Czechoslovak Mathematical Journal

Patricio Cumsille; Takéo Takahashi

Wellposedness for the system modelling the motion of a rigid body of arbitrary form in an incompressible viscous fluid

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 4, 961–992

Persistent URL: http://dml.cz/dmlcz/140434

Czechoslovak Mathematical Journal, 58 (133) (2008), 961–992

WELLPOSEDNESS FOR THE SYSTEM MODELLING THE MOTION OF A RIGID BODY OF ARBITRARY FORM IN AN INCOMPRESSIBLE VISCOUS FLUID

Patricio Cumsille, Santiago, Takéo Takahashi, Vandœuvre-lès-Nancy

(Received October 13, 2006)

Abstract. In this paper, we consider the interaction between a rigid body and an incompressible, homogeneous, viscous fluid. This fluid-solid system is assumed to fill the whole space $\mathbb{R}^d$, $d = 2$ or 3. The equations for the fluid are the classical Navier-Stokes equations whereas the motion of the rigid body is governed by the standard conservation laws of linear and angular momentum. The time variation of the fluid domain (due to the motion of the rigid body) is not known *a priori*, so we deal with a free boundary value problem.

We improve the known results by proving a complete wellposedness result: our main result yields a local in time existence and uniqueness of strong solutions for $d = 2$ or 3. Moreover, we prove that the solution is global in time for $d = 2$ and also for $d = 3$ if the data are small enough.

Keywords: Navier-Stokes equations, incompressible fluid, rigid bodies

MSC 2010: 35Q30, 76D03, 76D05

1. Introduction and statement of the main results

The aim of this work is to prove a result of wellposedness for a coupled system of nonlinear partial and ordinary differential equations modelling the motion of a rigid body immersed into a viscous incompressible fluid. The fluid flow is governed by the classical Navier-Stokes system, whereas the motion of the rigid body is governed by the balance equations for linear and angular momenta (Newton's laws).

For $d = 2$ or $d = 3$, we denote by $\mathscr{O}(t) \subset \mathbb{R}^d$, the domain occupied by the rigid body and we denote by $\mathscr{F}(t) = \mathbb{R}^d \setminus \mathscr{O}(t)$ the exterior domain occupied by the fluid

Patricio Cumsille's research was partially supported by CONICYT-FONDECYT grant (No. 3070040) and Takéo Takahashi's research was partially supported by Grant (JCJC06 137283) of the Agence Nationale de la Recherche.

at time t. For the sake of simplicity, we assume that the fluid is homogeneous and of density one. Moreover, we assume that the rigid body is also homogeneous.

By choosing a frame of coordinates whose origin initially coincides with the mass center of the rigid body, the domain occupied by the latter at instant t is given by

$$\mathscr{O}(t) = \{Q(t)y + h(t)\colon\ y \in \mathscr{O}(0)\}, \tag{1.1}$$

where $h(t)$ is the position of the mass center of the rigid body, and where $Q(t)$ is a rotation matrix associated to the angular velocity $\omega(t)$ of the rigid body. The matrix $Q(t)$ is the solution of the initial value problem

$$\begin{gathered} Q'(t)Q^*(t)y = \omega(t) \times y \quad \forall\, y \in \mathbb{R}^d, \\ Q(0) = \mathrm{Id}, \end{gathered} \tag{1.2}$$

where, for any matrix A, we have denoted by A^* the transpose matrix of A and by Id the identity matrix.

For planar motion (i.e. $d = 2$) we can assume that

$$\omega(t) = \tilde{\omega}(t)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and in that case,

$$Q(t) = \begin{pmatrix} \tilde{Q}(t) & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with

$$\tilde{Q}(t) = \begin{pmatrix} \cos\tilde{\theta}(t) & -\sin\tilde{\theta}(t) \\ \sin\tilde{\theta}(t) & \cos\tilde{\theta}(t) \end{pmatrix}, \tag{1.3}$$

and $\tilde{\theta}(t) = \int_0^t \tilde{\omega}(s)\,\mathrm{d}s$. The important quantities for $d = 2$ are $\tilde{\omega}$, $\tilde{Q}$ and $\tilde{\theta}$ and for simplicity of notation, we omit in the sequel the tilde in all these quantities. Therefore, for $d = 2$, ω is a *scalar* function and $Q(t)$ a rotation matrix of *order* 2 and of angle $\theta(t)$ (see (1.3)).

The system of equations modelling the motion of the fluid and of the rigid body can be written as

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \quad x \in \mathscr{F}(t),\ \ t \in (0,T), \tag{1.4}$$

$$\operatorname{div} u = 0, \quad x \in \mathscr{F}(t),\ \ t \in (0,T), \tag{1.5}$$

$$u(x,t) = h'(t) + \omega(t) \times [x - h(t)], \quad x \in \partial\mathscr{O}(t),\ \ t \in (0,T), \tag{1.6}$$

$$Mh''(t) = -\int_{\partial\mathscr{O}(t)} \sigma(u,p)n \, \mathrm{d}\Gamma + \int_{\mathscr{O}(t)} \varrho f(x,t) \, \mathrm{d}x, \quad t \in (0,T), \tag{1.7}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}(J\omega)(t) = -\int_{\partial\mathscr{O}(t)} [x - h(t)] \times \sigma(u,p)n \, \mathrm{d}\Gamma + \int_{\mathscr{O}(t)} [x - h(t)] \times \varrho f(x,t) \, \mathrm{d}x, \quad t \in (0,T), \tag{1.8}$$

$$u(x,0) = u_0(x), \quad x \in \mathscr{F}(0), \tag{1.9}$$

$$h(0) = 0 \in \mathbb{R}^d, \ h'(0) = h^{(1)} \in \mathbb{R}^d, \ \omega(0) = \omega^{(0)}. \tag{1.10}$$

In the above system the unknowns are $u(x,t)$ (the Eulerian velocity field of the fluid), $p(x,t)$ (the pressure field of the fluid), $h(t)$ (the position of the mass center of the rigid body) and $\omega(t)$ (the angular velocity of the rigid body). For $d = 3$, we have denoted by $a \times b$ the classical cross product for $a, b \in \mathbb{R}^3$ whereas for $d = 2$, for $a, b \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, we have denoted

$$a \times b = a_1 b_2 - a_2 b_1 \quad \text{and} \quad \alpha \times b = \alpha \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix}.$$

The boundary of the rigid body at instant t is denoted by $\partial\mathscr{O}(t)$ and the normal unit vector directed to the interior of the rigid body is denoted by $n(x,t)$. We have also denoted by $f(x,t)$ the applied body forces (per unit mass). The positive constant ν stands for the viscosity of the fluid. Furthermore, we have denoted by M (respectively, by ϱ) the mass (respectively, the density of the rigid body) and by J the inertia moment related to the mass center of the rigid body.

The formulae for M and J are

$$M = \int_{\mathscr{O}(t)} \varrho \, \mathrm{d}x = \int_{\mathscr{O}(0)} \varrho \, \mathrm{d}y,$$

$$J = \int_{\mathscr{O}(t)} \varrho |x - h(t)|^2 \, \mathrm{d}x = \int_{\mathscr{O}(0)} \varrho |y|^2 \, \mathrm{d}y \quad \text{if } d = 2,$$

and

$$J(t)_{kl} = \int_{\mathscr{O}(t)} \varrho \big[|x - h(t)|^2 \delta_{kl} - (x - h(t))_k (x - h(t))_l \big] \, \mathrm{d}x \quad \text{for } k, l \in \{1, \ldots, d\} \text{ if } d = 3,$$

where δ_{kl} is the Kronecker symbol.

Moreover, the notation $x \cdot y$ stands for the inner product of x and y and the notation $|x|$ stands for the corresponding norm. Finally, we have denoted by $\sigma(u,p)$ the Cauchy stress tensor field in the fluid defined by

$$\sigma_{kl}(u,p) = -p\delta_{kl} + \nu\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big), \quad \text{for } k, l \in \{1, \ldots, d\}. \tag{1.11}$$

We also define by $D(u)$ the matrix

$$D_{kl}(u) = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right).$$

The problem of interaction between a viscous incompressible fluid and a rigid body has been studied intensively in the recent years (see [2], [4], [6], [7], [11], [15], [18], [19], [20], [21], etc.). However, as far as we know, only few results concerning the existence and uniqueness of strong solutions for the problem (1.1), (1.2), (1.4)–(1.10) are available in the case where the system fills the whole space. In that case, we can mention the results of Takahashi and Tucsnak [22], and of Galdi and Silvestre [9]. In [22], the authors show the global in time existence and uniqueness of strong solutions in two spatial dimensions in the particular case where the rigid body is a disk. In [9], the authors prove the existence of local in time strong solutions for a rigid body having an arbitrary regular shape. Nevertheless, their result does not yield neither the uniqueness of solutions nor the global in time existence (even for small data).

On the other hand, due to the complexity of the problem, another related problem simpler than (1.1), (1.2), (1.4)–(1.10), in which the motion of the rigid body is prescribed as a constant rotation has also been investigated. In particular, a local in time existence and uniqueness result of *mild* solutions has been proved by Hishida [14], and recently a local in time existence result of strong solutions has been proved by Galdi and Silvestre [10]. Moreover, the authors prove that the solution is global in time, provided that the initial velocity u_0, in an appropriate norm, and the magnitude of ω do not exceed a certain constant depending only on the viscosity and on the regularity of $\mathscr{F}(0)$. However, the authors do not make any reference to uniqueness properties of the solution. Both works mentioned before deal with the problem by writing the equations of motion of the fluid-rigid body system in a frame attached to the rigid body. Furthermore, a local in time existence and uniqueness result of strong solutions has been very recently proved by Cumsille and Tucsnak [3]. There, the authors proved that the solution is global in time in two spatial dimensions provided that the velocity satisfies suitable *a priori* estimates. We remark that the work previously cited deals with the problem by making a new change of variables, instead of writing the equations of motion in a frame attached to the rigid body. We use here a similar idea to work with the problem (1.1), (1.2), (1.4)–(1.10).

In order to make the region occupied by the fluid time independent, it is quite natural to refer the equations of motion of the fluid-rigid body system in a frame attached to the rigid body, with origin in the center of mass of the latter, and coinciding with an inertial frame at time $t = 0$ (see [8] for details). More precisely, let us denote

$$\bar{u}(y,t) = Q^*(t)u(Q(t)y + h(t), t);$$
$$\bar{p}(y,t) = p(Q(t)y + h(t), t);$$
$$\bar{h}(t) = \int_0^t Q^*(s)h'(s)\,\mathrm{d}s;$$
$$\overline{J} = J(0); \quad \overline{\omega}(t) = \begin{cases} \omega(t) & \text{for } d = 2, \\ Q^*(t)\omega(t) & \text{for } d = 3; \end{cases}$$
$$\overline{f}_M(t) = -M\overline{\omega}(t) \times \bar{h}'(t); \quad \overline{f}_J(t) = \begin{cases} 0 & \text{for } d = 2, \\ (\overline{J}\overline{\omega}(t)) \times \overline{\omega}(t) & \text{for } d = 3; \end{cases}$$
$$\overline{f}(y,t) = Q^*(t)f(Q(t)y + h(t), t).$$

In this case, the equivalent system to the original one can be written as

$$\frac{\partial \bar{u}}{\partial t} - \nu\Delta\bar{u} + [(\bar{u} - \bar{h}' - \overline{\omega} \times y) \cdot \nabla]\bar{u} + \nabla\bar{p} + \overline{\omega} \times \bar{u} = \overline{f}, \quad (y,t) \in \mathscr{F}(0) \times (0,T), \tag{1.12}$$

$$\operatorname{div} \bar{u} = 0, \quad (y,t) \in \mathscr{F}(0) \times (0,T), \tag{1.13}$$

$$\bar{u}(y,t) = \bar{h}'(t) + \overline{\omega}(t) \times y, \quad (y,t) \in \partial\mathscr{O}(0) \times (0,T), \tag{1.14}$$

$$M\bar{h}''(t) = -\int_{\partial\mathscr{O}(0)} \sigma(\bar{u},\bar{p})n\,\mathrm{d}\Gamma + \int_{\mathscr{O}(0)} \varrho\overline{f}(y,t)\,\mathrm{d}y + \overline{f}_M(t), \quad t \in (0,T), \tag{1.15}$$

$$\overline{J}\overline{\omega}'(t) = -\int_{\partial\mathscr{O}(0)} y \times (\sigma(\bar{u},\bar{p})n)\,\mathrm{d}\Gamma + \int_{\mathscr{O}(0)} y \times \varrho\overline{f}(y,t)\,\mathrm{d}y + \overline{f}_J(t), \quad t \in (0,T), \tag{1.16}$$

$$\bar{u}(y,0) = u_0(y), \quad y \in \mathscr{F}(0), \tag{1.17}$$

$$\bar{h}(0) = 0, \quad \bar{h}'(0) = h^{(1)}, \quad \overline{\omega}(0) = \omega^{(0)}. \tag{1.18}$$

One of the main difficulties comes from the term $[(\overline{\omega} \times y) \cdot \nabla]\bar{u}$, whose coefficient becomes unbounded at large spatial distances. In order to overcome this difficulty, we use another change of variables which coincides with $Q(t)y + h(t)$ in a neighborhood of the rigid body and is equal to the identity far from the rigid body. By using this change of variables, we obtain a system of equations whose coefficients are bounded at large spatial distances, instead of the term $[(\overline{\omega}\times y)\cdot\nabla]\bar{u}$. This feature of our method allows us to improve the results of [9], in the sense that we get the uniqueness as well as the global character (in time) of the solution.

