Wladimir G. Boskoff; Bogdan D. Suceavă

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BARBILIAN'S METRIZATION PROCEDURE IN THE PLANE YIELDS EITHER RIEMANNIAN OR LAGRANGE GENERALIZED METRICS

Wladimir G. Boskoff, Constanţa, and Bogdan D. Suceavă, Fullerton

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Abstract. In the present paper we answer two questions raised by Barbilian in 1960. First, we study how far can the hypothesis of Barbilian's metrization procedure can be relaxed. Then, we prove that Barbilian's metrization procedure in the plane generates either Riemannian metrics or Lagrance generalized metrics not reducible to Finslerian or Langrangian metrics.

Keywords: Riemannian metrics, Finslerian metrics, Lagrangian metrics, Lagrange generalized metrics, Barbilian's metrization procedure, Apollonian metric

MSC 2010: 53B40, 53C60, 51K05, 30C62

1. Introduction

Barbilian's metrization procedure was introduced in its simplest form in 1934 in [2] in order to generalize a construction inspired from the study of the Klein-Beltrami model of non-Euclidean geometry. Later contributions on the topic of Barbilian spaces include P. J. Kelly's work [26] and major developments are due to D. Barbilian himself [3], [4], [5], [6]. Recently, many citations of the work [2], originally published in Časopis Mathematiky a Fysiky, have appeared, for example, in [7], [11], [13], [16], [18], [19], [20], [21], [22], [23], [24], [25]. The history of this subject is presented in [14], [15].

The following construction describes Barbilian's metrization procedure in its most general setting. It originates in [3] and it develops the idea from [2].

Consider two arbitrary sets K and J. The function $f\colon K\times J\to\mathbb{R}_+^*$ is called an influence of the set K over J if for any $A,B\in J$ the ratio $g_{AB}(P)=f(P,A)/f(P,B)$ has a maximum $M_{AB}\in\mathbb{R}$ when $P\in K$. Note that $g_{AB}\colon K\to\mathbb{R}_+^*$. In [3] it is pointed out that if we assume the existence of $\max g_{AB}(P)$, when $P \in K$, then there also exists $m_{AB} = \min\limits_{P \in K} g_{AB}(P) = 1/M_{BA}$.

In the particular case when T is a topological space, K a compact subset in T, and J some arbitrary subset, then any function $f\colon\ K \times J \to \mathbb{R}_+^*$ continuous in the first argument is an influence on J. It has been known since [3] that $d\colon\ J \times J \to \mathbb{R}_+$ given by

$$(1) \qquad d(A,B) = \ln \frac{\max\limits_{P \in K} g_{AB}(P)}{\min\limits_{P \in K} g_{AB}(P)}$$

is a semidistance, i.e.: (a) if $A = B$ then $d(A,B) = 0$; (b) d is symmetric; (c) d satisfies the triangle inequality.

The influence $f\colon\ K \times J \to \mathbb{R}_+^*$ is called effective if there is no pair $(A,B) \in J \times J$ such that the ratio $g_{AB}(P) = f(P,A)/f(P,B)$ is constant for all $P \in K$. In [3] it is shown that if $f\colon\ K \times J \to \mathbb{R}_+^*$ is an effective influence, then (1) is a distance.

In [16] an extension of Barbilian's metrization procedure is presented. The geometric motivation for extending the procedure is the fact that in the case when K is a circle in the plane and J is its interior, if we remove one point L from K, we can not apply the classical Barbilian's metrization procedure considering the influence of $K - \{L\}$ over J. Suppose that K and J are arbitrary sets and that they satisfy the *general extremum requirement*, that is for any A and B in J there exists $\sup g_{AB}(Q) < \infty$, when $Q \in K$. As we have seen in the case of maximum, presented above, if there exists $\sup\limits_{P \in K} g_{AB}(P) < \infty$ then there exists $\inf\limits_{P \in K} g_{AB}(P)$ and it equals $[\sup\limits_{P \in K} g_{BA}(P)]^{-1}$.

Theorem 1 [16]**.** *Suppose that g satisfies the general extremum requirement. Then the function $d^s\colon\ J \times J \to \mathbb{R}_+$ given by*

$$d^s(A,B) = \ln \frac{\sup\limits_{P \in K} g_{AB}(P)}{\inf\limits_{P \in K} g_{AB}(P)}$$

is a semidistance on J.

