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ON GRAPHS WITH THE LARGEST LAPLACIAN INDEX

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Abstract. Let G be a connected simple graph on n vertices. The Laplacian index of G, namely, the greatest Laplacian eigenvalue of G, is well known to be bounded above by n. In this paper, we give structural characterizations for graphs G with the largest Laplacian index n. Regular graphs, Hamiltonian graphs and planar graphs with the largest Laplacian index are investigated. We present a necessary and sufficient condition on n and k for the existence of a k-regular graph G of order n with the largest Laplacian index n. We prove that for a graph G of order $n \geqslant 3$ with the largest Laplacian index n, G is Hamiltonian if G is regular or its maximum vertex degree is $\Delta(G) = n/2$. Moreover, we obtain some useful inequalities concerning the Laplacian index and the algebraic connectivity which produce miscellaneous related results.

Keywords: eigenvalue, Laplacian index, algebraic connectivity, semi-regular graph, regular graph, Hamiltonian graph, planar graph

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1. Introduction

Throughout the paper, G denotes a connected simple graph on n vertices, unless specified otherwise. The spectrum of a graph G is the sequence of eigenvalues of the adjacency matrix $A(G)$, denoted by $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. The Laplacian matrix of G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees. $L(G)$ is singular and positive semidefinite with eigenvalues $\mu_n \geqslant \mu_{n-1} \geqslant \ldots \geqslant \mu_2 \geqslant \mu_1 = 0$, which are called Laplacian eigenvalues. The multiplicity of a Laplacian eigenvalue μ

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of G is denoted by $m_G(\mu)$. The greatest Laplacian eigenvalue (μ_n) of G is called the Laplacian index of G.

Characterizing graphs with a certain spectrum is an interesting topic, and many results have been reported (see [2]). For the Laplacian spectrum, the known works seem to be focused on the Laplacian index (see [8]). Here we recall the following important result, where $\overline{G}$ denotes the complement of the graph G and $\omega(\overline{G})$ the number of connected components of $\overline{G}$.

Theorem A ([8]). *For any graph G of order n we have $\mu_n \leqslant n$, and $m_G(n) = \omega(\overline{G}) - 1$.*

It is well known (see [3]) that G has the second smallest Laplacian value $\mu_2 = 0$ if and only if G is a disconnected graph. So μ_2 is called the *algebraic connectivity* of G. The following corollary is an immediate consequence of Theorem A.

Corollary 1.1. *G has the largest Laplacian eigenvalue $n \Leftrightarrow m_G(n) \geqslant 1 \Leftrightarrow \overline{G}$ is disconnected $\Leftrightarrow \mu_2(\overline{G}) = 0 \Leftrightarrow m_{\overline{G}}(0) \geqslant 2$.*

In [5], Gutman shows that among all trees with a fixed number of vertices, the star, and only the star, has the largest Laplacian index equaling its order.

In this paper, we will give structural characterizations for graphs on n vertices with the largest Laplacian index n. We will also investigate special classes of graphs, such as regular graphs, Hamiltonian graphs and planar graphs, with the largest Laplacian index n. Moreover, we will present some useful inequalities involving the Laplacian index and the algebraic connectivity, and then obtain miscellaneous related results.

We follow the standard terminology. The degree of a vertex u of a graph G is denoted by $d(u)$, and the maximum degree and the minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. G is said to be semi-regular if the degree of any vertex is either $\Delta(G)$ or $\delta(G)$. If $\Delta(G) = \delta(G) = k$, then G is k-regular. Clearly, $\mu_i = k - \lambda_i$ for k-regular graphs. Note that the set of semi-regular graphs includes the set of regular graphs.

A subgraph of G is a graph whose vertices and edges belong to G. A spanning subgraph of G is a subgraph containing all vertices of G. For any subset S of vertices of G, the induced subgraph $\langle S \rangle$ is the maximal subgraph with the vertex set S. A cycle in G is called a Hamiltonian cycle if it passes through every vertex of G exactly once. A graph that contains a Hamiltonian cycle is a Hamiltonian graph. If G can be embedded in the plane so that no two edges intersect at an inner point of any edge, then G is called a planar graph. Further, if G has such a plane embedding with all vertices lying on the exterior boundary, then G is called an outerplanar graph. An outerplanar graph is called a maximal outerplanar graph if no edge can be added without losing outerplanarity. Two graphs are homeomorphic if they can both be obtained from the same graph by inserting new vertices of degree two into some edges. The symbols P_n, C_n and $K_{s,r}$ denote the path of order n, the cycle of order n, and the complete bipartite graph with s vertices in one part and r vertices in another part, respectively. For terminology and notation not defined here, we refer the readers to [1–7].