In the rest of this work, we denote $\mathscr{F} = \mathscr{F}(0)$ and $\mathscr{O} = \mathscr{O}(0)$. As usual, for $m \in \mathbb{N}$ and $\alpha \in [1,\infty]$ we denote by $W^{m,\alpha}(\mathscr{F})$ the Sobolev spaces formed by the functions in $L^\alpha(\mathscr{F})$ which have distributional derivatives, up to the order m, in $L^\alpha(\mathscr{F})$, and by $H^m(\mathscr{F}) = W^{m,2}(\mathscr{F})$. We also denote by $\hat{H}^1(\mathscr{F})$ the homogeneous Sobolev space

$$\hat{H}^1(\mathscr{F}) = \{q \in L^2_{\text{loc}}(\overline{\mathscr{F}}) \mid \nabla q \in [L^2(\mathscr{F})]^d\},$$

where $q \in L^2_{\text{loc}}(\overline{\mathscr{F}})$ means that $q \in L^2(\mathscr{F} \cap B_0)$ for all open balls $B_0 \subset \mathbb{R}^d$ with $B_0 \cap \mathscr{F} \neq \emptyset$. We identify two functions of $\hat{H}^1(\mathscr{F})$ if they differ by a constant.

Moreover, we set

$$\mathscr{H}^m(\mathscr{F}(t)) = [H^m(\mathscr{F}(t))]^d, \qquad \mathscr{H}^m(\mathbb{R}^d) = [H^m(\mathbb{R}^d)]^d$$
$$\mathscr{L}^\alpha(\mathscr{F}(t)) = [L^\alpha(\mathscr{F}(t))]^d, \qquad \mathscr{L}^\alpha(\mathbb{R}^d) = [L^\alpha(\mathbb{R}^d)]^d.$$

We denote

$$\mathscr{F}_T = \{(x,t) \in \mathbb{R}^d \times [0,T];\, x \in \mathscr{F}(t)\}.$$

Consider a smooth mapping $X\colon \mathscr{F} \times [0,T] \to \mathbb{R}^d$ such that for all $t \in [0,T]$, $X(\cdot,t)$ is a C^∞-diffeomorphism from $\mathscr{F}$ onto $\mathscr{F}(t)$. Moreover, suppose that the mappings

$$(y,t) \mapsto D_t D_y^\alpha X(y,t), \quad \alpha \in \mathbb{N}^d,$$

exist, are continuous and compactly supported in $\mathscr{F}$ (such a mapping will be given in Section 2). For any $g\colon \mathscr{F}_T \to \mathbb{R}^d$, we denote by $g_X\colon \mathscr{F} \times [0,T] \to \mathbb{R}^d$ the mapping $g_X(y,t) = g(X(y,t),t)$, for all $t \geqslant 0$ and for all $y \in \mathscr{F}$. In order to analyze the problem (1.1), (1.2), (1.4)–(1.10), we need to introduce the following function spaces in variable domain:

$$L^2(0,T;\mathscr{H}^2(\mathscr{F}(t))) = \{u\colon\ u_X \in L^2(0,T;\mathscr{H}^2(\mathscr{F}))\},$$
$$H^1(0,T;\mathscr{L}^2(\mathscr{F}(t))) = \{u\colon\ u_X \in H^1(0,T;\mathscr{L}^2(\mathscr{F}))\},$$
$$C([0,T],\mathscr{H}^1(\mathscr{F}(t))) = \{u\colon\ u_X \in C([0,T],\mathscr{H}^1(\mathscr{F}))\},$$
$$L^2(0,T;\hat{H}^1(\mathscr{F}(t))) = \{p\colon\ p_X \in L^2(0,T;\hat{H}^1(\mathscr{F}))\}.$$

Finally, let us denote by $\mathscr{U}(\mathscr{F}_T)$ the space of strong solutions for the velocity, defined by

$$\mathscr{U}(\mathscr{F}_T) = L^2(0,T;\mathscr{H}^2(\mathscr{F}(t))) \cap C([0,T],\mathscr{H}^1(\mathscr{F}(t))) \cap H^1(0,T;\mathscr{L}^2(\mathscr{F}(t))). \tag{1.19}$$

We also define

$$\mathscr{U}_T = L^2(0,T;\mathscr{H}^2(\mathscr{F})) \cap C([0,T],\mathscr{H}^1(\mathscr{F})) \cap H^1(0,T;\mathscr{L}^2(\mathscr{F})). \tag{1.20}$$

The main results of this paper are the three following theorems.

Theorem 1.1. *Suppose that $\mathscr{O}$ is a bounded open connected subset of $\mathbb{R}^d$ with boundary $\partial\mathscr{O}$ of class C^3. Assume that*

$$f \in L^2_{\mathrm{loc}}(0,\infty;[W^{1,\infty}(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)]^d)$$

and that $u_0 \in \mathscr{H}^1(\mathscr{F})$ with

$$\begin{cases} \operatorname{div} u_0 = 0 & \text{in } \mathscr{F}, \\ u_0(x) = h^{(1)} + \omega^{(0)} \times x & \text{on } \partial\mathscr{O}. \end{cases}$$

Then, there exists $T_0 > 0$ such that the problem (1.1), (1.2), (1.4)–(1.10) *admits a unique strong solution (u,p,h,ω) in $[0,T_0)$, that is*

$$u \in \mathscr{U}(\mathscr{F}_T), \quad p \in L^2(0,T;\hat{H}^1(\mathscr{F}(t))), \quad h \in H^2(0,T;\mathbb{R}^d),$$

and

$$\begin{cases} \omega \in H^1(0,T;\mathbb{R}) & \text{if } d=2, \\ \omega \in H^1(0,T;\mathbb{R}^3) & \text{if } d=3, \end{cases}$$

for all $T \in (0,T_0)$.

Moreover, we can choose T_0 such that one of the following alternatives holds true:

(a) $T_0 = +\infty$,

(b) *the function $t \mapsto \|u(t)\|_{\mathscr{H}^1(\mathscr{F}(t))}$ is not bounded in $[0,T_0)$.*

Theorem 1.2. *Assume that the hypotheses in Theorem* (1.1) *hold true and suppose that $d=2$. Then, assertion* (a) *in Theorem* (1.1) *holds true, that is, the strong solution of the problem* (1.1), (1.2), (1.4)–(1.10) *is global in time.*

Theorem 1.3. *Assume that the hypotheses in Theorem* (1.1) *hold true and suppose that $d=3$. Moreover, suppose that*

$$f \in L^2(0,\infty,\mathscr{L}^2(\mathbb{R}^3)) \cap L^1(0,\infty,\mathscr{L}^2(\mathbb{R}^3)).$$

There exists a positive constant $c = c(\mathscr{O},\nu,\varrho)$ such that if

$$\|u_0\|^2_{\mathscr{L}^2(\mathbb{R}^3)} + \|D(u_0)\|^2_{[L^2(\mathbb{R}^3)]^9} + \|f\|^2_{L^2(0,\infty;\mathscr{L}^2(\mathbb{R}^3))} + \|f\|^2_{L^1(0,\infty;\mathscr{L}^2(\mathbb{R}^3))} < c, \tag{1.21}$$

then assertion (a) *in Theorem* (1.1) *holds true, that is, the strong solution of the problem* (1.1), (1.2), (1.4)–(1.10) *is global in time.*

As a direct consequence of Theorem 1.1, we recover the result [9, Theorem 4.1]:

Corollary 1.4. *Assume that the hypotheses in Theorem* (1.1) *hold true and suppose that* $d = 3$. *Let us denote by* B_R *the open ball centered at 0 and of radius* R *in* $\mathbb{R}^3$, *by* $\operatorname{diam}(\mathscr{O})$ *the diameter of* $\mathscr{O}$ *and* $\mathscr{F}_R = \mathscr{F} \cap B_R$ *for* $R > \operatorname{diam}(\mathscr{O})$. *Then there exists* T^* *and* $\bar{u} = \bar{u}(y,t)$, $\bar{p} = \bar{p}(y,t)$, $\bar{h} = \bar{h}(t)$, $\varpi = \varpi(t)$ *and* $Q = Q(t)$ *satisfying* (1.12)–(1.18) *and* (1.2) *almost everywhere in* $\mathscr{F} \times (0, T^*)$ *such that*

$$\bar{u}_i, \frac{\partial \bar{u}_i}{\partial y_j} \in L^\infty(0,T; L^2(\mathscr{F})), \quad \frac{\partial \bar{p}}{\partial y_i}, \frac{\partial^2 \bar{u}_i}{\partial y_j \partial y_k} \in L^2(0,T; L^2(\mathscr{F})),$$
$$\bar{h}_i \in H^2(0,T), \quad \varpi_i \in H^1(0,T), \quad Q_{ij} \in H^2(0,T)$$
$$\frac{\partial \bar{u}_i}{\partial t} \in L^2(0,T; L^2(\mathscr{F}_R)) \quad \forall R > \operatorname{diam}(\mathscr{O}),$$

for any $T \in (0, T^*)$. *Moreover,*

$$\bar{h}_i \in C^1([0,T]), \quad \varpi_i, Q_{ij} \in C([0,T]), \quad \bar{h}(0) = 0, \quad \bar{h}'(0) = h^{(1)}, \quad \varpi(0) = \omega^{(0)},$$
$$Q(0) = \mathrm{Id}, \quad \bar{u} \in C([0,T], H^1(\mathscr{F})) \quad \text{with } \bar{u}(\cdot, 0) = u_0(\cdot),$$

for any $T \in (0, T^*)$.

Remark 1.5. The existence of solutions for the problem (1.1), (1.2), (1.4)–(1.10), with initial data satisfying the same assumptions as in Theorem 1.1 has already been investigated in [22] assuming that the rigid body is an infinite cylinder of circular cross-section. Moreover, a similar problem was studied in [9], where the difference with our problem is that in [9], the authors suppose that there are prescribed external forces and torques acting on the rigid body and assume that only conservative forces act on the fluid. The novelty of our results consists in the fact that we obtain an existence and uniqueness result for strong solutions in the case of a rigid body of arbitrary and regular shape. On the other hand, we obtain a solution which is *unique and global in time* (assuming that the data are small enough if $d = 3$).

The plan of this paper is as follows: in Section 2 we introduce the change of variables, which plays a central role in Section 3, in order to prove the local existence of the problem (1.1), (1.2), (1.4)–(1.10). In Subsection 3.1 we study a linearized problem associated to (1.1), (1.2), (1.4)–(1.10), after of the change of variables. Regarding the problem (1.1), (1.2), (1.4)–(1.10), after of the change of variables, as a perturbation of the linearized problem of the previous subsection, we give in Subsection 3.2 the estimates needed in order to carry out a fixed point procedure, to prove that such a problem admits a unique local strong solution. In Subsection 3.3 we implement our fixed point procedure to conclude the proof of Theorem 1.1. In Section 4 we prove the global character of the solution. In Subsection 4.1 we give some preliminary results that are valid in two or in three spatial dimensions. In

Subsection 4.2 we prove that the solution is global in time in 2-D. Finally, in Subsection 4.3, we prove that the solution is also global in time in 3-D, if the initial velocity and the external force are small enough (in some appropriate norms).

2. The transformed equations

2.1. The change of variables

Let us consider a fixed pair (h, ω), with $h \in H^2(0,T;\mathbb{R}^d)$ and $\omega \in H^1(0,T;\mathbb{R})$ (resp. $\omega \in H^1(0,T;\mathbb{R}^3)$) for $d=2$ (resp. for $d=3$). We first remark that, by using a classical Sobolev embedding, we have that $h \in C^1([0,T],\mathbb{R}^d)$ and $\omega \in C([0,T],\mathbb{R})$ (resp. $\omega \in C([0,T],\mathbb{R}^3)$) for $d=2$ (resp. for $d=3$). Let $V\colon\ \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ be the rigid velocity field associated to (h,ω), defined by

$$V(x,t) = h'(t) + \omega(t) \times [x - h(t)] \quad \forall\, x \in \mathbb{R}^d,\ \ \forall\, t \in [0,T]. \tag{2.1}$$

Clearly, for all $t \in [0,T]$, $V(\cdot,t)$ is a C^∞ function and for all $x \in \mathbb{R}^d$, the function $V(x,\cdot)$ is in $H^1(0,T;\mathbb{R}^d)$. Moreover, by using a Sobolev embedding we have $V \in C(\mathbb{R}^d \times [0,T])$.

Let us denote by $\operatorname{diam}(\mathscr{O})$ the diameter of the set $\mathscr{O}$ and by B_r the open ball in $\mathbb{R}^d$, of radius $r>0$ and centered at the origin. Assume that

$$r > \operatorname{diam}(\mathscr{O}) + \|h\|_{L^\infty(0,T;\mathbb{R}^d)}. \tag{2.2}$$

With this choice of r and since $\mathscr{O}(t)$ is defined by (1.1), we have that $\mathscr{O}(t) \subset B_r$ for all $t \in [0,T]$.

