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COMMUTATIVE ZEROPOTENT SEMIGROUPS WITH FEW INVARIANT CONGRUENCES

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Abstract. Commutative semigroups satisfying the equation $2x + y = 2x$ and having only two G-invariant congruences for an automorphism group G are considered. Some classes of these semigroups are characterized and some other examples are constructed.

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Every congruence-simple (i.e., possesing just two congruence relations) commutative semigroup is finite and either two-element or a group of prime order. The class of (non-trivial) commutative semigroups having only trivial invariant congruences is considerably more opulent. These semigroups are easily divided into four pairwise disjoint subclasses (see 1.3). The fourth contains commutative semigroups that are nil of index two and have no irreducible elements. This subclass is a bit enigmatic and it is the purpose of the present note to construct various examples of the indicated semigroups (called zs-semigroups in the sequel). Among other, we show that if S is a non-trivial commutative zs-semigroup without non-trivial invariant congruences, then the group of automorphisms of S contains a non-commutative free subsemigroup.

1. Introduction

Let G be a multiplicative group. By a (unitary left G-) semimodule we mean a

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commutative semigroup S $(= S(+))$ together with a G-scalar multiplication $G \times S \to S$ such that $a(x+y) = ax+ay$, $a(bx) = (ab)x$ and $1x = x$ for all $a, b \in G$ and $x, y \in S$.

Let S be a semimodule. An element $w \in S$ is called absorbing if $Gw = w = S+w$. There exists at most one absorbing element in S and, if it exists, it will usually be denoted by the symbol o_S (or only o); we will also write $o \in S$.

A non-empty subset I of S is an ideal if $GI \subseteq I$ and $S + I \subseteq I$. The semimodule S will be called ideal-simple (or only id-simple) if $|S| \geqslant 2$ and $I = S$ whenever I is an ideal of S such that $|I| \geqslant 2$.

Lemma 1.1. *Let S be a semimodule and $w \in S$. The one-element set $\{w\}$ is an ideal of S if and only if $w = o_S$ is an absorbing element of S.*

Proof. Obvious. $\square$

A semimodule S will be called congruence-simple (or only cg-simple) if S has just two congruence relations (i.e., equivalences compatible with the addition and the scalar multiplication).

Proposition 1.2. *Every cg-simple semimodule is id-simple.*

Proof. If S is cg-simple, then S is non-trivial and, if I is an ideal of S, then $r = (I \times I) \cup \mathrm{id}_S$ is a congruence of S. Now, either $r = \mathrm{id}_S$ and $|I| = 1$ (see 1.1) or $r = S \times S$ and $I = S$. Thus S is id-simple. $\square$

Let S be a (commutative) semigroup/semimodule. We will say that S is

- ▷ a semigroup/semimodule with zero addition (a za-semigroup/za-semimodule) if $|S + S| = 1$ (then $o \in S$ and $S + S = o$);
- ▷ a zeropotent semigroup/semimodule (a zp-semigroup/zp-semimodule) if $2x + y = 2x$ for all $x, y \in S$ (then $o \in S$ and $2x = o$);
- ▷ a zp-semigroup/zp-semimodule without irreducible elements (a zs-semigroup/zs-semimodule) if S is a zp-semigroup/zp-semimodule and $S = S + S$;
- ▷ idempotent if $x + x = x$ for every $x \in S$;
- ▷ cancellative if $x + y \neq x + z$ for all $x, y, z \in S$, $y \neq z$.

The following basic classification of cg-simple semimodules is given in [1]:

Theorem 1.3. *Let S be a cg-simple semimodule. Then just one of the following four cases takes place:*

(1) *S is a two-element za-semimodule;*
(2) *S is idempotent;*
(3) *S is cancellative;*
(4) *S is a zs-semimodule.*

There exists only one two-element za-semimodule up to isomorphism. Cg-simple idempotent semimodules over a commutative group are fully characterized in [1] (see also [3], [4] and [5]) and cg-simple chains (and the corresponding groups) are studied in [6] and [7]. Some information on cg-simple cancellative semimodules is also available from [1] and various examples of non-trivial commutative zs-semigroups are collected in [2]. The aim of this note is to initiate a study of cg-simple zs-semimodules. The following starting result restricts our choice of groups in the zeropotent case:

Proposition 1.4. *Let no subsemigroup of a group G be a free semigroup of rank (at least) 2. Then there exist no cg-simple zs-semimodules over G.*

P r o o f. Let S be a non-trivial zs-semimodule and let $x, y, z \in S$ be such that $x = y + z \neq o$. Denote by A (B) the set of $a \in G$ ($b \in G$) such that $ax = y$ or $ax+v = y$, $v \in S$ ($bx = z$ or $bx+v = z$, respectively). Then $A \cap B = \emptyset$, $AA \cup AB \subseteq A$ and $BB \cup BA \subseteq B$. Now, if $a \in A$ and $b \in B$, then the subsemigroup of G generated by $\{a, b\}$ is free, a contradiction. Thus either $A = \emptyset$ or $B = \emptyset$ and we will assume $A = \emptyset$, the other case being similar.

Put $I = Gx \cup (Gx + S)$. Then I is an ideal of S, $y \notin I$ and $I \neq S$. On the other hand, $\{x, o\} \subseteq I$ and $|I| \geqslant 2$. Consequently, the semimodule S is not id-simple and, according to 1.2, it is not cg-simple either. □

Notice that among the groups from 1.4 we find all periodic groups and all locally nilpotent groups (but not all metabelian groups).

Now, let R be a subsemigroup of a group G and let $\mathbf{M} = \{A;\ A \subseteq G, A \neq \emptyset, AR \subseteq A\}$. The set $\mathbf{M}$ is closed under unions and non-empty intersections, $R \in \mathbf{M}$ and $G \in \mathbf{M}$. Now, we define an addition $+$ on $\mathbf{M}$ by $A + B = A \cup B$ if $A \cap B = \emptyset$ and $A + B = G$ otherwise.