2. Characterizations for graphs with the largest Laplacian index

From Theorem A it is easy to see that $0 \leqslant m_G(n) \leqslant n-1$ and that $m_G(n) = 0$ if and only if $\overline{G}$ is connected. Below we first give a necessary and sufficient condition for G to have $m_G(n) = k$ when $1 \leqslant k \leqslant n-1$.

Theorem 2.1. *For $1 \leqslant k \leqslant n-1$, $m_G(n) = k$ if and only if G has a spanning subgraph that is a complete $(k+1)$-partite graph.*

Proof. Suppose that $m_G(n) = k$. By Theorem A, $\overline{G}$ has $k+1$ connected components, say $\overline{G_1}, \overline{G_2}, \ldots, \overline{G_{k+1}}$. Thus G contains a subgraph $H = (V, E)$, where $V = V(\overline{G_1}) \cup V(\overline{G_2}) \cup \ldots \cup V(\overline{G_{k+1}})$, $E = \{\{u, v\} \mid u \in V(\overline{G_i}),\, v \in V(\overline{G_j}),\, i \neq j\}$. Clearly, H is a spanning subgraph that is a complete $(k+1)$-partite graph. Conversely, if G has a complete $(k+1)$-partite graph as a spanning subgraph, then $\overline{G}$ contains $k+1$ connected components. By Theorem A, $m_G(n) = k$. $\square$

From Theorem 2.1, we can easily obtain the following results.

Corollary 2.1. *If $m_G(n) = k$, then G contains K_{k+1} as a subgraph. In particular, $m_G(n) = n-1$ if and only if $G \cong K_n$.*

Corollary 2.2. *The chromatic number $\chi(G)$ satisfies $\chi(G) \geqslant m_G(n) + 1$.*

Corollary 2.3. *If G is a bipartite graph, then $m_G(n) \leqslant 1$. The equality holds if and only if G is complete.*

We recall that the join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from the disjoint union of G_1 and G_2 by adding the edges joining every vertex of G_1 with every vertex of G_2. We will use this graph operation to give a structural characterization for graphs on n vertices with the largest Laplacian index n in the next theorem.

Theorem 2.2. *The following statements are equivalent:*
(1) $\mu_n(G) = n$.
(2) *G has a spanning subgraph that is a complete bipartite graph.*
(3) *$G = G_1 \vee G_2$, where each G_i is a graph with at least one vertex.*

Proof. By Theorem 2.1 we see the equivalence of (1) and (2). So we only need to prove the equivalence of (2) and (3). Suppose that G has a spanning subgraph H that is a complete bipartite graph. Denote the two parts of H as V_1 and V_2. Let G_1 and G_2 be the two corresponding subgraphs of G induced by V_1 and V_2, respectively. Then $G = G_1 \vee G_2$. Thus we have proved that (2) implies (3). Conversely, suppose (3) is true. Letting V_1 and V_2 be the vertex sets of G_1 and G_2, respectively, then (2) immediately follows, since the complete bipartite graph with V_1 and V_2 as its two parts is a spanning subgraph of G. $\square$

From the above theorem we easily see that the complete graph K_n and all complete bipartite graphs on n vertices have the largest Laplacian index n.

Considering the maximum degree of G, we immediately get the following

Corollary 2.4. *If $\mu_n(G) = n$, then $\Delta(G) \geqslant \frac{1}{2}n$.*

In fact, it is well known (see [5]) that if $\delta(G) > \frac{1}{2}(n-2)$ then G is connected. So we can obtain Corollary 2.4 also by Corollary 1.1.

Applying Theorem 2.2 to trees and cycles, we can easily get the following results.