Let $\psi \in C^\infty(\mathbb{R}^d,\mathbb{R})$ be a cut-off function, whose support be contained in B_{2r} and such that $\psi \equiv 1$ on $\bar{B}_r$. We introduce the functions w defined in $\mathbb{R}^d \times [0,T]$ by

$$w(x,t) = h' \times (x - h(t)) + \frac{|x-h(t)|^2}{2}\omega, \tag{2.3}$$

and $\Lambda\colon\ \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ defined by

$$\Lambda(x,t) = \begin{cases} \begin{pmatrix} -\dfrac{\partial \psi}{\partial x_2} w + \psi V_1 \\ \dfrac{\partial \psi}{\partial x_1} w + \psi V_2 \end{pmatrix} & \text{if } d=2, \\ \begin{pmatrix} \psi V_1 + \dfrac{\partial \psi}{\partial x_2} w_3 - \dfrac{\partial \psi}{\partial x_3} w_2 \\ \psi V_2 + \dfrac{\partial \psi}{\partial x_3} w_1 - \dfrac{\partial \psi}{\partial x_1} w_3 \\ \psi V_3 + \dfrac{\partial \psi}{\partial x_1} w_2 - \dfrac{\partial \psi}{\partial x_2} w_1 \end{pmatrix} & \text{if } d=3. \end{cases} \tag{2.4}$$

With the previous definitions, it is easy to show that Λ satisfies the following lemma.

Lemma 2.1. *Let $r > 0$ satisfy* (2.2) *and let w, Λ be defined by* (2.3) *and* (2.4) *respectively. Then, we have*

(1) $\Lambda = 0$ *outside of* B_{2r}.

(2) $\operatorname{div} \Lambda = 0$ *in* $\mathbb{R}^d \times [0, T]$.

(3) $\Lambda(x, t) = V(x, t)$ *in* $\mathscr{O}_T = \{(x, t) \in \mathbb{R}^d \times [0, T];\ x \in \mathscr{O}(t)\}$,

(4) $\Lambda \in C(\mathbb{R}^d \times [0, T], \mathbb{R}^d)$. *Moreover, for all* $t \in [0, T]$, $\Lambda(\cdot, t)$ *is a* C^∞ *function and for all* $x \in \mathbb{R}^d$, *the function* $\Lambda(x, \cdot)$ *is in* $H^1(0, T; \mathbb{R}^d)$.

Next, consider the time dependent vector field $X(\cdot, t)$ satisfying

$$\begin{cases} \dfrac{\partial X}{\partial t}(y, t) = \Lambda(X(y, t), t), \ \ t \in \,]0, T], \\ X(y, 0) = y \in \mathbb{R}^d, \end{cases} \tag{2.5}$$

where Λ is defined by (2.4). We have the following result:

Lemma 2.2. *For all $y \in \mathbb{R}^d$, the initial-value problem* (2.5) *admits a unique solution $X(y, \cdot)\colon [0, T] \to \mathbb{R}^d$, which is a C^1 function in $[0, T]$. Moreover, we have the following properties*

(1) *For all $t \in [0, T]$, the mapping $y \mapsto X(y, t)$ is a C^∞-diffeomorphism from $\mathbb{R}^d$ onto itself and from $\mathscr{F}$ onto $\mathscr{F}(t)$.*

(2) *Denote by $Y(\cdot, t)$ the inverse of $X(\cdot, t)$. Then, for all $x \in \mathbb{R}^d$, the mapping $t \mapsto Y(x, t)$ is a C^1 function in $[0, T]$.*

(3) *For all $y \in \mathbb{R}^d$ and for all $t \in [0, T]$, the determinant of the jacobian matrix J_X of $X(\cdot, t)$ is equal to 1:*

$$\det J_X(y, t) = 1. \tag{2.6}$$

P r o o f. Since Λ is continuous and $\Lambda(\cdot, t)$ is a C^1 function, according to the classical result of Cauchy-Lipschitz-Picard, it follows that (2.5) admits a unique solution $X(\cdot, t)$, defined in $[0, T_1[$, with $T_1 \leqslant T$, which is C^1 in $[0, T_1[$. Moreover, since $\Lambda = 0$ outside of B_{2r}, Λ is bounded and therefore it follows that X does not blow-up before the time T. In particular, for any $t \in [0, T]$, $X(\cdot, t)$ is a bijection of $\mathbb{R}^d$ onto itself.

By using the regularity of Λ and a classical result (see, for instance, Hartman [12, Theorem 4.1, p. 100]), we get that for each $t \in [0, T]$, $X(\cdot, t)$ is a C^∞ function on $\mathbb{R}^d$. The fact that the inverse $Y(\cdot, t)$ of $X(\cdot, t)$ is a C^∞ function on $\mathbb{R}^d$ follows in a similar way. Thus, it follows that $X(\cdot, t)$ is a C^∞-diffeomorphism from $\mathbb{R}^d$ onto itself. Now, let us show that $X(\cdot, t)$ is a C^∞-diffeomorphism from $\mathscr{F}$ onto $\mathscr{F}(t)$. To do this, we remark that it is easy to verify that for each $y \in \mathscr{O}$, the function

$$X(y, t) = h(t) + Q(t)y, \tag{2.7}$$

is the solution of (2.5). We then conclude that for any $t \in [0,T]$, $X(\cdot,t)(\mathscr{O}) \subset \mathscr{O}(t)$ and in a similar way, $Y(\cdot,t)(\mathscr{O}(t)) \subset \mathscr{O}$. Therefore $X(\cdot,t)\colon \mathscr{F} \to \mathscr{F}(t)$ is a C^∞-diffeomorphism. The same conclusion follows for $Y(\cdot,t)\colon \mathscr{F}(t) \to \mathscr{F}$.

Finally, by using a classical result due to Liouville (see, for instance, Arnold [1, p. 249]) and the fact that $\operatorname{div}\Lambda = 0$, we obtain that X satisfies (2.6). □

In the sequel, we denote by J_X and J_Y the jacobian matrix of X and Y respectively:

$$J_X = \left(\frac{\partial X_i}{\partial y_j}\right)_{ij}, \quad J_Y = \left(\frac{\partial Y_i}{\partial x_j}\right)_{ij}.$$

2.2 The equations in the cylindrical domain

We first define the functions

$$U(y,t) = J_Y(X(y,t),t)\; u(X(y,t),t); \quad P(y,t) = p(X(y,t),t); \tag{2.8}$$

$$H(t) = \int_0^t Q^*(s)h'(s)\,\mathrm{d}s; \quad \Omega(t) = \begin{cases} \omega(t) & \text{for } d=2,\\ Q^*(t)\omega(t) & \text{for } d=3. \end{cases} \tag{2.9}$$

In order to write the equations satisfied by (U,P,H,Ω), we define for each $i \in \{1,\ldots,d\}$

$$\begin{aligned}(\mathbf{L}U)_i = {} & \sum_{j,k=1}^{d} \frac{\partial}{\partial y_j}\Big(g^{jk}\frac{\partial U_i}{\partial y_k}\Big) + 2\sum_{j,k,l=1}^{d} g^{kl}\Gamma^i_{jk}\frac{\partial U_j}{\partial y_l}\\ & + \sum_{j,k,l=1}^{d}\Big\{\frac{\partial}{\partial y_k}(g^{kl}\Gamma^i_{jl}) + \sum_{m=1}^{d} g^{kl}\Gamma^m_{jl}\Gamma^i_{km}\Big\}U_j;\end{aligned} \tag{2.10}$$

$$(\mathbf{M}U)_i = \sum_{j=1}^{d}\frac{\partial Y_j}{\partial t}\frac{\partial U_i}{\partial y_j} + \sum_{j,k=1}^{d}\Big\{\Gamma^i_{jk}\frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k}\frac{\partial^2 X_k}{\partial t\partial y_j}\Big\}U_j; \tag{2.11}$$

$$(\mathbf{N}U)_i = \sum_{j=1}^{d} U_j\frac{\partial U_i}{\partial y_j} + \sum_{j,k=1}^{d}\Gamma^i_{jk}U_jU_k; \tag{2.12}$$

$$(\mathbf{G}P)_i = \sum_{j=1}^{d} g^{ij}\frac{\partial P}{\partial y_j}; \tag{2.13}$$

where, for each $i, j, k \in \{1, \dots, d\}$, we have denoted

$$g^{ij}(y,t) = \sum_{k=1}^{d} \frac{\partial Y_i}{\partial x_k}(X(y,t),t)\frac{\partial Y_j}{\partial x_k}(X(y,t),t), \tag{2.14}$$

$$g_{ij}(y,t) = \sum_{k=1}^{d} \frac{\partial X_k}{\partial y_i}(y,t)\frac{\partial X_k}{\partial y_j}(y,t), \tag{2.15}$$

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{d} g^{kl}\Big\{\frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_l}\Big\}. \tag{2.16}$$

We also define

$$F(y,t) = J_Y(X(y,t),t)f(X(y,t),t); \quad F_M = -M\Omega \times H'; \tag{2.17}$$
$$F_J = \begin{cases} 0 & \text{for } d = 2, \\ \overline{J}\Omega \times \Omega & \text{for } d = 3. \end{cases}$$

(Recall that $\overline{J} = J(0)$.)

With the above notations, we can consider the following problem written in the fixed spatial domain $\mathscr{F} = \mathscr{F}(0)$:

$$\frac{\partial U}{\partial t} - \nu(\mathbf{L}U) + (\mathbf{M}U) + (\mathbf{N}U) + (\mathbf{G}P) = F, \quad \text{in } \mathscr{F} \times (0,T), \tag{2.18}$$

$$\operatorname{div} U = 0, \quad \text{in } \mathscr{F} \times (0,T), \tag{2.19}$$

$$U(y,t) = H'(t) + \Omega(t) \times y, \quad \text{on } \partial\mathscr{O} \times (0,T), \tag{2.20}$$

$$MH''(t) = -\int_{\partial\mathscr{O}} \sigma(U,P)n \,\mathrm{d}\Gamma + \int_{\mathscr{O}} \varrho F(y,t)\,\mathrm{d}y + F_M(t), \quad t \in (0,T), \tag{2.21}$$

$$\overline{J}\Omega'(t) = -\int_{\partial\mathscr{O}} y \times \sigma(U,P)n\,\mathrm{d}\Gamma + \int_{\mathscr{O}} y \times \varrho F(y,t)\,\mathrm{d}y + F_J(t), \quad t \in (0,T), \tag{2.22}$$

$$U(y,0) = u_0(y), \quad y \in \mathscr{F}, \tag{2.23}$$

$$H(0) = 0, \ \ H'(0) = h^{(1)}, \ \ \Omega(0) = \omega^{(0)}. \tag{2.24}$$

This system is the transformation of the system (1.1), (1.2), (1.4)–(1.10) by using the mapping X, as stated in the following proposition.

Proposition 2.3. *The quadruple (u, p, h, ω) satisfies*

$$u \in \mathscr{U}(\mathscr{F}_T), \quad p \in L^2(0,T;\hat{H}^1(\mathscr{F}(t))), \quad h \in H^2(0,T;\mathbb{R}^d),$$
$$\begin{cases} \omega \in H^1(0,T;\mathbb{R}) & \textit{if } d = 2, \\ \omega \in H^1(0,T;\mathbb{R}^3) & \textit{if } d = 3, \end{cases}$$

together with (1.1), (1.2), (1.4)–(1.10) *in* $[0,T]$ *if and only if the quadruple* (U,P,H,Ω) *defined by* (2.8)–(2.9) *satisfies*

$$U \in \mathscr{U}_T, \quad P \in L^2(0,T;\hat{H}^1(\mathscr{F})), \quad H \in H^2(0,T;\mathbb{R}^d),$$
$$\begin{cases} \Omega \in H^1(0,T;\mathbb{R}) & \text{if } d=2, \\ \Omega \in H^1(0,T;\mathbb{R}^3) & \text{if } d=3, \end{cases}$$

together with (2.18)–(2.24) *in* $[0,T]$.

For a proof of this proposition, see Propositions 4.5 and 4.6 in [21].

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is similar to the proof of the local in time existence of strong solutions given in [21] or in [3]. The main difference comes from the change of variables used to transform the system (1.1), (1.2), (1.4)–(1.10) into a system of equations written in a fixed spatial domain. For this reason, we only give the main steps of the proof without details.

3.1. The linearized problem

A first step in the proof of Theorem 1.1 is to get the wellposedness for the following linear system:

$$\frac{\partial U}{\partial t} - \nu \Delta U + \nabla P = Z, \quad \text{in } \mathscr{F} \times (0,T), \tag{3.1}$$

$$\operatorname{div} U = 0, \quad \text{in } \mathscr{F} \times (0,T), \tag{3.2}$$

$$U(y,t) = H'(t) + \Omega(t) \times y, \quad \text{on } \partial\mathscr{O} \times (0,T), \tag{3.3}$$

$$MH''(t) = -\int_{\partial\mathscr{O}} \sigma(U,P) n \, \mathrm{d}\Gamma + Z_M(t), \quad t \in (0,T), \tag{3.4}$$

$$\overline{J}\Omega'(t) = -\int_{\partial\mathscr{O}} y \times \sigma(U,P) n \, \mathrm{d}\Gamma + Z_J(t), \quad t \in (0,T), \tag{3.5}$$

$$U(y,0) = u_0(y), \quad y \in \mathscr{F}, \tag{3.6}$$

$$H(0) = 0, \quad H'(0) = h^{(1)}, \quad \Omega(0) = \omega^{(0)}. \tag{3.7}$$

To achieve this, we extend U to $\mathbb{R}^d$ by setting

$$U(y,t) = H'(t) + \Omega(t) \times y \quad \forall\, (y,t) \in \mathscr{O} \times (0,T).$$

An easy calculation yields that $D(U) = 0$ in $\mathscr{O} \times (0,T)$. It is thus natural to define the following function spaces:

$$\mathbb{H} = \{U \in \mathscr{L}^2(\mathbb{R}^d) \colon \operatorname{div} U = 0 \text{ in } \mathbb{R}^d, \;\; D(U) = 0 \text{ in } \mathscr{O}\}, \tag{3.8}$$

$$\mathbb{V} = \{U \in \mathscr{H}^1(\mathbb{R}^d) \colon \operatorname{div} U = 0 \text{ in } \mathbb{R}^d, \;\; D(U) = 0 \text{ in } \mathscr{O}\}. \tag{3.9}$$

We endow $\mathscr{L}^2(\mathbb{R}^d)$ and $\mathbb{H}$ with the inner product

$$(f,g)_{\mathscr{L}^2(\mathbb{R}^d)} = \int_{\mathscr{F}} f \cdot g \,\mathrm{d}y + \int_{\mathscr{O}} \varrho f \cdot g \,\mathrm{d}y \quad \forall f,g \in \mathscr{L}^2(\mathbb{R}^d),$$

where $\varrho > 0$ is the density of the rigid body.

