $2q$ and 1 respectively, where we have put $\tanh\theta = \sqrt{t}$. $M_{p,q}(t)$ is a tube of radius $\theta$ over a $(n-q-1)$-dimensional totally geodesic complex submanifold $CH^{n-q-1}$ of $CH^n$.

If $p = 0$ or $q = 0$, $M_{p,q}(t)$ is pseudo-Einstein and totally $\eta$-umbilical. $M_{0,n-1}(t)$ is called the *geodesic hypersphere* and the Ricci tensor $S$ is given by $S(X,Y) = (-2n + (2n-2)\coth^2\theta)g(X,Y) + 2n\eta(X)\eta(Y)$.

$M_{n-1,0}$ is a tube over a complex hyperbolic hyperplane and the Ricci tensor $S$ of $M_{n-1,0}(t)$ is given by $S(X,Y) = (-2n + (2n-2)\tanh^2\theta)g(X,Y) + 2n\eta(X)\eta(Y)$.

For fixed $t \in \mathbb{R}$, $t > 0$, we denote by $L(t)$ the Lorentz hypersurface of $H_1^{2n+1}$, given by

$$-|z_0|^2 + \sum_{j=1}^{n} |z_j|^2 = -1, \quad |z_0 - z_1|^2 = t.$$

We put $M_n^*(t) = \pi(L(t))$. Then $M_n^*(t)$ is a totally $\eta$-umbilical real hypersurface of $CH^n$ with two constant principal curvatures 1 and 2. We see that $M_n^*(t)$ is congruent to $M_n^*(1) = M_n^*$ for each $t > 0$. $M_n^*$ is a pseudo-Einstein real hypersurface with $S(X,Y) = -2g(X,Y) + 2n\eta(X)\eta(Y)$. We call $M_n^*$ a *self-tube*.

We notice that a complete and connected real hypersurface of $CH^n$, $n \geqslant 3$, is pseudo-Einstein if and only if it is totally $\eta$-umbilical (Montiel [7]).

The following theorem is a direct consequence of theorems in Montiel [7].

**Theorem B.** *Let $M$ be a totally $\eta$-umbilical real hypersurface of $CH^n$, $n \geqslant 3$. Then $M$ is locally congruent to one of the following spaces:*
(a) *a geodesic hypersphere $M_{0,n-1}(\tanh^2\theta)$ of radius $\theta > 0$,*
(b) *a tube $M_{n-1,0}(\tanh^2\theta)$ of radius $\theta > 0$ over a complex hyperbolic hyperplane,*
(c) *a self-tube $M_n^*$.*

For $r > 0$ and the unit normal vector field $N$, we define a map $\varphi_r\colon M_n^* \longrightarrow CH^n$ by $\varphi_r(x) = F(rN(x))$, where $F(rN(x))$ is the point of $CH^n$ reached at distance $r$ along the geodesic of $CH^n$ starting at $x$ with initial direction $rN(x)$. Then the real hypersurface $\varphi_r M_n^*(t)$ is congruent to $M_n^*$. Therefore, we say that $M_n^*$ is a "self-tube" (see [7, p. 526]).

**Example 3** ([2], [4], [6]). Let $M$ be a real hypersurface of a complex space form $M^n(c)$, $c \neq 0$, and let $T_0$ be the distribution defined by $T_0(x) = \{X \in T_x(M)\colon X \perp \xi\}$ for $x \in M$. If $T_0$ is integrable and its integral manifold is a totally geodesic submanifold $M^{n-1}(c)$, then $M$ is said to be *ruled real hypersurface*. Let $\gamma(t)$ ($t \in I$) be an arbitrary (regular) curve in $M^n(c)$. Then for every $t \in I$ there exists a totally geodesic submanifold $M^{n-1}(c)$ in $M^n(c)$ which is orthogonal to the plane $\tau_t$ spanned by $\{\gamma'(t), J\gamma'(t)\}$. Here we denote by $M_t^{n-1}(c)$ such a totally geodesic submanifold. Let $M = \{x \in M_t^{n-1}(c)\colon t \in I\}$. Then the construction of $M$ asserts that $M$ is a ruled real hypersurface in $M^n(c)$. Moreover, the construction of $M$ tells us that there are many ruled real hypersurfaces. The *holomorphic sectional curvature* $H$ of the ruled real hypersurface $M$ is $4c$ (see [3]).