Corollary 2.5 (Gutman [5]). *For trees G, $\mu_n(G) = n$ if and only if G is a star. In particular, for paths G, $\mu_n(G) = n$ if and only if $G \cong P_1$, P_2 or P_3.*

Corollary 2.6. *For cycles G, $\mu_n(G) = n$ if and only if $G \cong C_3$ or C_4.*

Let $S(n,i)$ denote a graph obtained by adding i edges to pendants of $K_{1,n-1}$.

Corollary 2.7. *For unicylic graph G of order n, $\mu_n(G) = n$ if and only if G is isomorphic to $S(n,1)$ or C_4.*

Corollary 2.8. *For bicylic graph G of order n, $\mu_n(G) = n$ if and only if G is isomorphic to $S(n,2)$ or $K_{2,3}$.*

Corollary 2.9. *For tricylic graph G of order n, $\mu_n(G) = n$ if and only if G is isomorphic to $S(n,3)$ or $K_{2,3}+e$ or $K_{2,4}$.*

In the next corollary, we present a method of constructing graphs G with the largest Laplacian index n, with $\Delta(G)$ running over $[\lceil \frac{1}{2}n \rceil, n-1]$.

Corollary 2.10. *For any positive integers n and k with $\frac{1}{2}n \leqslant k \leqslant n-1$, there exists a graph G with $\mu_n(G) = n$ and $\Delta(G) = k$.*

P r o o f. Consider $G = K_{k,n-k}$. Then $\Delta(G) = k$. By Theorem 2.2 we have $\mu_n(G) = n$. □

By virtue of Theorem 2.2 we can also easily give a structural characterization for regular graphs with the largest Laplacian index n as follows.

Corollary 2.11. *For $n \geqslant 2$, G is a regular graph with $\mu_n(G) = n$ if and only if $G = G_1 \vee G_2$, where G_i is an r_i-regular graph with n_i vertices, $n_i > 0$, $r_i \geqslant 0$ for $i = 1, 2$; $n_1 + n_2 = n$, and $n_1 + r_2 = n_2 + r_1$.*

Now, it is natural to consider the following question: Is it true that there exists a k-regular graph G with $\mu_n(G) = n$ whenever $\frac{1}{2}n \leqslant k \leqslant n-1$?

The answer is negative in general, since it is a well known fact that the number of vertices of odd degrees cannot be odd in any graph. However, we have obtained a necessary and sufficient condition on n and k for the existence of a k-regular graph G with $\mu_n(G) = n$, as follows.

Theorem 2.3. *There is a k-regular graph G with $\mu_n(G) = n$ if and only if nk is even and $\frac{1}{2}n \leqslant k \leqslant n-1$.*

Our proof of Theorem 2.3 needs the following lemma which has interest in its own right.

Lemma 2.1. *Let $p > 0$ be an even integer. For any integer k with $0 \leqslant k \leqslant p-1$, there is a k-regular simple graph G of order p.*

P r o o f. It is a well known fact that the complete graph K_p is 1-factorable (see p. 71 of [1]), i.e., there are edge-disjoint 1-regular spanning subgraphs $H_1, H_2, \ldots, H_{p-1}$ such that $G = H_1 \cup H_2 \cup \ldots \cup H_{p-1}$. Then, the desired graph is obtained by letting $G = H_1 \cup H_2 \cup \ldots \cup H_k$. □

Now the proof for Theorem 2.3 goes as follows.

P r o o f of Theorem 2.3. Necessity is directly seen from Corollary 2.4 and the fact we pointed out before Theorem 2.3. So we only need to show sufficiency. Assume that nk is even and $\frac{1}{2}n \leqslant k \leqslant n-1$. We may distinguish two cases according to the parity of k.

Case 1. k is even. Note that $0 \leqslant 2k - n \leqslant k - 1$. Then by Lemma 2.1, there is a $(2k-n)$-regular graph G_1 of order k. Let G_2 be the graph with $n-k$ isolated vertices. It is easy to check that $G = G_1 \vee G_2$ is k-regular. Hence G is a desired graph by Corollary 2.11.

