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NONCIRCULANT TOEPLITZ MATRICES ALL OF WHOSE POWERS ARE TOEPLITZ

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Abstract. Let a, b and c be fixed complex numbers. Let $M_n(a,b,c)$ be the $n\times n$ Toeplitz matrix all of whose entries above the diagonal are a, all of whose entries below the diagonal are b, and all of whose entries on the diagonal are c. For $1\leqslant k\leqslant n$, each $k\times k$ principal minor of $M_n(a,b,c)$ has the same value. We find explicit and recursive formulae for the principal minors and the characteristic polynomial of $M_n(a,b,c)$. We also show that all complex polynomials in $M_n(a,b,c)$ are Toeplitz matrices. In particular, the inverse of $M_n(a,b,c)$ is a Toeplitz matrix when it exists.

Keywords: Toeplitz matrix, Toeplitz inverse, Toeplitz powers, principal minor, Fibonacci sequence

MSC 2010: 15A15, 15A57, 11B39, 11B37

1. Introduction

For each positive integer n and for all $a,b,c\in\mathbb{C}$, let $M_n(a,b,c)$ denote the $n\times n$ Toeplitz matrix with all entries above the diagonal equal to a, all entries below the diagonal equal to b, and all entries on the diagonal equal to c. Thus, for example,

$$M_3(a,b,c)=\begin{bmatrix} c & a & a\\ b & c & a\\ b & b & c\end{bmatrix}.$$

Using the observation that each $k\times k$ principal minor of $M_n(a,b,c)$ is just $M_k(a,b,c)$, in Section 2, we show that $\det(M_n(a,b,c))$ satisfies a linear recurrence relation. We solve that relation to obtain a simple formula for the determinant of $M_n(a,b,c)$ and to obtain the characteristic polynomial of $M_n(a,b,c)$. We also study the sequence of principal minors for $M_n(a,b,c)$ for special choices of a, b and c. In Section 3, we show that every positive integer power of $M_n(a,b,c)$ is a Toeplitz matrix, and consequently, that every complex polynomial in $M_n(a,b,c)$ is a Toeplitz matrix. In particular, when $M_n(a,b,c)$ is invertible, its inverse is a Toeplitz matrix.

2. The principal minor sequence and the characteristic polynomial for $M_n(a,b,c)$

For a matrix A, $A(1|1]$ will denote the column vector obtained by deleting the first entry of the first column of A. $A[1|1)$ will denote the row vector obtained by deleting the first entry from the first row of A. $A(1)$ will denote the principal submatrix obtained from A by deleting the first row and the first column of A.

Lemma 1. *Let $a,b,c \in \mathbb{C}$. For each positive integer n, let $M_n = M_n(a,b,c)$. Then* $\det(M_1) = c$, $\det(M_2) = c^2 - ab$, *and for* $n \geqslant 3$,

$$\det(M_n) = (2c-a-b)\det(M_{n-1}) - (a-c)(b-c)\det(M_{n-2}).$$

P r o o f. Let $n \geqslant 3$. Let H be obtained from M_n by performing two elementary operations: Subtract the second row of M_n from the first row of M_n, and subtract the second column of the resulting matrix from the first column of the resulting matrix. Thus

$$\left[\begin{array}{c|ccccc} 2c-a-b & a-c & 0 & 0 & \dots & 0 \\ \hline b-c & & & & & \\ 0 & & & & & \\ 0 & & & M_{n-1} & & \\ \vdots & & & & & \\ 0 & & & & & \end{array}\right]$$

and $\det(M_n) = \det(H)$. Apply minor-cofactor expansion to the first row of H and then to the first column of the $H(1|2)$. □

Solving the second order linear recursion for the determinant in the previous lemma yields

Theorem 2. *Let $a,b,c \in \mathbb{C}$. For each positive integer n, let $M_n = M_n(a,b,c)$.*
If $a = b = c$, then $\det(M_1) = c$, *and* $\det(M_n) = 0$ *for* $n \geqslant 2$.
If $a = b \neq c$, then for $n \geqslant 1$,

$$\det(M_n) = [c + a(n-1)](c-a)^{n-1},$$

and M_n is nonsingular unless $n = 1 - c/a$, in which case, M_n is singular.