The study of 3.1)–(3.7) can be done as in [22] or [21] by using an approach based on the theory of semigroups. More precisely, we define

$$D(A) = \{U \in \mathbb{V};\ U|_{\mathscr{F}} \in \mathscr{H}^2(\mathscr{F})\}; \tag{3.10}$$

$$\mathscr{A}U = \begin{cases} -\nu\Delta U & \text{in } \mathscr{F}, \\ \dfrac{2\nu}{M}\displaystyle\int_{\partial\mathscr{O}} D(U)n \,\mathrm{d}\Gamma + 2\nu(\overline{J})^{-1}\left[\int_{\partial\mathscr{O}} y \times D(U)n \,\mathrm{d}\Gamma\right] \times y & \text{in } \mathscr{O}; \end{cases} \tag{3.11}$$

and

$$AU = \mathbb{P}\mathscr{A}U \quad \forall U \in D(A); \tag{3.12}$$

where $\mathbb{P}$ is the orthogonal projector from $\mathscr{L}^2(\mathbb{R}^d)$ onto $\mathbb{H}$, and where, in the expression of $\mathscr{A}U$, $D(U)$ represents the trace of the restriction of $D(U)$ to $\mathscr{F} = \mathbb{R}^d \setminus \mathscr{O}$.

As in [22], we can prove the following result.

Proposition 3.1. *The operator A defined by* (3.10), (3.11) *and* (3.12) *is self-adjoint and non-negative. Consequently, $-A$ is the generator of a contraction strongly continuous semigroup on $\mathbb{H}$. Moreover, there exists a constant $C = C(\nu,\varrho,\mathscr{F}) > 0$ such that for all $U \in D(A)$*

$$\|U\|_{\mathscr{H}^2(\mathscr{F})} \leqslant C\|(I+A)U\|_{\mathscr{L}^2(\mathbb{R}^d)}. \tag{3.13}$$

By using the above proposition, we obtain the following result, which can be proven in the same way as Corollary 4.3 from [22].

Corollary 3.2. *Let $T > 0$ and suppose that $Z \in L^2(0,T;\mathscr{L}^2(\mathscr{F}))$, $Z_M \in L^2(0,T;\mathbb{R}^d)$, $Z_J \in L^2(0,T;\mathbb{R})$ for $d = 2$, or $Z_J \in L^2(0,T;\mathbb{R}^3)$ for $d = 3$ and let $u_0 \in \mathscr{H}^1(\mathscr{F})$ be such that*

$$\operatorname{div} u_0 = 0, \quad \text{in } \mathscr{F},$$
$$u_0 = h^{(1)} + \omega^{(0)} \times y, \quad \text{on } \partial\mathscr{O}.$$

Then, the linear problem (3.1)–(3.7) *admits a unique strong solution (U,P,H,Ω) in $[0,T]$, i.e. $U \in \mathscr{U}_T$, $P \in L^2(0,T;\hat{H}^1(\mathscr{F}))$, $H \in H^2(0,T;\mathbb{R}^d)$, $\Omega \in H^1(0,T;\mathbb{R})$ if*

$d=2$, or $\Omega \in H^1(0,T;\mathbb{R}^3)$ if $d=3$. Moreover, there exists a positive constant C such that

$$\begin{aligned}(3.14)\quad &\|U\|_{\mathscr{U}_T} + \|\nabla P\|_{L^2(0,T;\mathscr{L}^2(\mathscr{F}))} + \|H'\|_{H^1(0,T;\mathbb{R}^d)} + \|\Omega\|_{H^1(0,T)} \\ &\quad \leqslant C(\|u_0\|_{\mathscr{H}^1(\mathbb{R}^d)} + \|Z\|_{L^2(0,T;\mathscr{L}^2(\mathscr{F}))} + \|Z_M\|_{L^2(0,T;\mathbb{R}^d)} + \|Z_J\|_{L^2(0,T)}).\end{aligned}$$

This constant C depends only on ϱ, ν, $\mathscr{F}$ and on T and it is non-decreasing with respect to T.

3.2. Proof the local existence and uniqueness result

Following the same approach than in [21] and in [3], we write the system (2.18)–(2.24) as the system

$$\begin{aligned}\frac{\partial U}{\partial t} - \nu\Delta U + \nabla P = \hat{Z}, \quad &\text{in } \mathscr{F}\times(0,T),\\ \operatorname{div} U = 0, \quad &\text{in } \mathscr{F}\times(0,T),\\ U(y,t) = H'(t) + \Omega(t)\times y, \quad &\text{on } \partial\mathscr{O}\times(0,T),\\ MH''(t) = -\int_{\partial\mathscr{O}} \sigma(U,P)n\,\mathrm{d}\Gamma + \hat{Z}_M(t), \quad &t\in(0,T),\\ \overline{J}\Omega'(t) = -\int_{\partial\mathscr{O}} y\times\sigma(U,P)n\,\mathrm{d}\Gamma + \hat{Z}_J(t), \quad &t\in(0,T),\\ U(y,0) = u_0(y), \quad &y\in\mathscr{F},\\ H(0)=0, \quad H'(0)=h^{(1)}, \quad &\Omega(0)=\omega^{(0)},\end{aligned}$$

with

$$\begin{aligned}\hat{Z}(y,t) &= F + \nu((\mathbf{L}-\Delta)U) - (\mathbf{M}U) - (\mathbf{N}U) + ((\nabla-\mathbf{G})P),\\ \hat{Z}_M(t) &= \int_{\mathscr{O}} \varrho F(y,t)\,\mathrm{d}y + F_M(t), \quad \hat{Z}_J = \int_{\mathscr{O}} y\times\varrho F(y,t)\,\mathrm{d}y + F_J(t),\end{aligned}$$

and where F, F_M and F_J are defined by (2.17).

We define the mapping $\mathscr{Z}$ from

$$L^2(0,T;L^2(\mathscr{F}))\times L^2(0,T;\mathbb{R}^d)\times L^2(0,T)$$

into itself by

$$\mathscr{Z}\begin{pmatrix} Z\\ Z_M\\ Z_J\end{pmatrix} = \begin{pmatrix} F + \nu((\mathbf{L}-\Delta)U) - (\mathbf{M}U) - (\mathbf{N}U) + ((\nabla-\mathbf{G})P)\\ \int_{\mathscr{O}} \varrho F(y,t)\,\mathrm{d}y + F_M\\ \int_{\mathscr{O}} y\times\varrho F(y,t)\,\mathrm{d}y + F_J\end{pmatrix},$$

where in the above expression, (U, P, H, Ω) is the solution of the linear system (3.1)–(3.7) associated with (Z, Z_M, Z_J) (see Corollary 3.2).

By estimating carefully the coefficients in $\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$, $\mathbf{G}$, F_M and F_J, we can prove that for small time T *depending only on* $\|u_0\|_{\mathscr{H}^1(\mathscr{F})}$, $h^{(1)}$ and $\omega^{(0)}$, the mapping $\mathscr{Z}$ admits a unique fixed point. Therefore, there exists a unique strong solution (U, P, H, Ω) of the system (2.18)–(2.24) on $[0, T]$. Moreover, since $\Omega(t)$ is a continuous mapping, it is well known that the initial-value problem

$$\begin{aligned} Q'(t)z &= Q(t)(\Omega(t) \times z) \quad \forall\, z \in \mathbb{R}^d, \\ Q(0) &= \mathrm{Id}, \end{aligned}$$

has a unique solution $Q \in C^1([0, T], \mathbb{R}^{d^2})$.

Finally, by using the inverse transformation

$$u(x,t) = J_X(Y(x,t),t)U(Y(x,t),t); \quad p(x,t) = P(Y(x,t),t);$$
$$h(t) = \int_0^t Q(s)H'(s)\,\mathrm{d}s; \quad \omega(t) = \begin{cases} \Omega(t) & \text{for } d = 2, \\ Q(t)\Omega(t) & \text{for } d = 3, \end{cases}$$

we obtain the existence and uniqueness of strong solutions on $[0, T]$ for the system (1.1), (1.2), (1.4)–(1.10).

3.3. End of the proof of Theorem 1.1

In the above section, we have shown that there exists a time $T > 0$ such that the problem (1.1), (1.2), (1.4)–(1.10) admits a unique strong solution (u, p, h, ω) in $[0, T]$. Let us define $T_0 > 0$ as follows:

$$T_0 := \sup\{T \in \mathbb{R}_+^*;\ (1.1),\ (1.2),\ (1.4)\text{–}(1.10) \text{ admits a unique strong solution in } [0,T]\}. \tag{3.15}$$

To finish the proof of Theorem 1.1, we have to prove that one of the alternatives (a) or (b) holds true. We act by contradiction: let us assume that $T_0 < \infty$ and that the function $t \mapsto \|u(t)\|_{\mathscr{H}^1(\mathscr{F}(t))}$ is bounded in $[0, T_0)$. It is not difficult to see that this implies that the mappings

$$t \mapsto |h'(t)|, \quad t \mapsto |\omega(t)|$$

are bounded in $[0, T_0)$. In particular, using that $T_0 < \infty$, we get that h is bounded in $[0, T_0)$ and therefore that there exists a uniform $r > 0$ so that (2.2) holds true. This fact, combined with the above subsections, implies that there exists $T_1 > 0$ such that, for all $t \in [0, T_0)$, there exists a unique strong solution of (1.1), (1.2), (1.4)–(1.10) in $[t, t + T_1]$. This contradicts (3.15) and completes the proof of Theorem 1.1.

4. Proof of the global existence of strong solutions

To get the global in time existence, we assume that $T_0 < \infty$ and we are going to show that the mappings

$$t \mapsto \|u(t)\|_{\mathscr{H}^1(\mathscr{F}(t))}, \quad t \mapsto |h'(t)|, \quad t \mapsto |\omega(t)|$$

are bounded in $[0, T_0)$ without any extra hypothesis if $d = 2$ and assuming that the initial velocity and the external force are small enough if $d = 3$. According to Theorem 1.1, this will imply Theorems 1.2 and 1.3.

4.1. Preliminary results

Let (u, p, h, ω) be the strong solution to the problem (1.1), (1.2), (1.4)–(1.10). It is natural to extend u to $\mathbb{R}^d$ by

$$u(x,t) = h'(t) + \omega(t) \times [x - h(t)] \quad x \in \mathscr{O}(t),\ t \in [0, T_0). \tag{4.1}$$

In that case, we have that $u(t) \in \mathscr{H}^1(\mathbb{R})^d)$ for all $t \in [0, T_0)$ and $D(u) = 0$ for $t \in [0, T_0)$ and $x \in \mathscr{O}(t)$. Moreover, we can easily check that

$$\int_{\mathscr{F}(t)} |D(u)|^2 \, \mathrm{d}x = \int_{\mathbb{R}^d} |D(u)|^2 \, \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x \geqslant \frac{1}{2} \int_{\mathscr{F}(t)} |\nabla u|^2 \, \mathrm{d}x.$$

A simple calculation shows that there exist two positive constants c_1 and c_2 depending only on $\mathscr{O}$ and on ϱ such that

$$c_1 |a|^2 \leqslant (Ja) \cdot a \leqslant c_2 |a|^2 \quad (a \in \mathbb{R}^3). \tag{4.2}$$

Lemma 4.1. *Let (u, p, h, ω) be the strong solution to the problem* (1.1), (1.2), (1.4)–(1.10) *as in Theorem* 1.1. *Then, there exists a positive constant $C = C(\mathscr{O}, \varrho)$ such that*

$$\begin{aligned} \sup_{t \in [0, T_0)} (\|u(t)\|^2_{\mathscr{L}^2(\mathscr{F}(t))} + |h'(t)|^2 + |\omega(t)|^2) + 2\nu \int_0^{T_0} \|\nabla u(t)\|^2_{[L^2(\mathscr{F}(t))]^{d^2}} \, \mathrm{d}t \\ \leqslant C(\|f\|^2_{L^1(0,T_0; \mathscr{L}^2(\mathbb{R}^d))} + \|u_0\|^2_{\mathscr{L}^2(\mathscr{F})} + |h^{(1)}|^2 + |\omega^{(0)}|^2). \end{aligned} \tag{4.3}$$