For this extension of Barbilian's metrization procedure we can drop the continuity requirement for the influence function. This result is presented in Section 3.

For a historical account on Barbilian's metrization procedure, see [14], [15]. Over the years, the paper [2] has been cited many times (e.g. [7], [8], [9], [10], [12], [13], [18], [19], [20], [21], [22], [23], [24], [25]). The geometric viewpoint is discussed in the monograph [11].

We aim in the present work at clarifying the role of Barbilian's metrization procedure in the context of its relations with various classes of metrics, such as for example the Riemann, Finsler, Lagrange or generalized Lagrange metrics. To this goal, we would like to remind here a few of the basic definitions.

In [17], p. 265, it is pointed out that the origin of Finsler geometry can be found in B. Riemann's historical Habilitation address, from 1854. More precisely, suppose that M is a real smooth finite dimensional manifold and let $\tau\colon TM \to M$ its tangent bundle. Let $(U,(x^i))$ be a local chart on M; our convention is that indices $i,j,k,\dots$ run from 1 to $n=\dim M$ and we are using Einstein's convention on summation. Associate to any section $v\in\tau^{-1}(U)$ the coordinates $(x^i(\tau(u)))$ and (y^i) such that for $\partial_i=\partial/\partial x^i$. A change of coordinates $(x^i,y^i)\to(x^{i'},y^{i'})$ on the smooth orientable manifold TM is

$$x^{i'}=x^{i'}(x^1,\dots,x^n),\quad y^{i'}=(\partial_j x^{i'})y^j,\quad \operatorname{rank}(\partial_j x^{i'})=n. \tag{2}$$

A Finsler structure or Finsler function of M is a function $F\colon TM\to[0,\infty)$, $(x,y)\to F(x,y)$ with the properties

(i) F is smooth on the slit tangent bundle $TM\setminus\{0\}$,
(ii) F is positively homogeneous of degree one in the y's, that is $F(x,\lambda y)=\lambda F(x,y)$, for all $\lambda>0$, and
(iii) the matrix

$$g_{ij}=\frac{1}{2}\frac{\partial^2F^2(x,y)}{\partial y^i\partial y^j}$$

is positive definite at every point of $TM\setminus\{0\}$.

The pair (M,F) is called a Finsler manifold and one says that $g_{ij}(x,y)$ is its Finsler metric. Notice that by the positive homogeneity it follows that $F^2(x,y)=g_{ij}(x,y)y^iy^j$. If $g_{ij}(x,y)$ do not depend on y's, the Finsler manifold (M,F) becomes a Riemannian manifold with the metric $\mathrm{d}s^2=g_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j$.

There are several generalizations of Finsler geometry (see [1], [27]). To us, for the present paper, of interest are Lagrange and generalized Lagrange geometries.

It is said that a set of matrices $g_{ij}(x,y)$ define a generalized Lagrange metric if they satisfy the following three requirements:

(i) A change of coordinates (2) implies

$$g_{ij}(x,y)=(\partial_i x^{i'})(\partial_j x^{j'})g_{i'j'}(x',y').$$

(ii) Symmetry: $g_{ij}(x,y)=g_{ji}(x,y)$.
(iii) Non-degeneracy: $\det(g_{ij}(x,y))\neq 0$.

A generalized Lagrange metric is said to be a Lagrange metric if there exists a smooth function $L\colon TM \to \mathbb{R}$ (called a Lagrangian) such that

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L(x,y)}{\partial y^i \partial y^j}.$$

The pair (M, L) is called a Lagrange manifold. Every Finsler manifold is a particular Lagrange manifold with $L = F^2$. A necessary and sufficient condition for a generalized Lagrange metric to be a Lagrange metric is that $C_{ijk} = \frac{1}{2}\partial g_{ij}/\partial y^k$ be totally symmetric.

These definition are used in Section 4.

2. How general is Barbilian's metrization procedure?

We have seen that Barbilian's metrization procedure appeared as a generalization of a configuration studied in the Klein-Beltrami model of hyperbolic geometry. Does that mean that the procedure generates only hyperbolic distances? Barbilian obtained in [3] the Poincaré metric in the disk. Too see this, consider $\mathbb{R}^2$ endowed with the Euclidean distance $\|\cdot\|$. Then, in [3] it is shown that for any circle K in $\mathbb{R}^2$, and for J the interior of K, by taking $f(P, A) = \|PA\|$, Barbilian's metrization procedure yields the Poincaré metric in the disk. Barbilian pointed out that for various choices of K, J and f a large array of distances could be obtained.