If $a \neq b$, then for $n \geqslant 1$,

$$\det(M_n) = \frac{b}{b-a}(c-a)^n - \frac{a}{b-a}(c-b)^n,$$

and M_n is nonsingular unless

$$b(c-a)^n = a(c-b)^n.$$

P r o o f. It is well known that the second order linear recurrence $a_k = pa_{k-1} + qa_{k-2}$ for $k \geqslant 3$, where p and q are constants, with initial conditions a_1 and a_2 specified, has a unique solution. The solution is obtained as follows. Let r_1 and r_2 be the roots of the quadratic $x^2 - px - q = 0$. When $r_1 \neq r_2$, the general solution is $a_k = s_1(r_1)^{k-1} + s_2(r_2)^{k-1}$ where s_1 and s_2 are constants chosen so that a_k has the specified initial values a_1 and a_2. When $r_1 = r_2$, let r denote the common root. If $r \neq 0$, then the general solution is $a_k = [a_1 + s(k-1)]r^{k-1}$ where $s = a_2/r - a_1$. When $r = 0$, it follows that $p = q = 0$, and we have $a_k = 0$ for $k \geqslant 3$.

¿From Lemma 1, we have $p = 2c - a - b = (c-a) + (c-b)$ and $q = -(c-a)(c-b)$. Thus the quadratic is

$$x^2 - ((c-a) + (c-b))x + (c-a)(c-b) = 0.$$

Clearly, the roots are $c-a$ and $c-b$, so the roots are distinct exactly when $a \neq b$. When $a = b$, the common value for the roots is $r = c - a$. It remains to examine the initial conditions. Direct substitution shows that $a_1 = \det(M_1) = c$, and $a_2 = \det(M_2) = c^2 - ab$. Using these initial conditions leads to the specified values of s_1 and s_2.

The singularity conditions follow from simple algebra. □

Theorem 3. *Let $a, b, c \in \mathbb{C}$. For $n \geqslant 1$, let $p_n(x)$ denote the characteristic polynomial of $M_n(a,b,c)$. Then $p_n(x)$ satisfies the recursion relationship*

$$p_n(x) = (2x - 2c + a + b)p_{n-1}(x) - (x - a + c)(x - b + c)p_{n-2}(x)$$

with $p_1(x) = x - c$ and $p_2(x) = c^2ab$. Alternatively, $p_n(x)$ can be expressed as

$$p_n(x) = x^n - \sum_{k=1}^{n}(-1)^k \binom{n}{k}[\det(M_k(a,b,c))]x^{n-k}.$$

When $a = b$,

$$p_n(x) = [x - c - a(n-1)](x - c + a)^{n-1}.$$

When $a \neq b$,

$$p_n(x) = \frac{b}{b-a}(x+a-c)^n - \frac{a}{b-a}(x+b-c)^n.$$

P r o o f. Since $p_n(x) = \det(xI_n - A) = \det(M_n(-a,-b,x-c))$, apply Theorem 2 and simplify. The recurrence relationship is obtained from Lemma 1. Finally, the coefficients in the sum of powers of x come from the well-known fact that the coefficient of x^{n-k} in the characteristic polynomial for the $n \times n$ matrix A is, up to a factor of $(-1)^k$, the sum of all $k \times k$ principal minors of A. Since each of the $k \times k$ principal minors of $M_n(a,b,c)$ has value $\det(M_k(a,b,c))$, and since there are $\binom{n}{k}$ such minors, the result follows. □

The following result is an immediate consequence of the well-known Gershgorin Circles Theorem:

Theorem 4. *Let* $a, b, c \in \mathbb{C}$. *Let* n *be a positive integer. If* λ *is an eigenvalue of* $M_n(a,b,c)$, *then*

$$|\lambda - c| \leqslant (n-1)\max\{|a|,|b|\}.$$

In particular, if $|c| > (n-1)\max\{|a|,|b|\}$, *then* $M_n(a,b,c)$ *is nonsingular.*

What can be said about the rank of $M_n(a,b,c)$ when the matrix is singular?