Lemma 1.5. $\mathbf{M}(+)$ *is a commutative zp-semigroup and* $o_{\mathbf{M}} = G$.

P r o o f. Easy to check. □

Moreover, we define a scalar multiplication on $\mathbf{M}$ by $(a, A) \to aA = \{ax;\ x \in A\}$, $a \in G$, $A \in \mathbf{M}$.

Lemma 1.6. $\mathbf{M}$ *is a zp-semimodule over the group* G.

P r o o f. Easy to check. □

Define a relation ξ on $\mathbf{M}$ by $(A, B) \in \xi$ iff $\{M \in \mathbf{M};\ A \cap M = \emptyset\} = \{M \in \mathbf{M};\ B \cap M = \emptyset\}$.

Lemma 1.7. *ξ is a congruence of the semimodule* $\mathbf{M}$.

Proof. Easy to check. $\square$

Lemma 1.8. *Let η be a congruence of $\mathbf{M}$ such that $\xi \subseteq \eta$ and $(R, G) \in \eta$. Then $\eta = \mathbf{M} \times \mathbf{M}$.*

Proof. Clearly, $(xR, G) = (xR, xG) \in \eta$ for every $x \in G$. Let $A \in \mathbf{M}$ and $a \in A$. If $aR \cap B \neq \emptyset$ for every $B \in \mathbf{M}$ such that $B \subseteq A$, then $(aR, A) \in \xi \subseteq \eta$, and so $(A, G) \in \eta$. On the other hand, if $B \in \mathbf{M}$ is maximal with respect to $B \subseteq A$ and $aR \cap B = \emptyset$, then $(A, B \cup aR) \in \xi$. Since $(G, B \cup aR) \in \eta$, we get $(A, G) \in \eta$ again. $\square$

Lemma 1.9. *$(R, G) \in \xi$ if and only if $G = RR^{-1}$ (then R is right uniform).*

Proof. If $(R, G) \in \xi$, then $R \cap A \neq \emptyset$ for every $A \in \mathbf{M}$. In particular, $R \cap xR \neq \emptyset$ for every $x \in G$, and hence $x \in RR^{-1}$. To show the other implication, we just proceed conversely. $\square$

Lemma 1.10.

(i) *If R is not right uniform, then $(R, G) \notin \xi$.*

(ii) *If G is not generated by R, then $(R, G) \notin \xi$.*

Proof. (i) There exist $a, b \in R$ such that $aR \cap bR = \emptyset$. Then $R \cap a^{-1}bR = \emptyset$, $ab^{-1}R \in \mathbf{M}$ and, of course, $G \cap a^{-1}bR = a^{-1}bR \neq \emptyset$. Thus $(R, G) \notin \xi$.

(ii) Use 1.9. $\square$

Lemma 1.11. *Assume that R is not right uniform. Then $(R, G) \notin \xi$ and, if κ is a congruence of $\mathbf{M}$ maximal with respect to $\xi \subseteq \kappa$ and $(R, G) \notin \kappa$, then $\mathbf{N} = \mathbf{M}/\kappa$ is a cg-simple zs-semimodule.*

Proof. $\mathbf{N}$ is non-trivial and it follows readily from 1.8 that $\mathbf{N}$ is a cg-simple zp-semimodule. Since R is not right uniform, there are right ideals A and B of R such that B is maximal with respect to $A \cap B = \emptyset$. Then $A + B = A \cup B$, $(A \cup B, R) \in \xi \subseteq \kappa$, $(A \cup B, G) \notin \kappa$ and $A/\kappa + B/\kappa \neq o_{\mathbf{N}}$. Thus $\mathbf{N}$ is not a za-semimodule, and hence $\mathbf{N}$ is a zs-semimodule by 1.3. $\square$

Proposition 1.12. *If R is not right uniform, then a factorsemimodule of $\mathbf{M}$ is a congruence-simple zs-semimodule.*

Proof. See 1.11. $\square$

Theorem 1.13. *There exists at least one cg-simple zs-semimodule over G if and only if the group G contains at least one subsemigroup that is a free semigroup of rank (at least) 2.*

Proof. The direct implication is shown in 1.4. As concerns the inverse implication, the existence of a cg-simple zs-semimodule is shown in 1.12. $\square$

2. Basic properties of zeropotent semimodules

Throughout this secion, let S be a zp-semimodule over a group G. First, define a relation $\preceq_S$ on S by $x \preceq_S y$ if and only if $x = y$ or $y = x + v$ for some $v \in S$.

Lemma 2.1.

(i) *The relation $\preceq_S$ is an ordering compatible with the addition and scalar multiplication.*

(ii) *o_S is a greatest element of the ordered set $(S, \preceq_S)$.*

(iii) *If $|S| \geqslant 2$, then $S \setminus (S + S)$ is the set of minimal elements of $(S, \preceq_S)$.*

(iv) *If $x, y, z \in S$ are such that $x \preceq_S y$ and $x \preceq_S z$, then $y + z = o$.*

Proof. Easy. $\square$

Proposition 2.2. *Assume that S is a non-trivial zs-semimodule. Then:*

(i) *The ordered set $(S, \preceq_S)$ has no minimal elements.*

(ii) *$S(+)$ is not finitely generated (and hence S is infinite).*

Proof. (i) This follows immediately from 2.1(iii).

(ii) If $S(+)$ were generated by a finite number m of elements, then S should contain at most 2^m elements, a contradiction with (i). $\square$

Lemma 2.3. *The following conditions are equivalent:*

(i) *If $x, y, z, u, v \in S$ are such that $x + y \neq o \neq z$ and $x + u = z = y + v$, then either $z = x + y$ or $z = x + y + w$ for some $w \in S$.*

(ii) *If $x, y, z \in S$ are such that $x + y \neq o \neq z$ and $x \preceq_S z$, $y \preceq_S z$, then $x + y \preceq_S z$.*

(iii) *If $x, y \in S$ are such that $x + y \neq o$, then $x + y = \sup(x, y)$ in $(S, \preceq_S)$.*

Proof. Easy. $\square$

The semimodule S will be called downwards-regular if the equivalent conditions of 2.3 are satisfied.