Case 2. k is odd. Since nk is even, n must be even. If $k = \frac{1}{2}n$, then $G = K_{k,k}$ is the desired graph. So we may assume that $k \geqslant \frac{1}{2}n + 1$. Then $2k - n - 1 \geqslant 1$. It is also clear that $2k - n - 1 < k - 1$. So by Lemma 2.1, there is a $(2k - n - 1)$-regular graph G_1 of order $k - 1$, because $k - 1$ is even. Note that $n - k + 1$ is even. Hence we may let G_2 be the 1-regular graph consisting of $\frac{1}{2}(n - k + 1)$ isolated graphs K_2. Then, $G = G_1 \vee G_2$ is a k-regular graph of order n. By Corollary 2.11, G is a graph as desired. □

We know that for k-regular graphs G of order n, $\lambda_i = k - \mu_i$. So the smallest of their least eigenvalues is $k - n$. Then Corollary 2.11 and Theorem 2.3 can be restated as follows.

Corollary 2.12. *For $n \geqslant 2$, G is a k-regular graph with the smallest eigenvalue $k - n$ if and only if $G = G_1 \vee G_2$, where G_i is an r_i-regular graph with n_i (> 0) vertices, $n_1 + n_2 = n$, and $n_1 + r_2 = n_2 + r_1 = k$.*

Corollary 2.13. *There is a k-regular graph G with the smallest eigenvalue $k - n$ if and only if nk is even and $\frac{1}{2}n \leqslant k \leqslant n - 1$.*

3. Laplacian index and algebraic connectivity

Lemma 3.1 ([6]). *For a symmetric partitioned matrix A, the eigenvalues of the quotient matrix Q_A interlace the eigenvalues of A. Moreover, if the interlacing is tight, then the partition is regular.*

It is obvious that $0 \leqslant m_G(n) \leqslant n - 1$ for any graph G of order n.

Theorem 3.1. *Let n and k be integers with $0 \leqslant k \leqslant n - 1$. If $m_G(n) = k$, then G has $k + 1$ vertices, say $v_1, v_2, \ldots, v_{k+1}$, such that*

$$\sum_{i=1}^{k+1} d(v_i) \geqslant \frac{(k+1)(n-k-1)}{n}\mu_2(G) + k(k+1), \tag{3.0}$$

$$\sum_{i=1}^{k+1} d(v_i) \geqslant -\frac{(k+1)(n-k-1)}{n}\mu_n(\overline{G}) + (n-1)(k+1). \tag{3.1}$$

The equality in (3.0) *holds if and only if the equality in* (3.1) *holds, and then G is semi-regular.*

P r o o f. First, we consider the case $k = n - 1$. By Theorem A, $\overline{G}$ has n connected components so that G is the complete graph K_n. Then it is easy to verify that the equalities in (3.0) and (3.1) hold. Next, we assume that $0 \leqslant k \leqslant n-2$. Since $m_G(n) = k$, then by Theorem A, $\overline{G}$ has at least $k+1$ connected components, from which we can get an independent set X consisting of $k+1$ vertices of $\overline{G}$. Let $Y = V(G) - X$. Then neither X nor Y is empty since $0 \leqslant k \leqslant n-2$. So, (X,Y) gives rise to a partition of $L(\overline{G})$ with the quotient matrix

$$Q_{L(\overline{G})} = \begin{pmatrix} \frac{\sum_{i=1}^{k+1}(n-1-d(v_i))}{k+1} & -\frac{\sum_{i=1}^{k+1}(n-1-d(v_i))}{k+1} \\ -\frac{\sum_{i=1}^{k+1}(n-1-d(v_i))}{n-k-1} & \frac{\sum_{i=1}^{k+1}(n-1-d(v_i))}{n-k-1} \end{pmatrix}.$$

By easy calculation one can see that $Q_{L(\overline{G})}$ has two eigenvalues

$$\delta_1 = \frac{n(n-1)}{n-k-1} - \frac{n}{(k+1)(n-k-1)} \sum_{i=1}^{k+1} d(v_i), \quad \delta_2 = 0,$$

and that $\delta_1 \geqslant \delta_2$. Then by using the Interlacing Lemma (Lemma 3.1) we obtain the inequality (3.1). We have $\mu_2(G) + \mu_n(\overline{G}) = n$, and so it is easy to see that the equality in (3.0) holds if and only if the equality in (3.1) holds.