Observe that $\operatorname{rank}(M_n(a,b,c))$ must be $n-1$ unless $M_{n-1}(a,b,c)$ is also singular. This leads to the following result:

Theorem 5. *Let* $a, b, c \in \mathbb{C}$. *For each positive integer* n, *let* $M_n = M_n(a,b,c)$

(i) *If* $a = b = c$, *then* $\operatorname{rank}(M_n) = 1$ *if* $c \neq 0$, *and* $\operatorname{rank}(M_n) = 0$ *if* $c = 0$

(ii) *If* $a = b \neq c$, *then* $\operatorname{rank}(M_n) = n$ *except when* $n = 1 - c/a$, *in which case,* $\operatorname{rank}(M_n) = n - 1$.

(iii) *If* $a \neq b$, *then* $\operatorname{rank}(M_n) = n$ *unless*

$$b(c-a)^n = a(c-b)^n. \tag{1}$$

If the equality holds, then $\operatorname{rank}(M_n) = n - 1$.

P r o o f. All but the last part of (iii) follow immediately from Theorem 2.

Suppose that equality (1) holds, that $a \neq b$, and that $ab = 0$. Then equality (1) forces $c = 0$, and the result follows from the fact that M_n is strictly triangular with either all entries below the diagonal or all entries above the diagonal nonzero.

Suppose that equality (1) holds, that $a \neq b$, and that $ab \neq 0$. Since $c-a$ and $c-b$ are distinct, it follows from equality (1) that $c - a \neq 0$ and $c - b \neq 0$. Thus

$$\left(\frac{c-a}{c-b}\right)^n = \frac{b}{a} \neq 0.$$

If $\operatorname{rank}(M_n) < n-1$, then $M_{n-1}(a,b,c)$ is singular, and hence,

$$\left(\frac{c-a}{c-b}\right)^{n-1} = \frac{b}{a} \neq 0.$$

Then

$$\frac{c-a}{c-b} = 1,$$

which implies $a = b$, a contradiction. □

When are a and b themselves the roots of the recursion relationship for the determinant?

Exactly when $\{a,b\} = \{c-a, c-b\}$. This is equivalent to $a+b=c$. We note several interesting cases when $a+b=c$.

Lemma 6. *Let $a \in \mathbb{C}$. For each positive integer n, let $N_n = M_n(a,-a,0)$. If $a=0$, then $\det(N_n) = 0$ for all $n \geqslant 1$. If $a \neq 0$, then*

$$\det(N_k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ a^k & \text{if } k \text{ is even.} \end{cases}$$

Finally, when $a \neq 0$ and k is odd, $\operatorname{rank}(N_k) = k-1$, and the null space of N_k is spanned by the vector $v = [\,1\ \ -1\ \ 1\ \ -1\ \ \ldots\ \ 1\ \ -1\ \ 1\,]^T$.

Proof. Applying Theorems 2 and 4 with $b=-a$ and $c=0$ yields the formulae for $\det(N_k)$ and the rank result. When k is odd, each odd numbered row of N_k contains an even number of -1 entries, followed by 0, followed by an even number of 1 entries. Consequently, the alternating sum in the dot product of the row with v is zero. When k is odd, the first and last entry of each even numbered row of N_k have opposite signs in the dot product with v, and hence cancel each other, leaving an even number of consecutive -1 entries and an even number of consecutive 1 entries; thus the remaining terms produce an alternating sum summing to zero. □