For every $x \in S$, let $\mathrm{Ann}_S(x) = \{y \in S;\ x + y = o\}$. Further, let $\mathrm{Ann}_S(S) = \{x + S;\ S + x = o\}$.

Lemma 2.4.

(i) *For every $x \in S$, the annihilator* $\mathrm{Ann}_S(x)$ *is an ideal of the additive semigroup* $S(+)$.

(ii) $\mathrm{Ann}_S(S)$ *is an ideal of the semimodule S.*

Proof. Obvious. $\square$

Define a relation $\dashv_S$ on S by $x \dashv_S y$ if and only if $\mathrm{Ann}_S(x) \subseteq \mathrm{Ann}_S(y)$.

Lemma 2.5.

(i) *The relation $\dashv_S$ is a quasiordering compatible with the addition and scalar multiplication.*

(ii) *If $x \preceq_S y$, then $x \dashv_S y$.*

(iii) $\pi_S = \ker(\dashv_S)$ *is a congruence of the semimodule S.*

(iv) $\pi_S = S \times S$ *if and only if S is a za-semimodule.*

Proof. Easy. $\square$

The semimodule S will be called separable if $\pi_S = \mathrm{id}_S$.

The semimodule S will be called upwards-regular if $\mathrm{Ann}_S(x+y) \subseteq \mathrm{Ann}_S(z)$ whenever $x, y, z \in S$ are such that $x + y \neq o \neq z$ and $\mathrm{Ann}_S(x) \cup \mathrm{Ann}_S(y) \subseteq \mathrm{Ann}_S(z)$.

In the sequel, let $\tau_S = \{(x, y) \in S \times S;\ x + y \neq o\}$ and $\sigma_S = \{(x, y);\ x + y = o\} = S \times S \setminus \tau_S$. Further, define μ_S (ν_S) by $(x, y) \in \mu_S$ ($(x, y) \in \nu_S$) if and only if $z \preceq_S x$, $z \preceq_S y$ ($z \dashv_S x$, $z \dashv_S y$, respectively) for at least one $z \in S$.

Lemma 2.6.

(i) *The relations τ_S, σ_S, μ_S and ν_S are symmetric.*

(ii) *The relations σ_S, μ_S and ν_S are reflexive.*

(iii) τ_S *is irreflexive.*

(iv) $\pi_S \subseteq \sigma_S$.

(v) $\mu_S \subseteq \nu_S \subseteq \sigma_S$.

Proof. Easy. $\square$

The semimodule S will be called (strongly) balanced if $\sigma_S = \nu_S$ ($\sigma_S = \mu_S$).

The semimodule S will be called transitive if the group G operates transitively on the set $S \setminus \{o_S\}$.

Proposition 2.7. *If S is non-trivial and transitive, then S is id-simple.*

Proof. Easy. $\square$

Proposition 2.8. *Assume that S is id-simple and either $S+S \neq S$ or $\operatorname{Ann}_S(S) \neq \{o_S\}$. Then:*

(i) *$S+S=\{o_S\}$, $\operatorname{Ann}_S(S)=S$ and S is a za-semimodule.*

(ii) *$x \preceq_S y$ if and only if either $x=y$ or $y=o_S$.*

(iii) *$\pi_S = S\times S = \dashv_S$.*

(iv) *G operates transitively on $R=S\setminus\{o_S\}$ (i.e., S is transitive).*

(v) *$\nu_S=(R\times R)\cup \mathrm{id}_S$ is a congruence of S.*

(vi) *G operates primitively on R if and only if id_S, ν_S and $S\times S$ are the only congruences of S.*

Proof. Easy. □

Proposition 2.9. *Assume that S is cg-simple and $|S|\geqslant 3$. Then:*

(i) *$\operatorname{Ann}_S(S)=\{o_S\}$ and S is separable.*

(ii) *$\dashv_S$ is a compatible ordering of S.*

Proof. It follows from 2.5(iii) that either $\pi_S=S\times S$ or $\pi_S=\mathrm{id}_S$. If $\pi_S=S\times S$, then S is a za-semimodule by 2.5(iv) and S is id-simple by 1.2. Now, it follows from 2.8(v) that $|R|=1$ and $|S|=2$, a contradiction. Consequently, $\pi_S=\mathrm{id}_S$ and $\dashv_S$ is transitive. The rest follows from 2.5. □

Proposition 2.10. *Assume that $|S|\geqslant 3$. Then S is cg-simple if and only if S is separable and id-simple.*

Proof. The direct implication follows from 1.2 and 2.9. Now, assume that S is separable and id-simple.

Let r be a congruence of S and $I=\{x;\ (x,o)\in r\}$. Then I is an ideal of S and $r=S\times S$ provided that $I=S$.

Let $(x,y)\in r$, $x\neq y$. Since S is separable, $(x,y)\notin \pi_S$ and we can assume that $x\dashv_S y$ is not true. Then $\operatorname{Ann}_S(x)\nsubseteq \operatorname{Ann}_S(y)$ and there is $z\in S$ such that $x+z=o\neq y+z$. Now, $y+z\in I$, $I\neq\{o\}$, $I=S$, since S is id-simple, and $r=S\times S$. □

Proposition 2.11. *Assume that S is transitive and $|S|\geqslant 3$. The following conditions are equivalent:*

(i) *S is cg-simple.*

(ii) *S is separable.*

Proof. (i) implies (ii) by 2.9(i) and (ii) implies (i) by 2.7 and 2.10. □

Proposition 2.12. *Assume that S is id-simple, take $w \in S$, $w \neq o$, and consider a congruence r of S maximal with respect to $(w, o) \notin r$. Then S/r is a cg-simple zp-semimodule.*

Proof. Clearly, $T = S/r$ is a non-trivial zp-semimodule. Now, let s be a congruence of S such that $r \subseteq s$, $r \neq s$, and put $I = \{x \in S;\ (x, o) \in s\}$. Then I is an ideal of S and $\{o, w\} \subseteq I$. Thus $I = S$, since S is id-simple, and we conclude that $s = S \times S$. It follows easily that T is cg-simple. □

Corollary 2.13. *Assume that S is id-simple and $S + S \neq \{o_S\}$. Then at least one factorsemimodule of S is a cg-simple zs-semimodule.*

Corollary 2.14. *Assume that S is transitive and $S + S \neq \{o_S\}$. Then at least one factorsemimodule of S is a cg-simple zs-semimodule.*

3. Examples of congruence-simple zs-semimodules

Example 3.1. Let S be a non-trivial commutative zs-semigroup and $G = \mathrm{Aut}(S)$ (the automorphism group of S). Then S becomes a G-semimodule. If S is separable and G operates transitively on $S \setminus \{o_S\}$, then S is a cg-simple semimodule.