P r o o f. By taking the inner product of (1.4) with u, we obtain that

$$\begin{aligned} \int_{\mathscr{F}(t)} \frac{\partial u}{\partial t} \cdot u \, \mathrm{d}x - \int_{\mathscr{F}(t)} \operatorname{div} \sigma(u, p) \cdot u \, \mathrm{d}x + \int_{\mathscr{F}(t)} [(u \cdot \nabla) u] \cdot u \, \mathrm{d}x \\ = \int_{\mathscr{F}(t)} f \cdot u \, \mathrm{d}x, \quad \text{a.e. in } (0, T_0). \end{aligned} \tag{4.4}$$

On the other hand, the Reynolds transport theorem combined with the fact that $\operatorname{div} u = 0$ in $\mathscr{F}(t)$ implies

$$(4.5)\quad \int_{\mathscr{F}(t)} \frac{\partial u}{\partial t}\cdot u \,\mathrm{d}x + \int_{\mathscr{F}(t)} [(u\cdot\nabla)u]\cdot u\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}(t)} |u|^2\,\mathrm{d}x, \quad \text{a.e. in } (0,T_0).$$

Moreover, by using (1.6), (1.7) and (1.8), we obtain that a.e. in $(0,T_0)$

$$(4.6)\quad \begin{aligned} -\int_{\mathscr{F}(t)} &\operatorname{div}\sigma(u,p)\cdot u\,\mathrm{d}x \\ &= 2\nu\int_{\mathscr{F}(t)} |D(u)|^2\,\mathrm{d}x - \int_{\partial\mathscr{O}(t)} (\sigma(u,p)n)\cdot u\,\mathrm{d}\Gamma \\ &= 2\nu\int_{\mathscr{F}(t)} |D(u)|^2\,\mathrm{d}x + \frac{M}{2}\frac{\mathrm{d}}{\mathrm{d}t}|h'(t)|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[(J\omega(t))\cdot\omega(t)] \\ &\quad - \int_{\mathscr{O}(t)} h'(t)\cdot\varrho f(x,t)\,\mathrm{d}x - \int_{\mathscr{O}(t)} [\omega(t)\times(x-h(t))]\cdot\varrho f(x,t)\,\mathrm{d}x. \end{aligned}$$

Therefore, by replacing (4.5) and (4.6) in (4.4), we get that

$$(4.7)\quad \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{\mathscr{L}^2(\mathscr{F}(t))} &+ 4\nu\|Du(t)\|^2_{[L^2(\mathscr{F}(t))]^{d^2}} + M\frac{\mathrm{d}}{\mathrm{d}t}|h'(t)|^2 + \frac{\mathrm{d}}{\mathrm{d}t}[(J\omega(t))\cdot\omega(t)] \\ &= 2(f(t),u(t))_{\mathscr{L}^2(\mathscr{F}(t))} + 2\int_{\mathscr{O}(t)} \varrho f(x,t)\cdot[h'(t)+\omega(t)\times(x-h(t))]\,\mathrm{d}x, \\ &\qquad \text{a.e. in } (0,T_0). \end{aligned}$$

The above inequality yields that

$$(4.8)\quad \begin{aligned} \|u(t)\|^2_{\mathscr{L}^2(\mathscr{F}(t))} &+ M|h'(t)|^2 + (J\omega(t))\cdot\omega(t) \\ &\leqslant 2\Big(\int_0^t \|f(s)\|_{\mathscr{L}^2(\mathbb{R}^d)}\,\mathrm{d}s\Big)^2 \\ &\quad + 2\|u_0\|^2_{\mathscr{L}^2(\mathscr{F})} + 2M|h^{(1)}|^2 + 2\overline{J}\omega^{(0)}\cdot\omega^{(0)}, \end{aligned}$$

for all $t\in[0,T_0)$. Combining (4.7) and the above inequality, we conclude that

$$(4.9)\quad \begin{aligned} 4\nu\int_0^t &\|D(u(s))\|^2_{[L^2(\mathscr{F}(s))]^{d^2}}\,\mathrm{d}s \\ &\leqslant 3\Big[\Big(\int_0^{T_0}\|f(s)\|_{\mathscr{L}^2(\mathbb{R}^d)}\,\mathrm{d}s\Big)^2 + \|u_0\|^2_{\mathscr{L}^2(\mathscr{F})} + M|h^{(1)}|^2 + \overline{J}\omega^{(0)}\cdot\omega^{(0)}\Big], \end{aligned}$$

for all $t\in[0,T_0)$. The result follows from (4.8) and (4.9). $\square$

The above lemma gives in particular an estimate on $\|u(t)\|^2_{\mathscr{L}^2(\mathscr{F}(t))}$. In order to obtain an estimate on $\|u(t)\|^2_{\mathscr{H}^1(\mathscr{F}(t))}$, we need to introduce some auxiliary functions.

First let us consider $\xi\colon \mathbb{R}^d \to \mathbb{R}$ a smooth function with compact support such that $\xi(x) = 1$ in a neighborhood of $\overline{\mathscr{O}}$. Then we set

$$\hat{\psi}(x,t) = \xi(Q^*(x - h(t))),$$

and we define $\hat{\Lambda}\colon \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ by

$$\hat{\Lambda}(x,t) = \begin{cases} \begin{pmatrix} -\dfrac{\partial \hat{\psi}}{\partial x_2} w + \hat{\psi} V_1 \\ \dfrac{\partial \hat{\psi}}{\partial x_1} w + \hat{\psi} V_2 \end{pmatrix} & \text{if } d = 2, \\ \begin{pmatrix} \hat{\psi} V_1 + \dfrac{\partial \hat{\psi}}{\partial x_2} w_3 - \dfrac{\partial \hat{\psi}}{\partial x_3} w_2 \\ \hat{\psi} V_2 + \dfrac{\partial \hat{\psi}}{\partial x_3} w_1 - \dfrac{\partial \hat{\psi}}{\partial x_1} w_3 \\ \hat{\psi} V_3 + \dfrac{\partial \hat{\psi}}{\partial x_1} w_2 - \frac{\partial \hat{\psi}}{\partial x_2} w_1 \end{pmatrix} & \text{if } d = 3, \end{cases} \tag{4.10}$$

with V and w defined by (2.1) and by (3.2). The function $\hat{\Lambda}$ satisfies the properties (2), (3) and (4) of Lemma 2.1 (with $\hat{\Lambda}$ instead of Λ). Define $\hat{X}$ as the solution of the initial boundary value problem

$$\begin{cases} \dfrac{\partial \hat{X}}{\partial t}(y,t) = \hat{\Lambda}(\hat{X}(y,t),t), \quad t \in \,]0,T], \\ \hat{X}(y,0) = y \in \mathbb{R}^d. \end{cases} \tag{4.11}$$

The function $\hat{X}$ satisfies all the properties of Lemma 2.2 (with $\hat{X}$ instead of X) and in particular that, for each $y \in \mathscr{O}$,

$$\hat{X}(y,t) = h(t) + Q(t)y. \tag{4.12}$$

Finally, we can estimate the function $\hat{\Lambda}$:

$$\|\hat{\Lambda}\|_{\mathscr{W}^{2,\infty}(\mathscr{F}(t))} \leqslant C(|h'(t)| + |\omega(t)|). \tag{4.13}$$

Similarly, by changing slightly the functions V and w defined by (2.1) and by (2.3), we can prove the following result.

Proposition 4.2. *Let* (H,Ω) *be the transformed of* (h,ω) *given by* (2.9) *and let* $r > \operatorname{diam}(\mathscr{O})$ *be fixed. Then, there exists a vector field* $\bar{\Lambda} \in C(\mathbb{R}^d \times [0,T_0))$ *satisfying the following properties:*

(1) $\bar{\Lambda} = 0$ *outside of* B_{2r}.

(2) $\operatorname{div} \bar{\Lambda} = 0$ *in* $\mathbb{R}^d \times [0,T_0)$.

(3) $\bar{\Lambda}(y,t) = H'(t) + \Omega(t) \times y$ *for all* $(y,t) \in \mathscr{O} \times [0,T_0)$.

(4) *For all* $t \in [0,T_0)$, $\bar{\Lambda}(\cdot,t)$ *is a* C^∞ *mapping in* $\mathbb{R}^d$ *and for all* $y \in \mathbb{R}^d$, $\bar{\Lambda}(y,\cdot)$ *is in* $H^1(0,T;\mathbb{R}^d)$ *for any* $T \in (0,T_0)$.

(5) *There exists a positive constant* $C = C(\mathscr{O},\varrho)$ *such that*

$$\|\bar{\Lambda}(t)\|_{[H^2(\mathbb{R}^2)]^2} \leqslant C(|h'(t)| + |\omega(t)|), \tag{4.14}$$

for each $t \in [0,T_0)$.

Next, we prove a technical result, which will be used in the proof of both Theorem 1.2 and Theorem 1.3.

Lemma 4.3. *Let* (u,p,h,ω) *be the strong solution to the problem* (1.1), (1.2), (1.4)–(1.10) *as in Theorem* 1.1, *and let us assume that* $f \in L^2(0,T_0;\mathscr{L}^2(\mathbb{R}^d))$. *Then, we have that for almost every* t *in* $(0,T_0)$,

$$\begin{aligned} -\int_{\mathscr{F}(t)} &\operatorname{div} \sigma(u,p) \cdot \Big(\frac{\partial u}{\partial t} + (\hat{\Lambda}\cdot\nabla)u - (u\cdot\nabla)\hat{\Lambda}\Big)\,\mathrm{d}x \\ &= \nu \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}(t)} |Du|^2\,\mathrm{d}x + M|h''(t)|^2 + \omega'(t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}(J\omega(t)) \\ &\quad - M[\omega(t)\times h'(t)]\cdot h''(t) \\ &\quad - \int_{\mathscr{O}(t)} \varrho f(x,t)\cdot[h''(t) + \omega'(t)\times(x-h(t)) - \omega(t)\times h'(t)]\,\mathrm{d}x \\ &\quad + 2\nu\int_{\mathscr{F}(t)} (Du) : ((\nabla u)(\nabla\hat{\Lambda}) - D((u\cdot\nabla)\hat{\Lambda}))\,\mathrm{d}x. \end{aligned} \tag{4.15}$$

P r o o f. We prove formula (4.15) only in the case $d=3$, the proof in the case $d=2$ is similar. We split the proof in two steps.

First Step. Let us consider

$$v \in \mathscr{U}(\mathscr{F}_{T_0})$$

satisfying

$$\operatorname{div} v = 0, \quad x \in \mathscr{F}(t), \ t \in (0,T_0),$$

and

$$v(x,t) = l(t) + k(t)\times(x-h(t)) \quad x \in \partial\mathscr{O}(t), \ t \in (0,T_0),$$

with $(l,k)\in H^1(0,T_0)$. We assume moreover that

$$\frac{\partial v}{\partial t}\Big|_{\mathscr{F}(t)} \in L^2(0,T_0;\mathscr{H}^1(\mathscr{F}(t))). \tag{4.16}$$

By using the change of variables $y\mapsto \hat{X}(y,t)$ introduced above, we easily get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}(t)} |Dv|^2\,\mathrm{d}x = \int_{\mathscr{F}(t)} \frac{\partial}{\partial t}|Dv|^2 + \hat{\Lambda}\cdot\nabla(|Dv|^2)\,\mathrm{d}x, \tag{4.17}$$

with $\hat{\Lambda}$ defined by (4.10). A simple calculation shows that

$$\frac{\partial}{\partial t}|Dv|^2 + \hat{\Lambda}\cdot\nabla(|Dv|^2) = 2Dv : D\Big(\frac{\partial v}{\partial t} + (\hat{\Lambda}\cdot\nabla)v\Big) - 2(Dv) : ((\nabla v)(\nabla\hat{\Lambda})). \tag{4.18}$$

Combining (4.17) and (4.18), we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}(t)} |Dv|^2\,\mathrm{d}x = \int_{\mathscr{F}(t)} Dv : D\Big(\frac{\partial v}{\partial t} + (\hat{\Lambda}\cdot\nabla)v\Big) - (Dv) : ((\nabla v)(\nabla\hat{\Lambda}))\,\mathrm{d}x$$

and thus

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}(t)} |Dv|^2\,\mathrm{d}x = &\int_{\mathscr{F}(t)} Dv : D\Big(\frac{\partial v}{\partial t} + (\hat{\Lambda}\cdot\nabla)v - (v\cdot\nabla)\hat{\Lambda}\Big)\\ &+ (Dv) : (D((v\cdot\nabla)\hat{\Lambda})) - (Dv) : ((\nabla v)(\nabla\hat{\Lambda}))\,\mathrm{d}x.\end{aligned} \tag{4.19}$$

By noticing that

$$\operatorname{div}\Big(\frac{\partial v}{\partial t} + (\hat{\Lambda}\cdot\nabla)v - (v\cdot\nabla)\hat{\Lambda}\Big) = 0,$$

we deduce from (4.19) that

$$\begin{aligned}\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}(t)} |Dv|^2\,\mathrm{d}x = &\int_{\mathscr{F}(t)} \sigma(v,p) : D\Big(\frac{\partial v}{\partial t} + (\hat{\Lambda}\cdot\nabla)v - (v\cdot\nabla)\hat{\Lambda}\Big)\\ &+ 2\nu(Dv) : (D((v\cdot\nabla)\hat{\Lambda})) - 2\nu(Dv) : ((\nabla v)(\nabla\hat{\Lambda}))\,\mathrm{d}x.\end{aligned} \tag{4.20}$$