Lemma 7. *Let $\varphi = (1+\sqrt{5})/2$, the golden ratio. For each positive integer n, let $P_n = M_n(\varphi, 1-\varphi, 1)$ and let $Q_n = M_n(-\varphi, \varphi-1, 0)$. Then $\det(P_n)$ is the $(n+1)$st Fibonacci number F_{n+1}, and $\det(Q_n)$ is the $(n-1)$st Fibonacci number F_{n-1}, where the Fibonacci sequence is given its classical indexing starting with $F_0 = 0$ and $F_1 = F_2 = 1$.*

Proof. For P_n, the choice of a, b and c yields $p = q = 1$ in the proof of Theorem 2. So the recursion for $\det(P_n)$ is

$$\det(P_n) = \det(Q_{n-1}) + \det(Q_{n-2}), \quad n \geqslant 3$$

with the initial conditions

$$\det(M_1) = 1 \quad \text{and} \quad \det(M_2) = 1 - \varphi(1-\varphi) = 2.$$

For Q_n, the choice of a, b and c again yields $p = q = 1$, so we again get the Fibonacci recursion. This time the initial conditions are

$$\det(M_1) = 0 \quad \text{and} \quad \det(M_2) = 0 - (-\varphi)(\varphi - 1) = 1.$$

□

The well-known matrix generator for the Fibonacci numbers is

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} F_k & F_{k-1} \\ F_{k-1} & F_{k-2} \end{bmatrix}$$

where $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is a Hessenberg Toeplitz matrix whose eigenvalues are φ and $-\varphi$. Thus the matrices $M_n(\varphi, 1-\varphi, 1)$ and $M_n(-\varphi, \varphi - 1, 0)$ provide another connection between matrices, the Fibonacci sequence, and the golden ratio.

Which principal minor sequences $s = (s_1, s_2, \ldots, s_n)$ can be obtained from a matrix of the form $M_n(a, b, c)$?

Clearly, we must have $s_1 = c$ and $s_2 = c^2 - ab$, and $s_k = ps_{k-1} + qs_{k-2}$ for $2 \leqslant k \leqslant n$ where $p = 2c - a - b$ and $q = -(a-c)(b-c)$. Since the initial conditions together with p and q completely determine the sequences, what we are really asking is which 4-tuples (s_1, s_2, p, q) can be realized by appropriate choices of a, b and c. Since $s_1 = c$ and $s_2 = c^2 - ab$, we must have $ab = s_1^2 - s_2$. Given p, we must have $a + b = 2c - p$. Finally, since

$$q = -(c-a)(c-b) = (c^2 - ab) - (2c - a - b)c = s_2 - ps_1,$$

the value for a realizable q is dependent on the choices for s_1, s_2 and p. Specifically, we have shown that:

Theorem 8. *Given $a_1, a_2, p, q \in \mathbb{C}$, the linear recursion $a_k = pa_{k-1} + qa_{k-2}$ for $k \geqslant 2$ with initial conditions a_1 and a_2 can be realized as the sequence of principal minors for a matrix $M_n(a, b, c)$ exactly when $q = a_2 - pa_1$. In this case, the linear recursion and the initial conditions are achieved by setting $c = a_1$, and by setting a and b to be the roots of $x^2 + (p - 2a_1)x + (a_1^2 - a_2) = 0$.*

As a special but interesting case, we determine matrices all of whose principal minors of every order have the value x where x is an arbitrary complex number.

Theorem 9. *For all positive integers n and for all $x \in \mathbb{C}$, all of the principal minors of $R_n = M_n(x, x-1, x)$ are equal to x.*

Proof. By Lemma 1, for $n \geqslant 3$,

$$\begin{aligned}\det(R_n) &= (2x - x - (x-1))\det(R_{n-1}) - (x - x)((x-1) - x)\det(R_{n-2}) \\ &= \det(R_{n-1})\end{aligned}$$

with $\det(R_1) = x$ and $\det(R_2) = x^2 - x(x-1) = x$. □

Remark 10. In [3] the inverse problem of constructing a matrix from its principal minors is considered. Under certain conditions, this problem has a solution that is produced by the algorithm pm2mat. When $x \notin \{0, 1\}$, the matrix $M_n(x, x-1, x)$ in Theorem 9 is (up to diagonal similarity and transposition) the output of the algorithm pm2mat in [3] when all principal minors are required to equal x.