Moreover, if the equality in (3.1) holds, then $\mu_1(\overline{G}) = 0 = \delta_2$ and $\mu_n(\overline{G}) = \delta_1$. Thus, the interlacing is tight and so the partition is regular. It follows that $d(v_1) = d(v_2) = \ldots = d(v_{k+1})$ and $d(v_{k+2}) = \ldots = d(v_n)$, i.e., G is semi-regular. □

Applying Theorem 3.1 to regular graphs, we have the following corollaries.

Corollary 3.1. *If G is r-regular and $m_G(n) = k$, then*

$$\mu_2(G) \leqslant \frac{n(r-k)}{n-k-1}, \tag{3.2}$$

$$\mu_n(\overline{G}) \geqslant \frac{n(n-r-1)}{n-k-1}. \tag{3.3}$$

The equality holds in (3.2) *if and only if the equality holds in* (3.3).

Moreover, we have another interesting result for regular graphs.

Corollary 3.2. *If G is r-regular, then $m_G(n) \leqslant r$. The equality holds if and only if $G \cong K_n$.*

P r o o f. Since $\mu_2(G) \geqslant \mu_1(G) = 0$, the inequality $m_G(n) \leqslant r$ follows from Corollary 3.2. If $G \cong K_n$, then $m_G(n) = n-1 = r$. If $m_G(n) = r$, then by Theorem 2.1 G has a spanning subgraph G_1 which is a complete $(r+1)$ partite graph. Let V_i denote the ith partite vertex of G_1, then $|V_i| = 1$ holds for $1 \leqslant i \leqslant r+1$. Otherwise, we would have $d(v_0) \geqslant r+1$ for some vertex $v_0 \in V(G)$, a contradiction. Hence $r = n-1$ and $G \cong K_n$. □

Let $\alpha(G)$, or α for short, denote the independence number of G, which is the largest number of vertices in a coclique (independent set of vertices) of G.

Theorem 3.2. *Let $\{v_1, v_2, \ldots, v_\alpha\}$ be a largest coclique of G. Then*

$$\mu_n(G) \geqslant \frac{n}{\alpha(n-\alpha)} \sum_{i=1}^{\alpha} d(v_i).$$

If G has a coclique that meets this bound, then G is semi-regular.

P r o o f. The coclique $\{v_1, v_2, \ldots, v_\alpha\}$ gives rise to a partition of $L(G)$ with the quotient matrix

$$Q_{L(G)} = \begin{pmatrix} \dfrac{\sum\limits_{i=1}^{\alpha} d(v_i)}{\alpha} & -\dfrac{\sum\limits_{i=1}^{\alpha} d(v_i)}{\alpha} \\ -\dfrac{\sum\limits_{i=1}^{\alpha} d(v_i)}{n-\alpha} & \dfrac{\sum\limits_{i=1}^{\alpha} d(v_i)}{n-\alpha} \end{pmatrix}.$$

Then $Q_{L(G)}$ has two eigenvalues $\delta_1 = n(\alpha(n-\alpha))^{-1} \sum\limits_{i=1}^{\alpha} d(v_i)$, $\delta_2 = 0$ with $\delta_1 \geqslant \delta_2$. Then by Lemma 3.2, the inequality follows. Moreover, if G has a coclique that meets this bound, then $\mu_1(G) = 0 = \delta_2$ and $\mu_n(G) = \delta_1$. The interlacing is tight and hence the partition is regular. So $d(v_1) = d(v_2) = \ldots = d(v_\alpha)$ and $d(v_{\alpha+1}) = \ldots = d(v_n)$, this implies that G is semi-regular. □

Applying Theorem 3.2 to regular graphs, we obtain the following simple upper bound for the independence number of a regular graph.

Corollary 3.3. *Let G be an r-regular graph and α its independence number. Then $\alpha \leqslant n-r$. If the equality holds then $\mu_n(G) = n$.*

P r o o f. Since G is r-regular, we have $\mu_n(G) \geqslant nr(n-\alpha)^{-1}$ by Theorem 3.2. From Theorem A we immediately have $n \geqslant \mu_n(G) \geqslant nr(n-\alpha)^{-1}$. Then the desired conclusion follows. □

Let $\beta(G)$, or β for short, denote the largest number of vertices in a clique of G.