## 3. Proof of the theorem

We prove our main theorem.

**Theorem 3.1.** *Let $M$ be a real hypersurface of a complex space form $M^n(c)$, $c \neq 0$, $n \geqslant 3$. Let $T_0$ denote the holomorphic distribution on $M$ defined by $T_0(x) = \{X \in T_x(M) \colon \eta(X) = 0\}$. If the shape operator $A$ of $M$ satisfies $g(AX,Y) = ag(X,Y)$ for any $X, Y \in T_0$, $a$ being a function, then $M$ is either totally $\eta$-umbilical or it is locally a ruled real hypersurface.*

To prove the theorem above, we prepare some lemmas.

Let $M$ be a real hypersurface of $M^n(c)$, $c \neq 0$, $n \geqslant 3$. Suppose that the shape operator $A$ satisfies $g(AX,Y) = ag(X,Y)$ for any $X, Y \in T_0$. We can choose a local field of orthonormal frames $\{e_1, \ldots, e_{2n-2}, \xi\}$ of $M$ such that the shape operator $A$ is represented by a matrix of the form

$$A = \begin{pmatrix} a & \ldots & 0 & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & a & h_{2n-2} \\ h_1 & \ldots & h_{2n-2} & b \end{pmatrix},$$

where we have put $h_i = g(Ae_i, \xi)$, $i = 1, \ldots, 2n-2$ and $b = g(A\xi, \xi)$.

We notice that $\{\varphi e_1, \ldots, \varphi e_{2n-2}, \xi\}$ is also a local field of orthonormal frames of $M$.

First of all, we consider the case $a \neq 0$.

**Lemma 3.2.** *Let $M$ be a real hypersurface of $M^n(c)$, $c \neq 0$, $n \geqslant 3$. Suppose that the shape operator $A$ of $M$ satisfies $g(AX,Y) = ag(X,Y)$, $a \neq 0$, for any $X, Y \in T_0$. Then $h_1, \ldots, h_{2n-2}$ satisfy*

$$h_i g(\varphi e_j, e_k) = h_j g(\varphi e_k, e_i) = h_k g(\varphi e_i, e_j)$$

*for any $i \neq j$, $j \neq k$, $k \neq i$.*

P r o o f. In the following, let $i, j, k$ and $l$ satisfy $i, j, k, l \leqslant 2n-2$. By the equation of Codazzi, we have

$$(\nabla_{e_i} A)e_j - (\nabla_{e_j} A)e_i = 2cg(e_i, \varphi e_j)\xi.$$

Since $Ae_i = ae_i + h_i\xi$ for $i = 1, \ldots, 2n-2$, we have

$$\begin{aligned}
&(\nabla_{e_i} A)e_j - (\nabla_{e_j} A)e_i \\
&\qquad = \nabla_{e_i} Ae_j - A\nabla_{e_i} e_j - \nabla_{e_j} Ae_i + A\nabla_{e_j} e_i \\
&\qquad = \nabla_{e_i}(ae_j + h_j\xi) - A\nabla_{e_i} e_j - \nabla_{e_j}(ae_i + h_i\xi) + A\nabla_{e_j} e_i \\
&\qquad = (e_i a)e_j + a\nabla_{e_i} e_j + (e_i h_j)\xi + h_j\varphi Ae_i - A\nabla_{e_i} e_j \\
&\qquad\quad - (e_j a)e_i - a\nabla_{e_j} e_i - (e_j h_i)\xi - h_i\varphi Ae_j + A\nabla_{e_j} e_i \\
&\qquad = 2cg(e_i, \varphi e_j)\xi
\end{aligned}$$

for any $i \neq j$. Thus, for any $k$ such that $k \neq i$ and $k \neq j$, we have