Moreover, in agreement with the above comment, $M_n(x, x-1, x)$ and $M_n(x-1, x, x)$ are the only choices of matrices of the form $M_n(a, b, c)$ with the property that all principal minors equal x. Indeed, it must be that $c = x$; enforcing the 2×2 and 3×3 principal minors be equal to x imposes that

$$ab = x(x-1) \quad \text{and} \quad a + b = 2x - 1$$

whose only solutions are $(a = x,\ b = x-1)$ or $(a = x-1,\ b = x)$.

Finally note, that by Theorem 8, the Fibonacci sequence cannot be obtained as $F_n = M_n(a, b, c)$ for any $a, b, c \in \mathbb{C}$, since this indexing corresponds to the 4-tuple $(1, 1, 1, 1)$, and $q \neq 1 - (1)(1)$.

3. Powers of $M_n(a, b, c)$ are Toeplitz matrices

We begin this section by recalling some definitions and by stating several elementary results.

The $n \times n$ matrix A is said to be *persymmetric* if $J_n A^T J_n = A$ where J_n is the $n \times n$ permutation matrix with ones on the cross-diagonal.

Observe that $J_n = J_n^T = J_n^{-1}$, and that if e_n denotes the $n \times 1$ vector of ones, then $J_n e_n = e_n$ and $e_n^T J = e_n^T$.

The $n \times n$ matrix $A = [a_{ij}]$ is said to be *Toeplitz* if there exist $2n - 1$ scalars

$$a_{-n+1},\ a_{-n+2},\ \ldots,\ a_{-1},\ a_0,\ a_1,\ \ldots,\ a_{n-2},\ a_{n-1}$$

such that $a_{ij} = a_{i-j}$. That is, the entries on each diagonal of a Toeplitz matrix descending from left to right have a common value.

Lemma 11. *Let A be a persymmetric matrix. Then A^k is persymmetric for every positive integer k. If A^{-1} exists, then A^k is persymmetric for every negative integer k.*

Note that every Toeplitz matrix is persymmetric. The following result is a partial converse.

Lemma 12 [4, Lemma 1]**.** *Let A be an $n \times n$ persymmetric matrix with $n \geqslant 2$. Then A is a Toeplitz matrix if and only if $A(1)$ is persymmetric.*

Theorem 13. *For all positive integers n, for all polynomials $p(x)$ in $\mathbb{C}[x]$, and for all $a, b, c \in \mathbb{C}$, the matrix $p(M_n(a,b,c))$ is a Toeplitz matrix. In particular, all positive integer powers of $M_n(a,b,c)$ are Toeplitz matrices. Further, if $M_n(a,b,c)$ is invertible, then its inverse is a Toeplitz matrix.*

P r o o f. Since the set of $n \times n$ Toeplitz matrices is a subspace of the set of $n \times n$ complex matrices, it suffices to prove that each positive integer power of $M_n(a,b,c)$ is a Toeplitz matrix in order to prove the result for polynomials in $M_n(a,b,c)$. Since the inverse of a matrix, when it exists, is a polynomial in the matrix, the result on inverses is clear. Since cI_n is a Toeplitz matrix, $M_n(a,b,c)$ is a Toeplitz matrix if and only if $M_n(a,b,c) - cI_n = M_n(a,b,0)$ is a Toeplitz matrix. Since the kth power of $M_n(a,b,c)$ is a polynomial in I_n and positive integer powers of $M_n(a,b,0)$, it suffices to prove that all positive integers powers of $M_n(a,b,0)$ are Toeplitz matrices. If $a \neq 0$, then $M_n(a,b,0) = aM_n(1,b/a,0)$, and consequently, the kth power of $M_n(a,b,c)$ is a Toeplitz matrix if and only if the kth power of $M_n(1,b/a,0)$ is a Toeplitz matrix. If $a = 0$, then $M_n(0,b,0) = bM_n(0,1,0)$, and all powers of the nilpotent matrix $M_n(0,1,0)$ are known to be Toeplitz matrices. Thus it suffices to prove that an arbitrary positive integer power of $N = M_n(1,b,0)$ is a Toeplitz matrix when $b \neq 0$.