Theorem 3.3. *Let $\{v_1, v_2, \ldots, v_\beta\}$ be a largest clique of G. Then*

$$\mu_2(G) \leqslant \frac{n}{\beta(n-\beta)}\Bigl(\sum_{i=1}^{\beta} d(v_i) + \beta(1-\beta)\Bigr).$$

If G has a clique that meets this bound, then G is semi-regular.

Proof. Note that $\beta(G) = \alpha(\overline{G})$, by Theorem 3.2 and Lemma 3.1, $\mu_2(G) = n - \mu_n(\overline{G}) \leqslant n - n(\beta(n-\beta))^{-1}\sum_{i=1}^{\beta}(n-1-d(v_i)) = n(\beta(n-\beta))^{-1}\Bigl(\sum_{i=1}^{\beta} d(v_i) + \beta(1-\beta)\Bigr)$.

If G has a clique that meets this bound, then $\overline{G}$ is semi-regular by Theorem 3.2 and so G is semi-regular. $\square$

Corollary 3.4. *If G is r-regular, then*

$$\mu_2(G) \leqslant \frac{n}{n-\beta}(r+1-\beta).$$

4. Hamiltonian graphs with the largest Laplacian index

In order to obtain the results of this section we first recall the well-known Dirac's theorem that gives a simple sufficient condition for Hamiltonian graphs.

Theorem 4.1 (Dirac, see [7]). *For a simple graph G with n ($n \geqslant 3$) vertices, if $d(u) \geqslant \frac{1}{2}n$ for every vertex u, then G is Hamiltonian.*

We prove the existence of Hamiltonian graphs G with the largest Laplacian index n in the next theorem.

Theorem 4.2. *For any integers n, k with $n \geqslant 3$ and $\lceil \frac{1}{2}n \rceil \leqslant k \leqslant n-1$, there exists a Hamiltonian graph G with $\mu_n(G) = n$ and $\Delta(G) = k$.*

Proof. When nk is even, by Theorem 2.3 there is a k-regular graph G of order n with $\mu_n(G) = n$. Then by Dirac's theorem (Theorem 4.1), G is Hamiltonian.

When nk is odd, then we can construct the desired graph G by adding some edges to the complete bipartite graph $K_{k,n-k}$ with bipartition (A, B), where $|A| = k$ and $|B| = n-k$. Let $A = \{a_1, \ldots, a_{n-k}, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_{n-k}\}$. (Note that $k > n-k$ since $k \geqslant \lceil \frac{1}{2}n \rceil$ and n is odd.) By adding the new edges joining a_i and a_{i+1}

for all $i = n-k, \ldots, k-1$, the resulting graph G is Hamiltonian since it contains a Hamiltonian cycle $a_1b_1a_ka_{k-1}\ldots a_{n-k}b_{n-k}a_{n-k-1}b_{n-k-1}\ldots b_2a_1$.

To show that $\Delta(G) = k$, we only need to show that $d(a_i) \leqslant k$ for all $a_i \in A$, since every $b_j \in B$ has the same degree k. We proceed by distinguishing whether $k = \lceil \frac{1}{2}n \rceil$ or not. If $k = \lceil \frac{1}{2}n \rceil$, then $k = \frac{1}{2}(n+1)$ since n is odd. Then $k = (n-k)+1$ and $d(a_i) \leqslant (n-k)+1 = k$ for all $a_i \in A$. If $k > \lceil \frac{1}{2}n \rceil$, then $k \geqslant \frac{1}{2}(n+1)+1$, and then $k \geqslant n-k+3$, hence $d(a_i) \leqslant (n-k)+2 < k$ for all $a_i \in A$. Thus we have proved $\Delta(G) = k$. Hence G is a desired graph by Theorem 2.2. □

Next we give some conditions for graphs to be Hamiltonian in terms of the largest Laplacian index.