On the other hand, since

$$v(\hat{X}(y,t),t) = l(t) + k(t)\times(Q(t)y) \quad \text{for } y\in\partial\mathscr{O},$$

we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[v(\hat{X}(y,\cdot),\cdot)](t) &= \Big[\frac{\partial v}{\partial t} + (\hat{\Lambda}\cdot\nabla)v\Big](X(y,t),t)\\ &= l'(t) + k'(t)\times(Q(t)y) + k(t)\times(\omega(t)\times Q(t)y)\\ &\qquad \text{for } y\in\partial\mathscr{O}.\end{aligned} \tag{4.21}$$

We also notice that

$$[(v\cdot\nabla)\hat{\Lambda}](x,t)=\omega(t)\times v\quad\text{for }x\in\partial\mathscr{O}(t). \tag{4.22}$$

Combining (4.21) and (4.22), we deduce that

$$\begin{aligned}\frac{\partial v}{\partial t}+(\hat{\Lambda}\cdot\nabla)v-(v\cdot\nabla)\hat{\Lambda}&=l'(t)+k'(t)\times(x-h(t))-\omega(t)\times l(t)\\&\quad+(k(t)\times\omega(t))\times(x-h(t))\quad\text{for }x\in\partial\mathscr{O}(t).\end{aligned} \tag{4.23}$$

Integrating (4.20) by parts and using (4.23), we deduce that

$$\begin{aligned}\nu\frac{\mathrm{d}}{\mathrm{d}t}&\int_{\mathscr{F}(t)}|Dv|^2\,\mathrm{d}x\\&=-\int_{\mathscr{F}(t)}\operatorname{div}\sigma(v,p)\cdot\Big(\frac{\partial v}{\partial t}+(\hat{\Lambda}\cdot\nabla)v-(v\cdot\nabla)\hat{\Lambda}\Big)\,\mathrm{d}x\\&\quad+\int_{\partial\mathscr{O}(t)}\sigma(v,p)n\cdot(l'(t)+k'(t)\times(x-h(t))-\omega(t)\times l(t)\\&\qquad\qquad+(k(t)\times\omega(t))\times(x-h(t)))\,\mathrm{d}\Gamma\\&\quad+\int_{\mathscr{F}(t)}2\nu(Dv):(D((v\cdot\nabla)\hat{\Lambda}))-2\nu(Dv):((\nabla v)(\nabla\hat{\Lambda}))\,\mathrm{d}x.\end{aligned} \tag{4.24}$$

Second Step. Let us use the change of variables (2.8)–(2.9) introduced in Section 2: we set

$$U(y,t)=J_Y(X(y,t),t)u(X(y,t),t).$$

Since the operator A defined by (3.10)–(3.12) is non-negative, we have that the mapping

$$U\mapsto\|U\|_{D(A)}:=\|(I+A)U\|_{\mathscr{L}^2(\mathbb{R}^d)},$$

is a norm in $D(A)$. Then, the dual space of $D(A)$ (endowed with the above norm), with respect to the pivot space $\mathbb{V}$, is $D(A)'=\mathbb{H}$ (see (3.8) and (3.9) for a definition of these spaces). Thus, since (u,p,h,ω) is the strong solution to the problem (1.1), (1.2), (1.4)–(1.10) we have that $U\in L^2(0,T;D(A))$ and that $U'\in L^2(0,T;D(A)')$. Therefore, according to a classical result (see, for instance, [23, pages 261–262]), there exists a sequence $U^N\in C^\infty([0,T],D(A))$ ($N\in\mathbb{N}$) such that

$$\begin{aligned}U^N&\to U\text{ strongly in }L^2(0,T;D(A)),\\(U^N)'&\to U'\text{strongly in }L^2(0,T;D(A)').\end{aligned} \tag{4.25}$$

Let us set $u^N(x,t) = J_X(Y(x,t),t)U^N(Y(x,t),t)$. We obtain from [16, Theorem 2.5] that

$$\frac{\partial u^N}{\partial t}(x,t) = J_X(Y(x,t),t)\Big[\frac{\partial U^N}{\partial t}(Y(x,t),t) + (\mathbf{M}U^N)(Y(x,t),t)\Big], \tag{4.26}$$

where $(\mathbf{M}U)$ is defined by (2.11). In particular, we deduce from (4.26) that

$$\frac{\partial u^N}{\partial t}\Big|_{\mathscr{F}(t)} \in L^2(0,T;\mathscr{H}^1(\mathscr{F}(t))).$$

We also have that

$$\operatorname{div} u^N = 0, \quad x \in \mathscr{F}(t),\ \ t \in (0,T_0),$$

and

$$u^N(x,t) = l^N(t) + k^N(t) \times (x - h(t)) \quad x \in \partial\mathscr{O}(t),\ \ t \in (0,T_0),$$

with $(l^N, k^N) \in H^1(0,T_0)$.

From the first step, we deduce that

$$\begin{aligned}
\nu \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{F}(t)} |Du^N|^2 \,\mathrm{d}x
&= -\int_{\mathscr{F}(t)} \operatorname{div} \sigma(u^N,p) \cdot \Big(\frac{\partial u^N}{\partial t} + (\hat{\Lambda}\cdot\nabla)u^N - (u^N\cdot\nabla)\hat{\Lambda}\Big)\,\mathrm{d}x \\
&\quad + \int_{\partial\mathscr{O}(t)} \sigma(u^N,p)n \cdot \big((l^N)'(t) \\
&\quad + (k^N)'(t) \times (x-h(t)) - \omega(t) \times l^N(t) + (k^N(t)\times\omega(t)) \times (x-h(t))\big)\,\mathrm{d}\Gamma \\
&\quad + \int_{\mathscr{F}(t)} 2\nu(Du^N) : (D((u^N\cdot\nabla)\hat{\Lambda})) - 2\nu(Du^N) : ((\nabla u^N)(\nabla\hat{\Lambda}))\,\mathrm{d}x.
\end{aligned} \tag{4.27}$$

Using again [16, Theorem 2.5], (4.25) implies that

$$\begin{gathered}
l^N \to h' \quad \text{in } H^1(0,T_0), \qquad k^N \to \omega \quad \text{in } H^1(0,T_0), \\
u^N \to u \quad \text{in } L^2(0,T_0;H^2(\mathscr{F}(t)))
\end{gathered}$$

and

$$\frac{\partial u^N}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{in } L^2(0,T_0;L^2(\mathscr{F}(t))).$$

Combining the above convergences with (4.27), we obtain the result. □

4.2. Proof of Theorem 1.2

We are now in a position to complete the proof of Theorem 1.2. Assume $d = 2$. We recall that we argue by contradiction: we assume that $T_0 < \infty$ and we are going to show that the mappings

$$t \mapsto \|u(t)\|_{\mathscr{H}^1(\mathscr{F}(t))}, \quad t \mapsto |h'(t)|, \quad t \mapsto |\omega(t)|,$$

are bounded in $[0, T_0)$.

Proof of Theorem 1.2. Let (u, p, h, ω) be the strong solution of the problem (1.1), (1.2), (1.4)–(1.10) given by Theorem 1.1. From Lemma 4.1, we already know that the mappings

$$t \mapsto \|u(t)\|_{\mathscr{L}^2(\mathscr{F}(t))}, \quad t \mapsto |h'(t)|, \quad t \mapsto |\omega(t)|,$$

are bounded in $[0, T_0)$. As a consequence, (4.13) yields that

$$\|\hat{\Lambda}\|_{\mathscr{W}^{2,\infty}(\mathscr{F}(t))} \leqslant C\, K_1, \tag{4.28}$$

where C is a positive constant and where K_1 is defined by

$$K_1 = (\|f\|^2_{L^1(0,T_0;\mathscr{L}^2(\mathbb{R}^2))} + \|u_0\|^2_{\mathscr{L}^2(\mathscr{F})} + |h^{(1)}|^2 + |\omega^{(0)}|^2)^{1/2}.$$

First, we take the inner product of the equation (1.4) with

$$\frac{\partial u}{\partial t} + (\hat{\Lambda} \cdot \nabla) u - (u \cdot \nabla)\hat{\Lambda}$$

and we obtain

$$\begin{aligned}
\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathscr{L}^2(\mathscr{F}(t))} &+ \int_{\mathscr{F}(t)} \frac{\partial u}{\partial t} \cdot [(\hat{\Lambda} \cdot \nabla) u - (u \cdot \nabla)\hat{\Lambda}]\,\mathrm{d}x \\
&- \int_{\mathscr{F}(t)} \operatorname{div} \sigma(u,p) \cdot \Big(\frac{\partial u}{\partial t} + (\hat{\Lambda} \cdot \nabla) u - (u \cdot \nabla)\hat{\Lambda}\Big)\,\mathrm{d}x \\
&= - \int_{\mathscr{F}(t)} [(u \cdot \nabla) u] \cdot \Big(\frac{\partial u}{\partial t} + (\hat{\Lambda} \cdot \nabla) u - (u \cdot \nabla)\hat{\Lambda}\Big)\,\mathrm{d}x \\
&+ \int_{\mathscr{F}(t)} f \cdot \Big(\frac{\partial u}{\partial t} + (\hat{\Lambda} \cdot \nabla) u - (u \cdot \nabla)\hat{\Lambda}\Big)\,\mathrm{d}x, \quad \text{a.e in } (0, T_0).
\end{aligned}$$

Combining the above equation with Lemmata 4.1 and 4.3, and with the relation (4.28), we obtain that for a.e. $t \in (0, T_0)$

$$\begin{aligned}
\frac{1}{2}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathscr{L}^2(\mathscr{F}(t))} + \nu \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{F}(t)} |D(u)|^2\,\mathrm{d}x + \frac{1}{2} M |h''(t)|^2 + \frac{1}{2} J |\omega'(t)|^2 \\
\leqslant C\big(K_1^4 + (K_1^2 + 1)\|\nabla u\|^2_{\mathscr{L}^2(\mathbb{R}^2)^4} + \|(u \cdot \nabla) u\|^2_{\mathscr{L}^2(\mathscr{F}(t))} \\
+ \|f(t)\|^2_{\mathscr{L}^2(\mathbb{R}^2)} + |\omega(t) \times h'(t)|^2\big).
\end{aligned} \tag{4.29}$$

The above inequality, the inequality (4.3) and (4.36) yield that for a.e. $t \in (0, T_0)$

$$(4.30)\quad \frac{1}{2}\Big\|\frac{\partial u}{\partial t}(t)\Big\|^2_{\mathscr{L}^2(\mathscr{F}(t))} + \nu\frac{\mathrm{d}}{\mathrm{d}t}\|D(u(t))\|^2_{[L^2(\mathbb{R}^2)]^4} + \frac{1}{2}M|h''(t)|^2 + \frac{1}{2}J|\omega'(t)|^2$$
$$\leqslant C\big(K_1^4 + (K_1^2+1)\|\nabla u\|^2_{\mathscr{L}^2(\mathbb{R}^2)^4} + \|f(t)\|^2_{\mathscr{L}^2(\mathbb{R}^2)} + \|(u\cdot\nabla)u(t)\|^2_{\mathscr{L}^2(\mathscr{F}(t))}\big),$$

and where $C = C(\mathscr{O}, \varrho, r)$ is a positive constant.

Next, we have to estimate $(u \cdot \nabla)u$ in terms of the left-hand side of (4.30). To do this, we use the change of variables $x = Q(t)y + h(t)$. Consider the functions $\bar{u}(y,t) = Q^*(t)u(Q(t)y + h(t), t)$ and $\bar{p}(y,t) = p(Q(t)y + h(t), t)$, with $Q(t)$ given by (1.2). By means of simple calculations, it is easy to see that $(u\cdot\nabla_x)u = Q(\bar{u}\cdot\nabla_y)\bar{u}$. Thus, it follows that

$$(4.31)\qquad \int_{\mathscr{F}(t)} |(u\cdot\nabla_x)u|^2\,\mathrm{d}x = \int_{\mathscr{F}} |(\bar{u}\cdot\nabla_y)\bar{u}|^2\,\mathrm{d}y.$$

Next, by applying the Hölder inequality combined with the continuous embedding of $H^{1/2}(\mathscr{F})$ in $L^4(\mathscr{F})$ and with an interpolation inequality (see, for instance, Lions and Magenes [17, p. 23]), we obtain that there exists a constant $C_1 = C_1(\mathscr{F}) > 0$ such that

$$(4.32)\quad \int_{\mathscr{F}} |(\bar{u}\cdot\nabla_y)\bar{u}|^2\,\mathrm{d}y \leqslant \|\bar{u}\|^2_{[L^4(\mathscr{F})]^2}\|\nabla\bar{u}\|^2_{[L^4(\mathscr{F})]^4}$$
$$\leqslant C_1\|\bar{u}\|_{\mathscr{L}^2(\mathscr{F})}\|\bar{u}\|_{\mathscr{H}^1(\mathscr{F})}\|\nabla\bar{u}\|_{[L^2(\mathscr{F})]^4}\|\nabla\bar{u}\|_{[H^1(\mathscr{F})]^4}$$
$$\leqslant C_1\|\bar{u}\|_{\mathscr{L}^2(\mathscr{F})}\|\nabla\bar{u}\|_{[L^2(\mathscr{F})]^4}\{\|\bar{u}\|_{\mathscr{L}^2(\mathscr{F})} + \|\nabla\bar{u}\|_{[L^2(\mathscr{F})]^4}\}$$
$$\times\Big\{\|\nabla\bar{u}\|_{[L^2(\mathscr{F})]^4} + \sum_{i=1}^{2}\|D^2\bar{u}_i\|_{[L^2(\mathscr{F})]^4}\Big\}.$$