Since A is a Toeplitz matrix, A and $A(1)$ are persymmetric, and A^k is persymmetric for every positive integer k. We will use induction on k to prove that A^k is a Toeplitz matrix. Specifically, for each k, we will prove that $A^k(1)$ is persymmetric and that $bJ_{n-1}(A^k[1|1))^T = A^k(1|1]$. Clearly, when $k = 1$, $A(1)$ is persymmetric by Lemma 11, and $bJ_{n-1}(A[1|1))^T = bJ_{n-1}(e_{n-1}^T)^T = be_{n-1} = A(1|1]$. Suppose that the induction hypothesis holds for k. Observe that

$$A = \begin{bmatrix} 0 & e_{n-1}^T \\ be_{n-1} & A(1) \end{bmatrix}$$

and that we can write A^k as

$$A^k = \begin{bmatrix} \alpha & u^T \\ v & M \end{bmatrix},$$

where $\alpha \in \mathbb{C}$, u and v are $(n-1)\times 1$ vectors and $M = A^k(1)$. By the induction hypothesis, $bJ_{n-1}u = v$, and M is persymmetric. Writing $A^{k+1} = AA^k = A^kA$ gives

$$A^{k+1} = \begin{bmatrix} e^T v & e^T M \\ \alpha be + A(1)M & beu^T + A(1)M \end{bmatrix} = \begin{bmatrix} bu^T e & \alpha e^T + u^T A(1) \\ bMe & ve^T + MA(1) \end{bmatrix}.$$

Since M is persymmetric, $JM^T = MJ$. Thus

$$bJ_{n-1}(A^{k+1}[1|1))^T = bJ(e^TM)^T = bJM^Te = bMJe = bMe = A^{k+1}(1|1].$$

Next,

$$\begin{aligned} J(A^{k+1}(1))^TJ = J(beu^T + A(1)M)^TJ &= bJue^TJ + JM^T(A(1))^TJ \\ &= (bJu)e^T + MJ(A(1))^TJ. \end{aligned}$$

Since $A(1)$ is persymmetric and since, by the induction hypothesis, $bJu = v$,

$$J(A^{k+1}(1))^TJ = ve^T + MA(1) = A^{k+1}(1).$$

Thus $A^{k+1}(1)$ is persymmetric. Thus the induction hypothesis holds for $k+1$. By the principle of induction, we have the desired result, that $A^k(1)$ is persymmetric for all positive integers k. By applying Lemma 11, we conclude that that A^k is a Toeplitz matrix for all positive integers k. □

Note added just prior to publication: Theorem 13 also follows from Theorem 1.3 of [5].

References

[1] *K. Griffin*: Solving the principal minor assignment problem and related computations. PhD. Dissertation. Washington State University, Washington, 2006.

[2] *K. Griffin, M. J. Tsatsomeros*: Principal minors, Part I: A method for computing all the principal minors of a matrix. Linear Algebra Appl. *419* (2006), 107–124.

[3] *K. Griffin, M. J. Tsatsomeros*: Principal minors, Part II: The principal minor assignment problem. Linear Algebra Appl. *419* (2006), 125–171.

[4] *N. M. Huang, R. E. Cline*: Inversion of persymmetric matrices having Toeplitz inverses. J. Assoc. Comput. Mach. *19* (1972), 437–444.

[5] *T. Shalom*: On algebras of Toeplitz matrices. Linear Algebra Appl. *96* (1987), 211–226.

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