$$\begin{aligned}
(3.1)\quad 0 &= ag(\nabla_{e_i} e_j - \nabla_{e_j} e_i, e_k) + ag(h_j\varphi e_i - h_i\varphi e_j, e_k) - g(\nabla_{e_i} e_j - \nabla_{e_j} e_i, Ae_k) \\
&= ah_j g(\varphi e_i, e_k) - ah_i g(\varphi e_j, e_k) + h_k g(e_j, \nabla_{e_i}\xi) - h_k g(e_i, \nabla_{e_j}\xi) \\
&= ah_j g(\varphi e_i, e_k) - ah_i g(\varphi e_j, e_k) + h_k g(e_j, \varphi Ae_i) - h_k g(e_i, \varphi Ae_j) \\
&= ah_j g(\varphi e_i, e_k) - ah_i g(\varphi e_j, e_k) + 2ah_k g(e_j, \varphi e_i).
\end{aligned}$$

By this equation, we obtain

$$(3.2)\qquad ah_k g(\varphi e_j, e_i) - ah_j g(\varphi e_k, e_i) + 2ah_i g(e_k, \varphi e_j) = 0,$$

$$(3.3)\qquad ah_i g(\varphi e_k, e_j) - ah_k g(\varphi e_i, e_j) + 2ah_j g(e_i, \varphi e_k) = 0.$$

Since $a \neq 0$, the equations (3.1) and (3.2) imply $h_i(\varphi e_j, e_k) = h_k g(\varphi e_i, e_j)$. Using (3.3), we have

$$h_i g(\varphi e_j, e_k) = h_j g(\varphi e_k, e_i) = h_k g(\varphi e_i, e_j).$$

**Lemma 3.3.** *Let $M$ be a real hypersurface of $M^n(c)$, $c \neq 0$, $n \geqslant 3$. Suppose that the shape operator $A$ of $M$ satisfies $g(AX, Y) = ag(X, Y)$, $a \neq 0$, for any $X, Y \in T_0$. If $h_i = 0$ for some $i$, then $h_1 = \ldots = h_{2n-2} = 0$.*

P r o o f. Suppose that there exists $h_i$ which satisfies $h_i = 0$. Then we have

$$h_j g(\varphi e_k, e_i) = h_k g(\varphi e_i, e_j) = 0$$

for any $j$ and $k$ such that $j \neq k$, $k \neq i$ and $i \neq j$. If there is a $h_j \neq 0$, then $g(\varphi e_k, e_i) = 0$ for any $k$ such that $k \neq i$ and $k \neq j$. Thus we have $e_i = \varphi e_j$ or $e_i = -\varphi e_j$. Since $h_k g(\varphi e_i, e_j) = 0$, we have $h_k = 0$ for any $k$ such that $k \neq i$ and $k \neq j$.

Let $l$ satisfy $l \neq i$, $l \neq j$ and $l \neq k$. Since $h_k = 0$ and $h_i = 0$, we have

$$h_j g(\varphi e_k, e_l) = h_k g(\varphi e_l, e_j) = 0,$$
$$h_j g(\varphi e_i, e_l) = h_i g(\varphi e_l, e_j) = 0.$$

Since $h_j \neq 0$, $e_l$ satisfies $g(\varphi e_k, e_l) = 0$ for any $k \neq j$, $k \neq i$ and $g(\varphi e_i, e_l) = 0$. Thus we obtain $e_l = \varphi e_j$ or $e_l = -\varphi e_j$. Then we have $e_i = e_l$ or $e_i = -e_l$. This is a contradiction. So we see that if there is an $h_i = 0$, then $h_1 = \ldots = h_{2n-2} = 0$. □

**Lemma 3.4.** *Let $M$ be a real hypersurface of $M^n(c)$, $c \neq 0$, $n \geqslant 3$. Suppose that the shape operator $A$ of $M$ satisfies $g(AX, Y) = ag(X, Y)$, $a \neq 0$, for any $X, Y \in T_0$. Then there exists $i$ such that $h_i = 0$.*