We show in this section that Euclidean and spherical distances can also be obtainted by Barbilian's metrization procedure.

Proposition 1. *Barbilian's metrization procedure yields the Euclidean distance in the interior of disks in* $\mathbb{R}^2$.

Proof. In the spirit of the theory reminded above, consider K to be any circle in $\mathbb{R}^2$ and J its interior. Let us take the function $f(M, A) = \mathrm{e}^{\frac{1}{2}\|MA\|}$, for all $M \in K$, and $A \in J$. Clearly

$$g_{AB}(M) = \mathrm{e}^{\frac{1}{2}(\|MA\| - \|MB\|)}.$$

By the triangle inequality, $\big|\|MA\| - \|MB\|\big| \leqslant \|AB\|$, which shows that the maximum and the minimum are attained and

$$\max_{M\in K} g_{AB}(M) = \mathrm{e}^{\frac{1}{2}\|AB\|}, \qquad \min_{M\in K} g_{AB}(M) = \mathrm{e}^{-\frac{1}{2}\|AB\|}.$$

Thus, the distance generated by Barbilian's metrization procedure is

$$d^B(A,B) = \ln \frac{\max\limits_{M\in K} g_{AB}(M)}{\min\limits_{M\in K} g_{AB}(M)} = \ln \mathrm{e}^{\|AB\|} = \|AB\|.$$

□

Proposition 2. *Barbilian's metrization procedure yields the spherical distance on a hemisphere in $S^2 \subset \mathbb{R}^3$.*

Proof. As in the previous proposition, we take K to be a great circle on the sphere and J one of the hemispheres bounded by K. Denote by (MA) the spherical distance on S^2. Then consider the influence $f\colon K \times J \to \mathbb{R}_+$ given by $f(M, A) = \mathrm{e}^{\frac{1}{2}(MA)}$. As in Proposition 1, we remark that

$$\frac{f(M,A)}{f(M,B)} = \mathrm{e}^{\frac{1}{2}[(MA)-(MB)]}.$$

By using the triangle inequality, we obtain the maximum and the minimum of the ratio $g_{AB}(M)$, which yields $d^B(A, B) = (AB)$. □

The spaces obtained in Propositions 1 and 2 are not complete. They enjoy the property that the sets K and J are geometrically related, more precisely that K is a simple closed curve and J a region delimited by K. However, we can produce the Euclidean metric in the whole plane as a complete space.

Proposition 3. *Barbilian's metrization procedure yields the Euclidean distance in a plane (π) in $\mathbb{R}^3$.*

Proof. Consider a plane (δ) parallel to the plane (π) and take $J = (\pi), K = (\delta)$. Let $M \in K$ and $A, B \in (\pi)$. Denote by $\mathrm{Pr}\colon (\delta) \to (\pi)$ the orthogonal projection on (π), $\mathrm{Pr}(M) = M'$, and by $\|\cdot\|$ the Euclidean distance. Consider the influence function $f\colon K \times J \to \mathbb{R}_+^*$, $f(M, A) = \exp o[\frac{1}{2} \cdot |(\mathrm{Pr} \times \mathrm{Id})(M, A) \cdot |] = \mathrm{e}^{\frac{1}{2}\|M'A\|}$. We have

$$\frac{f(M,A)}{f(M,B)} = \mathrm{e}^{\frac{1}{2}[\|M'A\|-\|M'B\|]}.$$

Since $-\|AB\| \leqslant \|M'A\| - \|M'B\| \leqslant \|AB\|$, with equality when M', A, B are collinear, we see that the induced distance in J is $d^B(A, B) = \|AB\|$. □

Proposition 4. *Barbilian's metrization procedure yields the spherical distance in a complete sphere in $\mathbb{R}^3$.*