Example 3.2. Let $(R, \leqslant)$ be a non-empty ordered set together with an irreflective and symmetric relation τ defined on R. For $x, y \in R$, let $x \vee y = \sup(x, y)$ provided that this supremum exists. Now, assume that the following three conditions are satisfied:

(α) If $x, y \in R$ are such that $(x, y) \in \tau$, then $x \vee y$ exists.
(β) If $(x, y) \in \tau$ and $(z, x \vee y) \in \tau$, then $(x, z) \in \tau$ and $(y, x \vee z) \in \tau$.
(γ) For every $x \in R$ there exist $y, z \in R$ such that $(y, z) \in \tau$ and $x = y \vee z$.

Further, let $o \notin R$, $S = R \cup \{o\}$, $x + y = x \vee y$ if $x, y \in R$, $(x, y) \in \tau$ and $x + y = o$ otherwise. Then S $(= S(+))$ becomes a commutative zs-semigroup.

Let G be a group operating on R (i.e., a mapping $G \times R \to R$ is defined such that $a(bx) = (ab)x$ and $1x = x$) and assume that $(ax, ay) \in \tau$ for every $(x, y) \in \tau$ and that $u \leqslant v$ implies $au \leqslant av$. Then $ax \vee ay = a(x \vee y)$ for $(x, y) \in \tau$ and S becomes a G-semimodule ($ao = o$). If G operates transitively on R, then S is a transitive semimodule. In such a case, by 2.14, at least one factorsemimodule of S is a cg-simple zs-semimodule. Furthermore, if S is transitive, then S is cg-simple iff it is separable (2.11). Finally, S is separable iff the following two conditions are satisfied:

(δ) For every $x \in R$ there exists $y \in R$ with $(x, y) \in \tau$.

(ε) For all $x, y \in R$, $x \neq y$, $(x, y) \notin \tau$, there exists $z \in R$ such that either $(x, z) \in \tau$, $(y, z) \notin \tau$ or $(x, z) \notin \tau$, $(y, z) \in \tau$.

(Notice that (δ) is true provided that S is transitive.)

Example 3.3 (cf. 3.2)**.** Let T $(= T(\wedge, \vee))$ be a distributive lattice with a smallest element 0_T and a greatest element 1_T such that $|T| \geqslant 3$. Consider the basic order $\leqslant$ defined on T and also the ordered set $(R, \leqslant)$, $R = T \setminus \{0_T, 1_T\}$. Assume that the following two conditions are satisfied:

(μ) If $x, y \in R$ and $x \wedge y = 0_T$, then $x \vee y \neq 1_T$.

(ν) For every $x \in R$ there exist $y, z \in R$ such that $y \wedge z = 0_T$ and $y \vee z = x$.

Put $S = T \setminus \{1_T\}$ and define an addition on S by $x + y = x \vee y$ if $x \wedge y = 0_T$ and $x + y = 1_T$ otherwise. Then S $(= S(+))$ is a commutative zs-semigroup. Further, let a group G operate on R ($a(bx) = (ab)x$ and $1x = x$) in such a way that $x \leqslant y$ implies $ax \leqslant ay$. Then S becomes a G-semimodule ($a1_T = 1_T$). If G operates transitively on R, then S is a cg-simple zs-semimodule iff the following is true:

(σ) For all $x, y \in R$, $x \neq y$, $x \wedge y \neq 0_T$, there exists $z \in R$ such that either $x \wedge z = 0_T \neq y \wedge z$ or $x \wedge z \neq 0_T = y \wedge z$.

Example 3.4. Let I be an infinite set with $|I| \geqslant \aleph_1$ and let $\aleph$ be an infinite cardinal number such that $\aleph < |I|$. Denote by $\mathbf{J}$ the set $\{A;\ A \subseteq I, |A| = \aleph\} \cup \{I\}$ and define an operation $\oplus$ on $\mathbf{J}$ by $A \oplus B = A \cup B$ if $A \cap B = \emptyset$ and $A \oplus B = I$ otherwise. Then $\mathbf{J}$ is a non-trivial commutative zs-semigroup and $\mathbf{J}$ becomes a G-semimodule, $G = \operatorname{Aut}(\mathbf{J}(\oplus))$. It is easy to check that the semimodule $\mathbf{J}$ is transitive, separable and upwards-regular, but neither downwards-regular non balanced. By 2.11, $\mathbf{J}$ is cg-simple.

Example 3.5. Let I be an infinite set, $\mathbf{K}$ a (non-principal) maximal ideal of the Boolean algebra of subsets of I such that $K \in \mathbf{K}$ for every $K \subseteq I$, $|K| = |I|$, and let $\mathbf{L} = \{A \in \mathbf{K};\ |A| = |I|\} \cup \{I\}$. Define an addition $\oplus$ on $\mathbf{L}$ by $A \oplus B = A \cup B$ if $A \cap B = \emptyset$ and $A \oplus B = I$ otherwise and put $G = \operatorname{Aut}(\mathbf{L}(\oplus))$. Then $\mathbf{L}$ $(= \mathbf{L}(\oplus))$ is a non-trivial separable commutative zs-semigroup and G operates transitively on $\mathbf{L} \setminus \{o\}$. Consequently, $\mathbf{L}$ is a cg-simple zs-semimodule over G.