On the other hand, we can consider (u, p) as the solution of a resolvent Stokes problem at some fixed time $t > 0$:

$$u - \nu\Delta u + \nabla p = \tilde{f} + u \quad \text{in } \mathscr{F}(t),$$
$$\operatorname{div} u = 0 \quad \text{in } \mathscr{F}(t),$$
$$u|_{\partial\mathscr{O}(t)} = h'(t) + \omega(t)\times[x - h(t)] \quad x \in \partial\mathscr{O}(t),$$

where

$$(4.33)\qquad \tilde{f}(x,t) = -\frac{\partial u}{\partial t}(x,t) - (u\cdot\nabla)u(x,t) + f(x,t).$$

Thus, by means of simple calculations, it clearly follows that $(\bar{u}, \bar{p})$ satisfies the similar resolvent Stokes problem

$$\begin{gathered} \bar{u} - \nu\Delta\bar{u} + \nabla\bar{p} = \tilde{F} + \bar{u} \quad \text{in } \mathscr{F}, \\ \operatorname{div} \bar{u} = 0 \quad \text{in } \mathscr{F}, \\ \bar{u}|_{\partial\mathscr{O}} = \bar{\Lambda}|_{\partial\mathscr{O}}, \end{gathered}$$

where

$$\tilde{F}(y,t) = Q^*(t)\tilde{f}(Q(t)y + h(t), t),$$

and where $\bar{\Lambda}(y,t)$ is given by Proposition 4.2. Therefore, as a consequence of [5, Theorem 2.1], we obtain that there exists a constant $C_2 = C_2(\nu, \mathscr{F}) > 0$ such that

$$\sum_{i=1}^{2} \|D^2\bar{u}_i\|_{[L^2(\mathscr{F})]^4} \leqslant C_2(\|\tilde{F}\|_{\mathscr{L}^2(\mathscr{F})} + \|\bar{u}\|_{\mathscr{L}^2(\mathscr{F})} + \|\bar{\Lambda}\|_{\mathscr{H}^2(\mathbb{R}^2)}). \tag{4.34}$$

Finally, from (4.31) and (4.32), combined with the above inequality, we deduce that

$$\begin{aligned} &\|(u\cdot\nabla_x)u\|^2_{\mathscr{L}^2(\mathscr{F}(t))} \\ &\quad \leqslant C_1\|u\|_{\mathscr{L}^2(\mathscr{F}(t))}\|\nabla u\|^2_{[L^2(\mathscr{F}(t))]^4}(\|u\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|\nabla u\|_{[L^2(\mathscr{F}(t))]^4}) \\ &\qquad + C_1C_2\|u\|_{\mathscr{L}^2(\mathscr{F}(t))}\|\nabla u\|_{[L^2(\mathscr{F}(t))]^4}(\|u\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|\nabla u\|_{[L^2(\mathscr{F}(t))]^4}) \\ &\qquad\qquad \times (\|\tilde{f}\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|\bar{u}\|_{\mathscr{L}^2(\mathscr{F})} + \|\bar{\Lambda}\|_{\mathscr{H}^2(\mathbb{R}^2)}). \end{aligned}$$

Combining the above inequality with the definition (4.33) of $\tilde{f}$ and with the estimate (4.14) of $\bar{\Lambda}$, we obtain that for a.e. $t \in (0, T_0)$

$$\begin{aligned} &\|(u\cdot\nabla)u\|^2_{\mathscr{L}^2(\mathscr{F}(t))} \\ &\quad \leqslant C_1\|u\|_{\mathscr{L}^2(\mathscr{F}(t))}\|\nabla u\|^2_{[L^2(\mathscr{F}(t))]^4}(\|u\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|\nabla u\|_{[L^2(\mathscr{F}(t))]^4}) \\ &\qquad + C_1C_2\|u\|_{\mathscr{L}^2(\mathscr{F}(t))}\|\nabla u\|_{[L^2(\mathscr{F}(t))]^4}(\|u\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|\nabla u\|_{[L^2(\mathscr{F}(t))]^4}) \\ &\qquad \times \Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|(u\cdot\nabla)u\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|f\|_{\mathscr{L}^2(\mathbb{R}^2)} + \|u\|_{\mathscr{L}^2(\mathscr{F}(t))} + CK_1\Big), \end{aligned}$$

which implies that for any $\varepsilon > 0$ there exists a positive constant C_ε depending on $\mathscr{O}$, ν and ϱ such that a.e. in $(0, T_0)$

$$\begin{aligned} \|(u\cdot\nabla)u\|^2_{\mathscr{L}^2(\mathscr{F}(t))} \leqslant C_\varepsilon\big(&\|u\|_{\mathscr{L}^2(\mathscr{F}(t))}\|\nabla u\|^2_{[L^2(\mathscr{F}(t))]^4}(\|u\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|\nabla u\|_{[L^2(\mathscr{F}(t))]^4}) \\ &+ \|u\|^2_{\mathscr{L}^2(\mathscr{F}(t))}\|\nabla u\|^2_{[L^2(\mathscr{F}(t))]^4}(\|u\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|\nabla u\|_{[L^2(\mathscr{F}(t))]^4})^2 \\ &+ \|f\|^2_{\mathscr{L}^2(\mathbb{R}^2)} + \|u\|^2_{\mathscr{L}^2(\mathscr{F}(t))} + K_1^2\big) + \varepsilon\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathscr{L}^2(\mathscr{F}(t))}. \end{aligned}$$

Therefore, by replacing the above estimate (with ε small enough) in (4.30), we get that for a.e. $t \in (0, T_0)$

$$\begin{aligned}\frac{1}{8}\Big\|\frac{\partial u}{\partial t}(t)\Big\|^2_{\mathscr{L}^2(\mathscr{F}(t))} &+ \frac{1}{2}\nu\frac{\mathrm{d}}{\mathrm{d}t}\|D(u(t))\|^2_{[L^2(\mathbb{R}^2)]^4} + \frac{1}{4}M|h''(t)|^2 + \frac{1}{4}J|\omega'(t)|^2 \\ &\leqslant C_3\big(K_1^4 + (K_1^2+1)\|\nabla u\|^2_{\mathscr{L}^2(\mathbb{R}^2)^4} + \|f(t)\|^2_{\mathscr{L}^2(\mathbb{R}^2)} \\ &\quad + \|u\|_{\mathscr{L}^2(\mathscr{F}(t))}\|\nabla u\|^2_{[L^2(\mathscr{F}(t))]^4}(\|u\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|\nabla u\|_{[L^2(\mathscr{F}(t))]^4}) \\ &\quad + \|u\|^2_{\mathscr{L}^2(\mathscr{F}(t))}\|\nabla u\|^2_{[L^2(\mathscr{F}(t))]^4}(\|u\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|\nabla u\|_{[L^2(\mathscr{F}(t))]^4})^2 \\ &\qquad + \|u\|^2_{\mathscr{L}^2(\mathscr{F}(t))} + K_1^2\big),\end{aligned}$$

for some positive constant $C_3 = C_3(\mathscr{O}, \varrho, \nu, r)$.

Hence, by integrating the above inequality with respect to t, and applying Lemma 4.1, we obtain that for all $t \in [0, T_0)$

$$\begin{aligned}\frac{1}{8}\int_0^t \Big\|\frac{\partial u}{\partial t}(s)\Big\|^2_{\mathscr{L}^2(s)}\,\mathrm{d}s &+ \frac{1}{4}\nu\|\nabla u(t)\|^2_{[L^2(\mathbb{R}^2)]^4} \\ &+ \frac{1}{4}M\int_0^t |h''(s)|^2\,\mathrm{d}s + \frac{1}{4}J\int_0^t |\omega'(s)|^2\,\mathrm{d}s \\ &\leqslant K_2 + K_3\int_0^t \Big(\frac{1}{4}\nu\|\nabla u(s)\|^2_{[L^2(\mathbb{R}^2)]^4}\Big)(2\nu\|\nabla u(s)\|^2_{[L^2(\mathbb{R}^2)]^4})\,\mathrm{d}s,\end{aligned} \tag{4.35}$$

where

$$K_2 = \frac{1}{4}\nu\|\nabla u_0\|^2_{[L^2(\mathbb{R}^2)]^4} + C_4\big(K_1^2(K_1^2+1)T_0 + K_1^2(K_1^4+1) + \|f\|^2_{L^2(0,T_0;\mathscr{L}^2(\mathbb{R}^2))}\big),$$
$$K_3 = C_4K_1^2,$$

and where $C_4 = C_4(\mathscr{O}, \varrho, \nu, r)$ is a positive constant.

Finally, by applying the Grönwall lemma in (4.35), we get that for all $t \in [0, T_0)$

$$\begin{aligned}\frac{1}{8}\int_0^t \Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathscr{L}^2(\mathscr{F}(s)}\,\mathrm{d}s &+ \frac{1}{4}\nu\|\nabla u(t)\|^2_{[L^2(\mathbb{R}^2)]^4} + \frac{1}{4}M\int_0^t |h''(s)|^2\,\mathrm{d}s + \frac{1}{4}J\int_0^t |\omega'(s)|^2\,\mathrm{d}s \\ &\leqslant K_2\exp 2\Big\{K_3\int_0^{T_0}(2\nu\|\nabla u(s)\|^2_{[L^2(\mathbb{R}^2)]^4})\,\mathrm{d}s\Big\} \\ &\leqslant K_2\exp\{2CK_3K_1\}.\end{aligned}$$

The above estimate combined with Lemma 4.1 implies the result. □

Remark 4.4. In the above proof, assuming that $T_0 < \infty$, we have shown that the H^1-norm of $u(t)$ does not blow-up at T_0, which, according to Theorem 1.1, implies the global in time existence of u for $d = 2$. This proof also yields that

$$\frac{\partial u}{\partial t}, \quad \text{and} \quad h'',\ \omega'$$

are bounded in $L^2(0, T_0; \mathscr{L}(\mathscr{F}(t)))$, and in $L^2(0, T_0)$ respectively.

4.3. Proof of Theorem 1.3

In this section we prove Theorem 1.3: if the H^1-norm of the initial velocity u_0 is small enough, and if the external force is also small enough (in some appropriate norm), then the solution of the problem (1.1), (1.2), (1.4)–(1.10), given by Theorem 1.1, is global in time. We still argue by contradiction and assume that $T_0 < \infty$.

Proof of Theorem 1.3. We begin the proof as in the proof of Theorem 1.2: let (u, p, h, ω) be the strong solution of the problem (1.1), (1.2), (1.4)–(1.10) given by Theorem 1.1. From Lemma 4.1, we already know that the mappings

$$t \mapsto \|u(t)\|_{\mathscr{L}^2(\mathscr{F}(t))}, \quad t \mapsto |h'(t)|, \quad t \mapsto |\omega(t)|,$$

are bounded in $[0, T_0)$. The relation (4.28) also holds true, the constant K_1 being defined by

$$K_1 = (\|f\|^2_{L^1(0,\infty;\mathscr{L}^2(\mathbb{R}^3))} + \|u_0\|^2_{\mathscr{L}^2(\mathscr{F})} + |h^{(1)}|^2 + |\omega^{(0)}|^2)^{1/2}.$$

We can also estimate $\hat{\Lambda}$ in a different way. From the Sobolev-Gagliardo-Nirenberg inequality, the following relation

$$|h'(t)| + |\omega(t)| \leqslant C\|\nabla u(t)\|_{[L^2(\mathbb{R}^3)]^9} \tag{4.36}$$

holds true and the above inequality and (4.13) imply that

$$\|\hat{\Lambda}\|_{\mathscr{W}^{2,\infty}(\mathscr{F}(t))} \leqslant C\|\nabla u(t)\|_{[L^2(\mathbb{R}^3)]^9}. \tag{4.37}$$

Both relations (4.28) and (4.37) are used in the sequel. We first take the inner product of the equation (1.4) with

$$\frac{\partial u}{\partial t} + (\hat{\Lambda} \cdot \nabla)u - (u \cdot \nabla)\hat{\Lambda}$$

and we obtain

$$\begin{aligned}
&\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathscr{L}^2(\mathscr{F}(t))} + \int_{\mathscr{F}(t)} \frac{\partial u}{\partial t} \cdot [(\hat{\Lambda} \cdot \nabla)u - (u \cdot \nabla)\hat{\Lambda}]\,\mathrm{d}x \\
&\qquad - \int_{\mathscr{F}(t)} \operatorname{div}\sigma(u,p) \cdot \Big(\frac{\partial u}{\partial t} + (\hat{\Lambda} \cdot \nabla)u - (u \cdot \nabla)\hat{\Lambda}\Big)\,\mathrm{d}x \\
&\quad = - \int_{\mathscr{F}(t)} [(u \cdot \nabla)u] \cdot \Big(\frac{\partial u}{\partial t} + (\hat{\Lambda} \cdot \nabla)u - (u \cdot \nabla)\hat{\Lambda}\Big)\,\mathrm{d}x \\
&\qquad + \int_{\mathscr{F}(t)} f \cdot \Big(\frac{\partial u}{\partial t} + (\hat{\Lambda} \cdot \nabla)u - (u \cdot \nabla)\hat{\Lambda}\Big)\,\mathrm{d}x, \quad \text{a.e in } (0, T_0).
\end{aligned}$$