Proof. Consider two concentric spheres S_1 and S_2 in $\mathbb{R}^3$, and let their common center be O. We take $S_1 = K$ and $S_2 = J$, and $A, B \in J$ and $M \in K$. Denote by $\{M'\} = OM \cap J$ and define Pr to be the radial projection from S_1 to S_2 given by $\mathrm{Pr}(M) = M'$. Denote by $(\cdot)$ the spherical distance, and consider the influence function $f\colon K \times J \to \mathbb{R}_+^*$, $f(M, A) = \exp o[\frac{1}{2}((\mathrm{Pr} \times \mathrm{Id})(M, A))] = \mathrm{e}^{\frac{1}{2}(M'A)}$. Then the argument is concluded similarly as in the previous Proposition. □

3. A discontinuous influence that yields a distance

It is known that the influence functions continuous in the first variable generate semidistances and distances [3]. In this section we show by example that it is still possible to obtain a distance by Barbilian's metrization procedure if we have an influence function $f\colon\ K \times J \to \mathbb{R}_+^*$ which is not continuous. As far as we know, such an example does not appear elsewhere in the literature. However, the origin of this problem may be traced back to [3] and especially [5], where the problem is specifically mentioned of how much the hypotheses of Barbilian's metrization procedure can be relaxed.

To this goal, consider $K =$ the circle given by the equation $x^2 + y^2 = 1$ in the two-dimensional real plane, and consider J the set delimited by K. Define the influence f as follows. For any $P \in K$ and $A \in J \cap \{(x, y) \in \mathbb{R}^2 \colon\ y \geqslant 0\}$ let

$$f(P, A) = \|PA\| + 1.$$

For any $P \in K$ and $A \in J \cap \{(x, y) \in \mathbb{R}^2 \colon\ y < 0\}$ let

$$f(P, A) = \|PA\|.$$

We claim that with this construction the function

$$g_{AB}(P) = \frac{f(P, A)}{f(P, B)},$$

for any $A, B \in J$, admits supremum and infimum. This means that the general extremum requirement is satisfied, in the terminology of [3], and therefore Barbilian's metrization procedure yields a semidistance. To see this last claim, consider $A, B \in J$. Then we have

$$\frac{f(P, A)}{f(P, B)} \leqslant \frac{\|PA\| + 1}{\|PB\|} < \frac{3}{\|PB\|} < \infty.$$

Furthermore, since the circle $x^2 + y^2 = 1$ is not included in the geometric locus of the points P for which $f(P, A)/f(P, B) =$ constant, the semidistance constructed above is actually a distance.

This example shows that the continuity condition stated in [3] for the example recalled in the first section is not essential. The essential condition is the so-called general extremum requirement, more precisely the existence of a supremum and of an infimum.

This analysis leads us to the weakest hypothesis necessary to have the construction from Barbilian's metrization procedure. More precisely, if we consider K as the punctured circle, $K = S^1 - \{p\}$, for some arbitrary point of $x^2 + y^2 = 1$, the remaining set is not compact in the topology induced from $\mathbb{R}^2$. If we use the previous noncontinous influence on a noncompact set, and if we drop Barbilian's original extremum requirement [3] regarding the existence of a maximum, replacing it with the existence of a supremum, we obtain *the most general extent of Barbilian's metrization procedure.*

4. What classes of metrics does Barbilian metrization procedure yield?

We mention here the following result, which is a particular form of the result from [5], part 2, paragraph 7, and a version of the argument used by P. A. Hästö in [22], in the proof of Lemma 3.5.

Lemma 1. *Let K be a simple closed curve in $\mathbb{R}^2$ and J the interior region delimited by K. Consider the influence $f(M,A) = \|MA\|$, where by $\|MA\|$ we denote the Euclidean distance. Consider*

$$g_{AB}(M) = \frac{f(M,A)}{f(M,B)} = \frac{\|MA\|}{\|MB\|}$$

and consider the distance induced on J by the Barbilian's metrization procedure, $d^B(A,B)$. Suppose furthermore that for $M \in K$ the extrema $\max g_{AB}(M)$ and $\min g_{AB}(M)$ for any A and B in J are attained each in a unique point of K. Then:

(a) *For any $A \in J$ and any line d passing through A there exist exactly two circles tangent to K and to d in A.*

(b) *The metric induced by the Barbilian distance has the form*

$$\mathrm{d}s^2 = \frac{1}{4}\Big(\frac{1}{R} + \frac{1}{r}\Big)^2 (\mathrm{d}x_1^2 + \mathrm{d}x_2^2), \tag{3}$$

where R and r are the radii of the circles described in (a)*.*

It would be interesting to understand what classes of metrics are generated by the metric given by (3). We prove here the following classification result.