Example 3.6. Let I be an infinite set and $\mathbf{I}$ the set of infinite subsets of I. Define an operation $\boxplus$ on $\mathbf{I}$ by $A \boxplus B = A \cup B$ if $A \cap B$ is finite and $A \boxplus B = I$ otherwise. Then $\mathbf{I}$ $(= \mathbf{I}(\boxplus))$ is a non-trivial commutative zs-semigroup and r is a congruence of $\mathbf{I}$, where $(A, B) \in r$ iff the symmetric difference $(A \cup B) \setminus (A \cap B)$ is finite. Then $\mathbf{J} = \mathbf{I}/r$ is a non-trivial (commutative) zs-semigroup. Moreover, if $|I| = \aleph_0$ and $G = \operatorname{Aut}(\mathbf{J})$, then $\mathbf{J}$ is a separable, upwards- and downwards-regular transitive G-semimodule ($\mathbf{J}$ is not balanced). Consequently, $\mathbf{J}$ is a cg-simple zs-semimodule.

Assume that $|I| \geqslant \aleph_1$ and put $\mathbf{P} = \{A \in \mathbf{I};\ |A| = \aleph_0\} \cup \{I\}$. Then $\mathbf{P}$ is a subsemigroup of $\mathbf{I}$ and $\mathbf{Q} = \mathbf{P}/r$ is a non-trivial (commutative) zs-semigroup. Moreover, if $H = \mathrm{Aut}(\mathbf{Q})$, then $\mathbf{Q}$ is a transitive H-semimodule and it is easy to check that $\mathbf{Q}$ is an upwards- and downwards-regular strongly balanced cg-simple zs-semimodule.

4. Fractional left ideals and zeropotent semimodules

In this section, let R be a subsemigroup of a group G such that $1 \in R$. We denote by $\mathbf{F}$ $(= \mathbf{F}(G,R))$ the set of fractional left R-ideals of G. That is, $A \in \mathbf{F}$ iff $A \subseteq G$, $A \neq \emptyset$, $RA \subseteq A$ and $A \subseteq Ra$ for some $a \in G$. The set $(\mathbf{G}(G,R) =)$ $\mathbf{G} = \mathbf{F} \cup \{\emptyset\}$ is closed under arbitrary intersections and G operates on $\mathbf{G}$ via $a * A = Aa^{-1}$, $A \in \mathbf{G}$, $a \in G$. The set $(\mathbf{P}(G,R) =)$ $\mathbf{P} = \{Ra;\ a \in G\}$ of principal fractional left R-ideals is contained in $\mathbf{F}$ and we put $(\mathbf{Q}(G,R) =)$ $\mathbf{Q} = \mathbf{P} \cup \{\emptyset\}$. Notice that G operates transitively on $\mathbf{P}$.

Construction 4.1. *Assume that the following condition is satisfied:*

(f1) *If $a \in G$ is such that $R \cap aR = \emptyset$, then $R \cap Ra = Rb$ for some $b \in G$ (then $b \in R$).*

Now, define an addition on the set $\mathbf{Q}$ in the following way:

(1) $Ra + Rb = Ra \cap Rb$ for all $a, b \in G$ such that $R \cap ab^{-1}R = \emptyset$ (by (f1), we have $Ra \cap Rb \in \mathbf{P}$);

(2) $Ra + Rb = \emptyset$ for all $a, b \in G$ such that $R \cap ab^{-1}R \neq \emptyset$;

(3) $Ra + \emptyset = \emptyset = \emptyset + Ra$ for every $a \in G$;

(4) $\emptyset + \emptyset = \emptyset$.

Now, we have obtained a groupoid $\mathbf{Q} = \mathbf{Q}(+)$.

Lemma 4.1.1. *$A + B = B + A$, $A + A = \emptyset$ and $A + \emptyset = \emptyset$ for all $A, B \in \mathbf{Q}$.*

Proof. Obvious. □

Lemma 4.1.2. *For every $a \in G$, the transformation $A \to a * A$ $(= Aa^{-1})$ is an automorphism of $\mathbf{Q}(+)$.*

Proof. Easy to check. □

Lemma 4.1.3. $\mathbf{Q}$ *is a semigroup if and only if the following condition is satisfied:*

(f2) *If* $a, b, c \in G$ *are such that* $R \cap aR = \emptyset = R \cap bc^{-1}R$ *and* $R \cap Ra = Rc$, *then* $R \cap dR = \emptyset = R \cap ab^{-1}R$, *where* $Ra \cap Rb = Rd$.

P r o o f. (i) Let $\mathbf{Q}(+)$ be associative. Then $(R + Ra) + Rb = R + (Ra + Rb)$. But $(R + Ra) + Rb = (R \cap Ra) + Rb = Rc + Rb = Rc \cap Rb = R \cap Ra \cap Rb \neq \emptyset$, and hence $R \cap ab^{-1}R = \emptyset$, $Ra \cap Rb = Rd$ by (f1), $R + Rd \neq \emptyset$ and $R \cap dR = \emptyset$.

(ii) Let (f2) be satisfied. First, if $a, b \in G$ are such that $(R + Ra) + Rb \neq \emptyset$, then (f2) implies $(R + Ra) + Rb = R + (Ra + Rb)$. Next, if $a, b, c \in G$ are such that $(Ra+Rb)+Rc \neq \emptyset$, then $(R+Rba^{-1})+Rca^{-1} = a*((Ra+Rb)+Rc) \neq \emptyset$, and hence $(R+Rba^{-1})+Rca^{-1} = R+(Rba^{-1}+Rca^{-1}) = a*(Ra+(Rb+Rc))$. Consequently, $(Ra + Rb) + Rc = a^{-1} * (a * ((Ra + Rb) + Rc)) = a^{-1} * (a * (Ra + (Rb + Rc))) = Ra + (Rb + Rc)$. Finally, if $a, b, c \in G$ are such that $Ra + (Rb + Rc) \neq \emptyset$, then $(Rc+Rb)+Ra = Ra+(Rb+Rc) \neq \emptyset$, and therefore $Ra+(Rb+Rc) = (Rc+Rb)+Ra = Rc + (Rb + Ra) = (Ra + Rb) + Rc$ by the commutativity of the addition and the preceding part of the proof. The rest is clear. □

Assume that (f2) is true. It follows from 4.1.1, 4.1.2 and 4.1.3 that $\mathbf{Q}$ becomes a non-trivial transitive zp-semimodule over the group G.