P r o o f. Suppose that $h_1 \neq 0, \ldots, h_{2n-2} \neq 0$, and $i$, $j$, $k$ and $l$ are different from each other. By Lemma 3.1, we have

$$h_i g(\varphi e_j, e_k) = h_j g(\varphi e_k, e_i) = h_k g(\varphi e_i, e_j), \tag{3.4}$$
$$h_j g(\varphi e_k, e_l) = h_k g(\varphi e_l, e_j) = h_l g(\varphi e_j, e_k), \tag{3.5}$$
$$h_k g(\varphi e_l, e_i) = h_l g(\varphi e_i, e_k) = h_i g(\varphi e_k, e_l), \tag{3.6}$$
$$h_l g(\varphi e_i, e_j) = h_i g(\varphi e_j, e_l) = h_j g(\varphi e_l, e_i). \tag{3.7}$$

By (3.5) and (3.7), we obtain

$$h_i g(\varphi e_j, e_k) = \frac{h_i h_k}{h_l} g(\varphi e_l, e_j) = -\frac{h_i h_k}{h_l} \times \frac{h_l}{h_i} g(\varphi e_i, e_j) = -h_k g(\varphi e_i, e_j).$$

Since $h_i g(\varphi e_j, e_k) = h_k g(\varphi e_i, e_j)$, we have $h_i g(\varphi e_j, e_k) = 0$. Since $h_i \neq 0$, we have $g(\varphi e_j, e_k) = 0$ for any $j$ and $k$ such that $i \neq j$, $j \neq k$ and $k \neq i$. Here, we fix the index $i$. Then we obtain $e_k = \varphi e_i$ or $e_k = -\varphi e_i$ for any $k \neq i$. This is a contradiction. Consequently, we see that there is a $h_i$ such that $h_i = 0$. □

P r o o f of Theorem 3.1. From Lemmas 3.2, 3.3 and 3.4, if $a \neq 0$, we have $h_i = 0$ for all $i$, and hence $A = aI + b\eta \otimes \xi$. Thus $M$ is a totally $\eta$-umbilical real hypersurface.

We next suppose that $a = 0$. Then $g(AX, Y) = 0$ for any $X, Y \in T_0$. Using the basic formulas from the Preliminaries, we easily check that, for any $X, Y \in T_0$, we have

$$g(\nabla_X Y, \xi) = -g(Y, \varphi AX) = g(AX, \varphi Y) = 0.$$

From here we see that always $\nabla_X Y \in T_0$ and the distribution $T_0$ is integrable. Moreover, $\tilde{\nabla}_X Y = \nabla_X Y$, and hence the integral manifold of $T_0$ is a totally geodesic complex submanifold of $M^n(c)$. Consequently, $M$ is locally a ruled real hypersurface. This completes the proof of our theorem. □

From Theorem A and Theorem 3.1 we have

**Theorem 3.5.** *Let $M$ be a real hypersurface of a complex projective space $CP^n$, $n \geqslant 3$. If the shape operator $A$ of $M$ satisfies $g(AX, Y) = ag(X, Y)$ for any $X, Y \in T_0$, $a$ being a function, then $M$ is locally congruent to a geodesic hypersphere or a ruled real hypersurface.*

From Theorem B and Theorem 3.1, we have the following theorem.

**Theorem 3.6.** *Let $M$ be a real hypersurface of a complex hyperbolic space $CH^n$, $n \geqslant 3$. If the shape operator $A$ of $M$ satisfies $g(AX, Y) = ag(X, Y)$ for any $X, Y \in T_0$, $a$ being a function, then $M$ is locally congruent to one of the following spaces:*

(a) *a ruled real hypersurface,*
(b) *a geodesic hypersphere $M_{0,n-1}(\tanh^2\theta)$ of radius $\theta > 0$,*
(c) *a tube $M_{n-1,0}(\tanh^2\theta)$ of radius $\theta > 0$ over a complex hyperbolic hyperplane,*
(d) *a self-tube $M_n^*$.*

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