Theorem 2. *Barbilian's metrization procedure for K and J two subsets of the Euclidean plane $\mathbb{R}^2$, and $f(M,A) = \|MA\|$, yields either a Riemannian metric or a Lagrange generalized metric not reducible to a Finslerian or a Lagrangian metric.*

P r o o f. According to Lemma 1, from a metric that has the form

$$\mathrm{d}s^2 = \frac{1}{4}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)^2 \,\mathrm{d}\sigma^2$$

we get $\mathrm{d}s^2 = g_{11}(\mathrm{d}x^2 + \mathrm{d}y^2)$, which means $g_{11} = g_{22}$ and $g_{12} = g_{21} = 0$.

If $g_{11} = g_{11}(x, y)$, we obtain a Riemannian metric.

On the other hand, if $g_{11} = g_{11}(x, y, \dot{x}, \dot{y})$, the metric is a Lagrange generalised metric (see [27]).

The metric is reducible to a Lagrangian metric if and only if the Cartan tensor $C_{ijk} = \frac{1}{2}\partial g_{ij}/\partial x^k$ is totally symmetric (see [27], section 4.1, Theorem 1.1). For us, for $k = 1$ we have $\partial/\partial\dot{x}$, and for $k = 2$ we have $\partial/\partial\dot{y}$. Since

$$\frac{\partial g_{11}}{\partial \dot{y}} \neq \frac{\partial g_{12}}{\partial \dot{x}} = 0,$$

the symmetry of the Cartan tensor can never occur. Therefore, in this second case the metric is irreducible to a Riemannian, Finslerian or Lagrangian metric. □

This theorem gives the final answer to a problem raised by Barbilian originally in [5]. More precisely, we have seen what metrics can naturally be obtained by Barbilian's metrization procedure.

Let us discuss here two examples. For the beginning, consider $\mathbb{R}^2$ endowed with the Euclidean distance $\|\cdot\|$. It is known from [3] that for any circle K of radius ϱ in $\mathbb{R}^2$, and for J the interior region delimited by K, a Barbilian distance is obtained in J by taking the influence $f(P, A) = \|PA\|$. By a straightforward computation, one can easily see that this yields a Riemannian metric.

For the second example, consider $K =$ the ellipse given by $x^2/a^2 + y^2/b^2 = 1$, and by J the region in the plane described by $x^2/a^2 + y^2/b^2 < 1$. It is difficult to compute the metric induced by (3). However, if for a point A from J and two lines of slopes m_1 and m_2 passing through A we have $g_{11}(x, y, m_1) \neq g_{11}(x, y, m_2)$, the metric is, by Theorem 2, a Lagrange generalised metric not reducible to a Riemannian, Finslerian or Lagrangian metric. Specifically, choosing $A = O(0, 0)$ and m_1 in the direction corresponding to the $x-$axis, we get $g_{11} = g_{22} = 4/b$. If we take m_2 in the direction corresponding to the $y-$axis, then $g_{11} = g_{22} \neq 4/b$, because the two circles of radii $b/2$ can not be tangent to the ellipse. Consequently, the metric induced on J by Theorem 2 is a Lagrange generalized metric, and this fact can be established even without computing specifically the metric.

In the view of Theorem 2, it might be of interest to note that Zair Ibragimov has shown that the Barbilian metric is Riemannian in zero, one or all points of the domain (he uses the term "conformal" for Riemannian). See [24], [25] for this fact and related implications.