Lemma 4.1.4. $\mathbf{Q}$ *is a (non-trivial) zs-semimodule if and only if the following condition is satisfied:*

(f3) $R \cap aR = \emptyset$ *for at least one* $a \in G$.

P r o o f. Use the transitivity of $\mathbf{Q}$. □

Proposition 4.1.5. *Assume that the conditions* (f1), (f2) *and* (f3) *are satisfied. Then:*

(i) $\mathbf{Q} = \mathbf{Q}(+, *)$ *is a non-trivial transitive zs-semimodule over* G.

(ii) $\mathbf{Q}$ *is ideal-simple.*

(iii) *If* $Ra \preceq_{\mathbf{Q}} Rb$, *then* $Rb \subseteq Ra$.

(iv) $\mathrm{Ann}_{\mathbf{Q}}(Ra) = \{Rb;\ R \cap ab^{-1}R \neq \emptyset\} \cup \{\emptyset\}$.

(v) *If* $Rb \subseteq Ra$, *then* $Ra \dashv_{\mathbf{Q}} Rb$.

(vi) $\mathrm{Ann}_{\mathbf{Q}}(\mathbf{Q}) = \{\emptyset\}$.

(vii) $\mathbf{Q}$ *is balanced.*

P r o o f. See 4.1.1, 4.1.2, 4.1.3 and 4.1.4 to show (i), …, (vi). Finally, if $R \cap ab^{-1}R \neq \emptyset$, then $ab^{-1}r \in R$ for some $r \in R$ and we have $Ra \cup Rb \subseteq Rr^{-1}b$. Now, using (v), we show easily that $\mathbf{Q}$ is balanced. □

Finally, assume that the conditions (f1), (f2) and (f3) are satisfied (see 4.1.5) and consider two more conditions:

(f4) For every $a \in R \setminus R^{-1}$ there exists $b \in G$ such that $R \cap bR = \emptyset$ and $Ra = R \cap Rb$.

(f5) For every $a \in (RR^{-1}) \setminus R^{-1}$ there exists $b \in G$ such that $R \cap bR = \emptyset \neq R \cap abR$.

Lemma 4.1.6. *The following conditions are equivalent:*

(i) *If $a, b \in G$, then $Ra \preceq_{\mathbf{Q}} Rb$ if and only if $Rb \subseteq Ra$ (see 4.1.5(iii)).*

(ii) *The condition* (f4) *is satisfied.*

Proof. Easy to check. □

Lemma 4.1.7. *If* (f4) *is true, then* $\mathbf{Q}$ *is downwards-regular and strongly balanced.*

Proof. Use 4.1.6. □

Lemma 4.1.8. $\mathrm{Ann}_{\mathbf{Q}}(R) = \{Rb;\ b \in RR^{-1}\} \cup \{\emptyset\}$.

Proof. Easy to check. □

Lemma 4.1.9. *The following conditions are equivalent:*

(i) *If $a, b \in G$, then $Ra \dashv_{\mathbf{Q}} Rb$ if and only if $Rb \subseteq Ra$ (see 4.1.5(v)).*

(ii) *The condition* (f5) *is satisfied.*

Proof. (i) implies (ii). Let $a \in G$ be such that $R \cap abR = \emptyset$ whenever $b \in G$ is such that $R \cap bR = \emptyset$. It follows from 4.1.7(iv) and 4.1.8 that $R_a \dashv_{\mathbf{Q}} R$. Now, by (i), $R \subseteq Ra$, and hence $a \in R^{-1}$.

(ii) implies (i). Let $a, b \in G$ be such that $Ra \dashv_{\mathbf{Q}} Rb$ and $Ra \neq Rb$. Then $Rc \dashv_{\mathbf{Q}} R$, $c = ab^{-1}$ and $Rc \neq R$. Now, assume that $R \nsubseteq Rc$. Then $c \notin R^{-1}$ and, by (f5), $R \cap dR = \emptyset \neq R \cap cdR$ for some $d \in G$. Consequently, $Rd^{-1} \in \mathrm{Ann}_{\mathbf{Q}}(Rc) \subseteq \mathrm{Ann}_{\mathbf{Q}}(R)$ and $Rd^{-1} = Re$ for some $e \in RR^{-1}$ (4.1.8). Thus $d^{-1} = rs^{-1}$, $r, s \in R$, $dR = sr^{-1}R$ and $s \in R \cap dR$, a contradiction. It follows that $R \subseteq Rc$ and $Rb \subseteq Ra$. □

Lemma 4.1.10. *If* (f5) *is true, then* $\mathbf{Q}$ *is separable and upwards-regular.*

Proof. Use 4.1.9. □

4.2. Consider the conditions (f1), …, (f5) defined in 4.1.

Lemma 4.2.1.

(i) *If* (f1) *is true, then $G = RR^{-1} \cup R^{-1}R$ (and hence the group G is generated by R).*

(ii) *If* $G = RR^{-1} \cup R^{-1}R$ *and every left ideal of* R *is principal, then* (f1) *is true.*
(iii) (f3) *is true if and only if* $G \neq RR^{-1}$.

P r o o f. Easy to see. $\square$

Corollary 4.2.2. *If* G *is generated by* R, R *is left uniform, not right uniform and every left ideal of* R *is principal, then the conditions* (f1) *and* (f3) *are satisfied.*

5. Zeropotent semimodules and fractional left ideals

In this section, let S be an ideal-simple zeropotent G-semimodule such that $\mathrm{Ann}_S(S) = \{o_S\}$ (or, equivalently, $S + S \neq \{o_S\}$). For $u, v \in S$, put $(u : v) = \{a \in G;\ au \preceq_S v\}$ and $[u : v] = \{a \in G;\ au \dashv_S v\}$.