Proposition 4.1. *Let $n > 3$ be an even number and let G be a graph of order n with $\Delta(G) = \frac{1}{2}n$. If $\mu_n(G) = n$ then G is Hamiltonian.*

P r o o f. If $\mu_n(G) = n$, then by Theorem 2.2 G contains a spanning subgraph $K_{a,b}$. Since $\Delta(G) = \frac{1}{2}n$ (n even), $a = b = \frac{1}{2}n$. Then by Dirac's theorem, G is Hamiltonian. □

Proposition 4.2. *Let G be a regular graph of order $n \geqslant 3$. If $\mu_n(G) = n$ then G is Hamiltonian.*

P r o o f. If $\mu_n(G) = n$, then by Theorem 2.2 G contains a spanning subgraph $K_{a,b}$. Let G be k-regular. Then $k \geqslant \max\{a,b\} \geqslant \lceil \frac{1}{2}n \rceil$. By Dirac's theorem G is Hamiltonian. □

Summarizing Propositions 4.1 and 4.2, we get the following result.

Theorem 4.3. *For graphs G of order $n \geqslant 3$ with the largest Laplacian index n, G is Hamiltonian if G is regular or its maximum vertex degree satisfies $\Delta(G) = \frac{1}{2}n$.*

Note that for k-regular graphs G, the condition $\mu_n(G) = n$ in Proposition 4.2 can be replaced by $\lambda_n(G) = k - n$. Then we can give a sufficient condition, in terms of eigenvalues, for a regular graph to be Hamiltonian.

Theorem 4.4. *For k-regular graphs G of order $n \geqslant 3$, if $k - n$ is an eigenvalue of G, then G is Hamiltonian.*

5. Planar graphs with the largest Laplacian index

Considering connected graphs of order n, we denote by $\mathbb{P}_n$, $\mathrm{O}\mathbb{P}_n$, $\mathrm{MO}\mathbb{P}_n$ the sets of planar graphs, outerplanar graphs, and maximal outerplanar graphs, respectively.

To prove our results, we need the following well known theorems for planar graphs.

Theorem 5.1 (Kuratowski, see [7]). *$G \in \mathbb{P}_n$ if and only if G contains no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Theorem 5.2 (see [7]). *$G \in \mathrm{O}\mathbb{P}_n$ if and only if G contains no subgraph homeomorphic to K_4 or $K_{2,3}$.*

Now we establish the following theorems.

Theorem 5.3. *For $G \in \mathbb{P}_n$, $\mu_n(G) = n$ if and only if either $\Delta(G) = n - 1$, or $\Delta(G) = n - 2$ and G has at least two vertices of degree $n - 2$, which are not adjacent.*

Proof. Since $G \in \mathbb{P}_n$, G contains no $K_{3,3}$ by Theorem 5.1. Hence by Theorem 2.2, $\mu_n(G) = n$ if and only if G contains either $K_{1,n-1}$ or $K_{2,n-2}$. Then the conclusion of Theorem 5.3 is easily obtained. □

Theorem 5.4. *For $G \in \mathrm{O}\mathbb{P}_n$ ($n \geqslant 5$), $\mu_n(G) = n$ if and only if $\Delta(G) = n - 1$.*

Proof. Since $G \in \mathrm{O}\mathbb{P}_n$ and $n \geqslant 5$, by Theorem 5.2 G contains no $K_{2,3}$. Hence by Theorem 2.2, $\mu_n(G) = n$ if and only if G contains a spanning subgraph $K_{a,b}$ with $\min\{a, b\} = 1$. Then the theorem immediately follows. □

Now we give the following corollary of Theorem 5.3 without a proof, since it can be easily seen by contradiction.

Corollary 5.1. *Let G be a graph of order n with $\Delta(G) < n - 2$. If $\mu_n(G) = n$, then $G \notin \mathbb{P}_n$.*

Similarly we have the following corollary of Theorem 5.4.

Corollary 5.2. *Let G be a graph of order n with $\Delta(G) < n - 1$ ($n \geqslant 5$). If $\mu_n(G) = n$, then $G \notin \mathrm{O}\mathbb{P}_n$.*

Note that $\mathrm{MO}\mathbb{P}_n \subset \mathrm{O}\mathbb{P}_n$ and that the maximum degree of the maximal outerplanar graph of order 4 is 3. We obtain the following corollary easily from Theorem 5.4.

Corollary 5.3. *For $G \in \mathrm{MO}\mathbb{P}_n$, $\mu_n(G) = n$ if and only if $\Delta(G) = n - 1$.*

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