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References

[1] *M. Anastasiei*: Certain generalizations of Finsler metrics. Contemp. Math. *196* (1996), 161–169.
[2] *D. Barbilian*: Einordnung von Lobayschewskys Massenbestimmung in einer gewissen allgemeinen Metrik der Jordansche Bereiche. Časopis Mathematiky a Fysiky *64* (1934–35), 182–183.
[3] *D. Barbilian*: Asupra unui principiu de metrizare. Stud. Cercet. Mat. *10* (1959), 68–116.
[4] *D. Barbilian*: Fundamentele metricilor abstracte ale lui Poincaré şi Carathéodory ca aplicaţie a unui principiu general de metrizare. Stud. Cercet. Mat. *10* (1959), 273–306.
[5] *D. Barbilian*: J-metricile naturale finsleriene. Stud. Cercet. Mat. *11* (1960), 7–44.
[6] *D. Barbilian and N. Radu*: J-metricile naturale finsleriene şi funcţia de reprezentare a lui Riemann. Stud. Cercet. Mat. *12* (1962), 21–36.
[7] *A. F. Beardon*: The Apollonian metric of a domain in $\mathbb{R}^n$, in Quasiconformal mappings and analysis. Springer-Verlag, 1998, pp. 91–108.
[8] *W. G. Boskoff*: *S*-Riemannian manifolds and Barbilian spaces. Stud. Cercet. Mat. *46* (1994), 317–325.
[9] *W. G. Boskoff*: Finslerian and induced Riemannian structures for natural Barbilian spaces. Stud. Cercet. Mat. *47* (1995), 9–16.
[10] *W. G. Boskoff*: The connection between Barbilian and Hadamard spaces. Bull. Math. Soc. Sci. Math. Roum., Nouv. Sr. *39* (1996), 105–111.
[11] *W. G. Boskoff*: Hyperbolic geometry and Barbilian spaces, Istituto per la Ricerca di Base. Hardronic Press, 1996.
[12] *W. G. Boskoff*: A generalized Lagrange space induced by the Barbilian distance. Stud. Cercet. Mat. *50* (1998), 125–129.
[13] *W. G. Boskoff and P. Horja*: The characterization of some spectral Barbilian spaces using the Tzitzeica construction. Stud. Cercet. Mat. *46* (1994), 503–514.
[14] *W. G. Boskoff and B. D. Suceavă*: Barbilian spaces: the history of a geometric idea. Historia Mathematica *34* (2007), 221–224.
[15] *W. G. Boskoff and B. D. Suceavă*: The history of Barbilian metrization procedure and Barbilian spaces. Mem. Secţ. Ştiinţ. Acad. Română Ser. IV *28* (2005), 7–16.
[16] *W. G. Boskoff, M. G. Ciucă and B. D. Suceavă*: Distances induced by Barbilian's metrization procedure. Houston J. Math. *33* (2007), 709–717.
[17] *S. S. Chern, W. H. Chen and K. S. Lam*: Lectures on Differential Geometry. World Scientific, 1999.
[18] *F. W. Gehring and K. Hag*: The Apollonian metric and quasiconformal mappings. Contemp. Math. *256* (2000), 143–163.
[19] *P. A. Hästö*: The Appollonian metric: uniformity and quasiconvexity. Ann. Acad. Sci. Fennicae *28* (2003), 385–414.
[20] *P. A. Hästö*: The Apollonian metric: limits of the approximation and bilipschitz properties. Abstr. Appl. Anal. (2003), 1141–1158.
[21] *P. A. Hästö*: The Apollonian metric: quasi-isotropy and Seittenranta's metric. Comput. Methods Funct. Theory *4* (2004), 249–273.
[22] *P. A. Hästö*: The Apollonian inner metric. Comm. Anal. Geom. *12* (2004), 927–947.
[23] *P. A. Hästö and H. Lindén*: Isometries of the half-Apollonian metric. Complex Var. Theory Appl. *49* (2004), 405–415.

[24] *Z. Ibragimov*: On the Apollonian metric of domains in $\overline{\mathbb{R}}^n$. Complex Var. Theory Appl. *48* (2003), 837–855.
[25] *Z. Ibragimov*: Conformality of the Apollonian metric. Comput. Methods Funct. Theory *3* (2003), 397–411.
[26] *P. J. Kelly*: Barbilian geometry and the Poincaré model. Amer. Math. Monthly *61* (1954), 311–319.
[27] *R. Miron, M. Anastasiei and I. Bucătaru*: The Geometry of Lagrange Spaces Handbook of Finsler geometry. Kluwer Acad. Publ., Dordrecht, 2003, pp. 969–1122.

Authors' addresses: Wladimir G. Boskoff, Department of Mathematics and Computer Science, Ovidius University, Constanţa, Romania, e-mail: `boskoff@univ-ovidius.ro`; Bogdan D. Suceavă, Department of Mathematics, California State University, Fullerton, CA 92834-6850, U.S.A., e-mail: `bsuceava@fullerton.edu`.