Lemma 5.1.
(i) $(u : v) \subseteq [u : v]$ *for all* $u, v \in S$.
(ii) $(u : o) = [u : o] = G$ *for every* $u \in S$.
(iii) $(o : w) = [o : w] = \emptyset$ *for every* $w \in S$, $w \neq o$.
(iv) $(u : av) = a(u : v)$ *and* $(au : v) = (u : v)a^{-1}$ *for all* $a \in G$ *and* $u, v \in S$.
(v) $[u : av] = a[u : v]$ *and* $[au : v] = [u : v]a^{-1}$ *for all* $a \in G$ *and* $u, v \in S$.
(vi) $(au : au) = a(u : u)a^{-1}$ *for all* $a \in G$ *and* $u \in S$.
(vii) $[au : au] = a[u : u]a^{-1}$ *for all* $a \in G$ *and* $u \in S$.

P r o o f. The inclusion $(u : v) \subseteq [u : v]$ follows from 2.5(ii) and the remaining assertions can be checked readily. $\square$

Lemma 5.2.
(i) $(u : v_1)(v_2 : u) \subseteq (v_2 : v_1)$ *and* $[u : v_1][v_2 : u] \subseteq [v_2 : v_1]$ *for all* $u, v_1, v_2 \in S$.
(ii) $(u : u)(v : u) \subseteq (v : u)$ *and* $[u : u][v : u] \subseteq [v : u]$ *for all* $u, v \in S$.
(iii) $(u : u)(u : u) \subseteq (u : u)$ *and* $[u : u][u : u] \subseteq [u : u]$ *for every* $u \in S$.

P r o o f. Easy to check directly. $\square$

Lemma 5.3. *Let* $u_1, u_2, u, v_1, v_2, v \in S$.
(i) *If* $u_1 \preceq_S u_2$, *then* $(u_2 : v) \subseteq (u_1 : v)$.
(ii) *If* $v_1 \preceq_S v_2$, *then* $(u : v_1) \subseteq (u : v_2)$.
(iii) *If* $v_2 \preceq_S u_1$, *then* $(u_1 : v_1)(u_2 : v_2) \subseteq (u_2 : v_1)$.

P r o o f. Easy to check directly. $\square$

Lemma 5.4. *Let* $u_1, u_2, u, v_1, v_2, v \in S$.

(i) *If* $u_1 \dashv_S u_2$, *then* $[u_2 : v] \subseteq [u_1 : v]$.

(ii) *If* $v_1 \dashv_S v_2$, *then* $[u : v_1] \subseteq [u : v_2]$.

(iii) *If* $v_2 \dashv_S u_1$, *then* $[u_1 : v_1][u_2 : v_2] \subseteq [u_2 : v_1]$.

Proof. Easy to check directly. □

Lemma 5.5. $(u : v) \neq \emptyset \neq [u : v]$ *for all* $u, v \in S$, $u \neq o$.

Proof. Denote by I the set of $z \in S$ such that $au \preceq_S z$ for some $a \in G$. Then $\{o, u\} \subseteq I$ and I is an ideal of S. Since S is id-simple, we get $I = S$, $v \in I$, and therefore $(u : v) \neq \emptyset$. Since $(u : v) \subseteq [u : v]$, we have $[u : v] \neq \emptyset$, too. □

In the remaining part of this section, fix an element $w \in S$, $w \neq o_S$. It follows from 5.1(i), 5.2(iii) and 5.5 that both $R_1 = (w : w)$ and $R_2 = [w : w]$ are subsemigroups of G and $1 \in R_1 \subseteq R_2$. We put $\mathbf{F}_i = \mathbf{F}(G, R_i)$, $\mathbf{G}_i = \mathbf{G}(G, R_i)$, $\mathbf{P}_i = \mathbf{P}(G, R_i)$ and $\mathbf{Q_i} = \mathbf{Q}(G, R_i)$, $i = 1, 2$ (see the preceding section).

Lemma 5.6.

(i) $R_1^* = R_1 \cap R_1^{-1} = \{a \in G;\ aw = w\}$.

(ii) $R_2^* = R_2 \cap R_2^{-1} = \{a \in G;\ (w, aw) \in \pi_S\}$.

(iii) *If* S *is separable, then* $R_1^* = R_2^*$.

Proof. (i) If $aw = w$, then $a^{-1}w = w$, $a, a^{-1} \in R_1$ and $a \in R_1^*$. Conversely, if $a \in R_1^*$, then $a, a^{-1} \in R_1$. Now, if $w \neq aw$, then $w = aw + u = a^{-1}w + v$, $u, v \in S$, and we get $aw = w + av$, $w = w + z$, $z = av + u$, $w = w + 2z = w + o = o$, a contradiction.

(ii) Easy to check.

(iii) Since S is separable, we have $\pi_S = \mathrm{id}_S$ and the assertion follows by combinating (i) and (ii). □

Lemma 5.7. *Let* $v \in S$. *Then:*

(i) $R_1(v : w) \subseteq (v : w)$.

(ii) $R_2[v : w] \subseteq [v : w]$.

(iii) $(w : v) \neq \emptyset = [w : v]$.

(iv) $(v : w) \subseteq R_1 a^{-1}$ *for every* $a \in (w : v)$.

(v) $[v : w] \subseteq R_2 a^{-1}$ *for every* $a \in [w : v]$.

(vi) *If* $v \neq o_S$, *then* $(v : w) \neq \emptyset \neq [v : w]$.

(vii) $(v : w)(v : v) \subseteq (v : w)$.

(viii) $[v : w][v : v] \subseteq [v : w]$.

P r o o f. (i) If $a \in R_1$ and $b \in (v : w)$, then $aw = w$, $bv \preceq_S w$, and so $abv \preceq_S aw = w$ and $ab \in (v : w)$.