Combining the above equation with Lemmata 4.1 and 4.3, and with relations (2.8) and (4.37), we obtain that for a.e. $t \in (0, T_0)$

$$(4.38)\quad \frac{1}{2}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathscr{L}^2(\mathscr{F}(t))} + \nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}(t)} |D(u)|^2\,\mathrm{d}x + \frac{1}{2}M|h''(t)|^2 + \frac{c_2}{2}|\omega'(t)|^2 \\ \leqslant C\big(K_1(K_1+1)\|\nabla u(t)\|^2_{[L^2(\mathscr{F}(t))]^9} + \|(u\cdot\nabla)u\|^2_{\mathscr{L}^2(\mathscr{F}(t))} + \|f(t)\|^2_{\mathscr{L}^2(\mathbb{R}^3)} \\ + |\omega(t)\times h'(t)|^2 + |(J\omega(t))\times\omega(t)|^2\big),$$

where $C = C(\mathscr{O}, \varrho, r)$ is a positive constant and where c_2 is the constant defined by (4.2). The above inequality, the inequality (4.3) and the inequality (4.36) yield that for a.e. $t \in (0, T_0)$

$$(4.39)\quad \frac{1}{2}\Big\|\frac{\partial u}{\partial t}(t)\Big\|^2_{\mathscr{L}^2(\mathscr{F}(t))} + \nu\frac{\mathrm{d}}{\mathrm{d}t}\|D(u(t))\|^2_{[L^2(\mathbb{R}^3)]^9} + \frac{1}{2}M|h''(t)|^2 + \frac{1}{2}c_2|\omega'(t)|^2 \\ \leqslant C\big(K_1(K_1+1)\|\nabla u(t)\|^2_{[L^2(\mathscr{F}(t))]^9} + \|f(t)\|^2_{\mathscr{L}^2(\mathbb{R}^3)} \\ + \|(u\cdot\nabla)u(t)\|^2_{\mathscr{L}^2(\mathscr{F}(t))}\big),$$

and where $C = C(\mathscr{O}, \varrho, r)$ is a positive constant.

As in the proof of Theorem 1.2, we have to estimate $(u\cdot\nabla)u$ in terms of the left-hand side of (4.39). We consider the function $\bar{u}(y,t) = Q^*(t)u(Q(t)y + h(t), t)$ and we use similar arguments as those of the proof of Theorem 1.2, (see (4.31), (4.32)) to finally obtain

$$(4.40)\quad \|(u\cdot\nabla_x)u\|^2_{\mathscr{L}^2(\mathscr{F}(t))} \leqslant C_1\|\nabla\bar{u}\|^3_{[L^2(\mathbb{R}^3)]^9}\left(\|\nabla\bar{u}\|_{[L^2(\mathbb{R}^3)]^9} + \sum_{i=1}^{3}\|D^2\bar{u}_i\|_{[L^2(\mathscr{F})]^9}\right),$$

for some constant $C_1 = C_1(\mathscr{F}) > 0$ and a.e. in $(0, T)$.

On the other hand, if we define $\overline{v} = \bar{u} - \bar{\Lambda}$, then for any fixed time $t > 0$, $(\overline{v}(t), \bar{p}(t))$ is a solution of the stationary Stokes equations:

$$-\nu\Delta\overline{v} + \nabla\bar{p} = \tilde{F} \quad \text{in } \mathscr{F},$$
$$\operatorname{div}\overline{v} = 0 \quad \text{in } \mathscr{F},$$
$$\overline{v} = 0 \quad \text{on } \partial\mathscr{O},$$

where

$$\tilde{f}(x,t) = -\frac{\partial u}{\partial t}(x,t) - (u\cdot\nabla)u(x,t) + f(x,t),$$

where

$$\tilde{F}(y,t) = Q^*(t)\tilde{f}(Q(t)y + h(t), t),$$

and where $\bar{\Lambda}(y,t)$ is given by Proposition 4.2. Therefore, as consequence of [13, Lemma 1], we obtain that there exists a positive constant C depending only on ν and on $\mathscr{F}$ such that

$$\sum_{i=1}^{3} \|D^2 \overline{v}_i\|_{[L^2(\mathscr{F})]^9} \leqslant C(\|\tilde{F}\|_{\mathscr{L}^2(\mathscr{F})} + \|\nabla \overline{v}\|_{\mathscr{L}^2(\mathscr{F})}).$$

The above inequality yields that the existence of a constant $C_2 = C_2(\nu, \mathscr{F}) > 0$ such that

$$\sum_{i=1}^{3} \|D^2 \bar{u}_i\|_{[L^2(\mathscr{F})]^9} \leqslant C_2(\|\tilde{F}\|_{\mathscr{L}^2(\mathscr{F})} + \|\nabla \bar{u}\|_{\mathscr{L}^2(\mathscr{F})} + \|\bar{\Lambda}\|_{\mathscr{H}^2(\mathbb{R}^3)}).$$

Proceeding as in the proof of Theorem 1.2, we get that a.e. in $(0, T_0)$

$$\begin{aligned}
&\|(u\cdot\nabla)u\|^2_{\mathscr{L}^2(\mathscr{F}(t))} \\
&\quad \leqslant C_1\|\nabla u\|^4_{[L^2(\mathbb{R}^3)]^9} + C_1C_2\|\nabla u\|^3_{[L^2(\mathbb{R}^3)]^9} \\
&\qquad \times \Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|(u\cdot\nabla)u\|_{\mathscr{L}^2(\mathscr{F}(t))} + \|f\|_{\mathscr{L}^2(\mathbb{R}^3)} + \|\nabla u\|_{[L^2(\mathbb{R}^3)]^9}\Big).
\end{aligned}$$

The above relation implies that

$$\begin{aligned}
\text{(4.41)}\qquad &\Big\|\frac{\partial u}{\partial t}(t)\Big\|^2_{\mathscr{L}^2(\mathscr{F}(t))} + |h''(t)|^2 + |\omega'(t)|^2 + \nu\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|^2_{[L^2(\mathbb{R}^3)]^9} \\
&\quad \leqslant C_3\big(K_1(K_1+1)\|\nabla u(t)\|^2_{[L^2(\mathbb{R}^3)]^9} + \|f\|^2_{\mathscr{L}^2(\mathbb{R}^3)} \\
&\qquad + \|\nabla u(t)\|^4_{[L^2(\mathbb{R}^3)]^9} + \|\nabla u(t)\|^6_{[L^2(\mathbb{R}^3)]^9}\big),
\end{aligned}$$

for some positive constant $C_3 = C_3(\mathscr{O}, \varrho, \nu)$, and therefore,

$$\begin{aligned}
&\int_0^t \Big\|\frac{\partial u}{\partial t}(s)\Big\|^2_{\mathscr{L}^2(\mathscr{F}(s))}\,\mathrm{d}s + \int_0^t |h''(s)|^2 + \int_0^t |\omega'(s)|^2\,\mathrm{d}s + \nu\|\nabla u(t)\|^2_{[L^2(\mathbb{R}^3)]^9} \\
&\quad \leqslant \nu\|\nabla u_0\|^2_{[L^2(\mathbb{R}^3)]^9} + C_4(K_1^2(K_1^2+1) + \|f\|^2_{L^2(0,\infty;\mathscr{L}^2(\mathbb{R}^3))}) \\
&\qquad + C_4\int_0^t 2\nu\|\nabla u(s)\|^6_{[L^2(\mathbb{R}^3)]^9}\,\mathrm{d}s,
\end{aligned}$$

for some positive constant $C_4 = C_4(\mathscr{O}, \varrho, \nu)$.

Let us assume that

$$\text{(4.42)}\qquad \nu\|\nabla u_0\|^2_{[L^2(\mathbb{R}^3)]^9} + C_4(K_1^2(K_1^2+1) + \|f\|^2_{L^2(0,\infty;\mathscr{L}^2(\mathbb{R}^3))}) + C_4CK_1^2 < \nu,$$

where C is the constant appearing in (4.3). Then it can be easily checked that

$$\|\nabla u(t)\|_{[L^2(\mathbb{R}^3)]^9} \leqslant 1 \quad \forall\, t \in [0, T_0).$$

On the other hand, the relation (1.21) implies (4.42) for c small enough. Therefore, from Lemma 4.1 and from the above inequality, we have that the mapping $t \mapsto \|u(t)\|_{\mathscr{H}^1(\mathscr{F}(t))}$ is bounded in $[0, T_0)$ provided (1.21) holds true for c small enough. We conclude the global existence and uniqueness to the problem (1.1), (1.2), (1.4)–(1.10) for small data. □

Remark 4.5. We note that (4.41) combined with the fact that the mapping

$$t \mapsto \|D(u(t))\|_{[L^2(\mathbb{R}^3)]^9}$$

is bounded in $[0, T_0)$ implies that $\partial u/\partial t$ is bounded in $L^2(0, T_0; \mathscr{L}^2(\mathscr{F}(t)))$, and that h'', ω' are bounded in $L^2(0, T_0)$.

References

[1] *V. I. Arnold*: Ordinary Differential Equations. Springer, Berlin, 1992; translated from the third Russian edition.

[2] *C. Conca, J. San Martín, M. Tucsnak*: Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid. Commun. Partial Differ. Equations *25* (2000), 1019–1042.

[3] *P. Cumsille, M. Tucsnak*: Wellposedness for the Navier-Stokes flow in the exterior of a rotating obstacle. Math. Methods Appl. Sci. *29* (2006), 595–623.

[4] *B. Desjardins, M. J. Esteban*: Existence of weak solutions for the motion of rigid bodies in a viscous fluid. Arch. Ration. Mech. Anal. *146* (1999), 59–71.

[5] *R. Farwiq, H. Sohr*: The stationary and non-stationary Stokes system in exterior domains with non-zero divergence and non-zero boundary values. Math. Methods Appl. Sci. *17* (1994), 269–291.

[6] *E. Feireisl*: On the motion of rigid bodies in a viscous fluid. Appl. Math. *47* (2002), 463–484.

[7] *E. Feireisl*: On the motion of rigid bodies in a viscous compressible fluid. Arch. Ration. Mech. Anal. *167* (2003), 281–308.

[8] *G. P. Galdi*: On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications. Handbook of Mathematical Fluid Dynamics, Vol. I. Elsevier, Amsterdam, 2002, pp. 653–791.

[9] *G. P. Galdi, A. L. Silvestre*: Strong solutions to the problem of motion of a rigid body in a Navier-Stokes liquid under the action of prescribed forces and torques. Nonlinear problems in mathematical physics and related topics, I. Int. Math. Ser. Vol. 1. Kluwer/Plenum, New York, 2002, pp. 121–144.

[10] *G. P. Galdi, A. L. Silvestre*: Strong solutions to the Navier-Stokes equations around a rotating obstacle. Arch. Ration. Mech. Anal. *176* (2005), 331–350.

[11] *C. Grandmont, Y. Maday*: Existence for an unsteady fluid-structure interaction problem. M2AN, Math. Model. Numer. Anal. *34* (2000), 609–636.

[12] *P. Hartman*: Ordinary Differential Equations. Birkhäuser, Boston, 1982.

[13] *J. G. Heywood*: The Navier-Stokes equations: On the existence, regularity and decay of solutions. Indiana Univ. Math. J. *29* (1980), 639–681.

[14] *T. Hishida*: An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle. Arch. Rational Mech. Anal. *150* (1999), 307–348.

[15] *K.-H. Hoffmann, V. N. Starovoitov*: On a motion of a solid body in a viscous fluid. Two-dimensional case. Adv. Math. Sci. Appl. *9* (1999), 633–648.
[16] *A. Inoue, M. Wakimoto*: On existence of solutions of the Navier-Stokes equation in a time dependent domain. J. Fac. Sci. Univ. Tokyo Sect. IA Math. *24* (1977), 303–319.
[17] *J.-L. Lions, E. Magenes*: Non-homogeneous boundary value problems and applications. Vol. I. Springer, Berlin-Heidelberg-New York, 1972.
[18] *J. A. San Martín, V. Starovoitov, M. Tucsnak*: Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid. Arch. Ration. Mech. Anal. *161* (2002), 113–147.
[19] *D. Serre*: Chute libre d'un solide dans un fluide visqueux incompressible. Existence. Japan J. Appl. Math. *4* (1987), 99–110. (In French.)
[20] *A. L. Silvestre*: On the slow motion of a self-propelled rigid body in a viscous incompressible fluid. J. Math. Anal. Appl. *274* (2002), 203–227.
[21] *T. Takahashi*: Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain. Adv. Differ. Equ. *8* (2003), 1499–1532.
[22] *T. Takahashi, M. Tucsnak*: Global strong solutions for the two dimensional motion of an infinite cylinder in a viscous fluid. J. Math. Fluid Mech. *6* (2004), 53–77.
[23] *R. Temam*: Navier-Stokes equations. Theory and numerical analysis, 3rd ed., with an appendix by F. Thomasset. North-Holland, Amsterdam-New York-Oxford, 1984.

Authors' addresses: Patricio Cumsille, Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Avda. Andrés Bello s/n, Casilla 447, Chillán, Chile, e-mail: `pcumsille@ubiobio.cl`; Takahashi Takéo (corresponding author), Institut Elie Cartan UMR 7502, INRIA, Nancy-Université, CNRS, Project-Team CORIDA » BP239, 54506, Vandœuvre-lès-Nancy, Cedex, France, e-mail: `Takeo.Takahashi@iecn.u-nancy.fr`.