(ii) Similar to (i).

(iii) See 5.5.

(iv) By 5.2(i), $(v : w)(w : v) \subseteq (w : w) = R_1$, and so $(v : w) \subseteq R_1(w : v)^{-1}$.

(v) Similar to (iv).

(vi) See 5.5.

(vii) Use 5.2(i).

(viii) Similar to (vii). □

Using the foregoing lemma, we get mappings $(\varphi_w =)\ \varphi : S \to \mathbf{G}_1$ and $(\psi_w =)\ \psi : S \to \mathbf{G}_2$ defined by $\varphi(v) = (v : w)$ and $\psi(v) = [v : w]$ for every $v \in S$ (5.7(i), (ii)).

Lemma 5.8.

(i) *$\varphi(S \setminus \{o\}) \subseteq \mathbf{F}_1$.*

(ii) *$\varphi(av) = \varphi(v)a^{-1} = a * \varphi(v)$ for all $a \in G$ and $v \in S$.*

(iii) *If $u \preceq_S v$, then $\varphi(v) \subseteq \varphi(u)$.*

P r o o f. (i) This follows from 5.5.

(ii) We have $\varphi(av) = (av : w) = (v : w)a^{-1} = \varphi(v)a^{-1} = a * \varphi(v)$ by 5.1(iv).

(iii) If $a \in \varphi(v)$, then $av \preceq_S w$, and, of course, $au \preceq_S av$. Thus $au \preceq_S w$ and $a \in \varphi(u)$. □

Lemma 5.9.

(i) *$\psi(S \setminus \{o\}) \subseteq \mathbf{F}_2$.*

(ii) *$\psi(av) = \psi(v)a^{-1} = a * \psi(v)$ for all $a \in G$ and $v \in S$.*

(iii) *If $u \dashv_S v$, then $\psi(v) \subseteq \psi(u)$.*

P r o o f. Similar to that of 5.8. □

Lemma 5.10.

(i) *$\varphi(v) \subseteq \psi(v)$ for every $v \in S$.*

(ii) *$\varphi(w) = R_1$ and $\psi(w) = R_2$.*

(iii) *$\varphi(o_S) = \emptyset = \psi(o_S)$.*

P r o o f. Obvious. □

Lemma 5.11. *Assume that S is transitive. Then:*

(i) *φ is a bijection of S onto $\mathbf{Q}_1$.*

(ii) *$u \preceq_S v$ if and only if $\varphi(v) \subseteq \varphi(u)$.*

Proof. (i) Let $u, v \in S$ be such that $\varphi(u) = \varphi(v)$. If $u = o$ or $v = o$, then $\varphi(u) = \emptyset = \varphi(v)$ and $u = o = v$ by 5.8(i). Hence, assume that $u \neq o \neq v$. Then $u = aw$ and $v = bw$ for some $a, b \in G$. Now, $R_1 a^{-1} = \varphi(aw) = \varphi(u) = \varphi(v) = \varphi(bw) = R_1 b^{-1}$, $R_1 a^{-1} b = R_1$, $a^{-1} b \in R_1^*$, $w = a^{-1} bw$ and, finally, $u = aw = bw = v$ (use 5.8(ii) and 5.6(i)). We have proved that φ is an injective mapping.

If $a \in G$, then $\varphi(a^{-1} w) = (w : w)a = R_1 a$ by 5.1(iv). It follows that φ is a projective mapping. Consequently, $\varphi : S \to \mathbf{Q}_1$ is a bijection.

(ii) If $u \preceq_S v$, then $\varphi(v) \subseteq \varphi(u)$ by 5.8(iii). Conversely, if $\varphi(v) \subseteq \varphi(u)$, $v \neq o$, $u = aw$, $v = bw$, then $R_1 b^{-1} = \varphi(v) \subseteq \varphi(u) = R_1 a^{-1}$, $R_1 \subseteq R_1 a^{-1} b = \varphi(b^{-1} aw) = (b^{-1} aw : w)$, $1 \in (b^{-1} aw : w)$, $b^{-1} aw \preceq_S w$ and, finally, $u = aw \preceq_S bw = v$. □

Lemma 5.12. *Assume that S is transitive. Then:*

(i) *ψ is a projection of S onto $\mathbf{Q}_2$.*

(ii) $\ker(\psi) = \pi_S$.

(iii) *$u \dashv_S v$ if and only if $\psi(v) \subseteq \psi(u)$.*

Proof. Similar to that of 5.11. □

Corollary 5.13. *Assume that S is transitive. Then $\psi : S \to \mathbf{Q}_2$ is a bijection if and only if S is separable.*

Lemma 5.14. *If S is downwards-regular, then $\varphi(u + v) = \varphi(u) \cap \varphi(v)$ for all $u, v \in S$ such that $u + v \neq o_S$.*

Proof. The inclusion $\varphi(u + v) \subseteq \varphi(u) \cap \varphi(v)$ is clear from the definitions. Conversely, if $a \in \varphi(u) \cap \varphi(v)$, then $au \preceq_S w$, $av \preceq_S w$, and hence $a(u + v) \preceq_S w$, since S is downwards-regular. Thus $a \in \varphi(u + v)$. □

Lemma 5.15. *If S is upwards-regular, then $\psi(u+v) = \psi(u) \cap \psi(v)$ for all $u, v \in S$ such that $u + v \neq o_S$.*

Proof. Similar to that of 5.14. □

Theorem 5.16. *Let S be a transitive zeropotent G-semimodule such that $S+S \neq \{o_S\}$ (see 2.8). Let $w \in S$, $w \neq o_S$, $R_1 = \{a \in G;\ aw \preceq_S w\}$ and $R_2 = \{a \in G;\ aw \dashv_S w\}$. Then:*

(i) *S is ideal-simple, $S+S=S$ and $\mathrm{Ann}_S(S) = \{o_S\}$.*

(ii) *Both R_1 and R_2 are subsemigroups of G and $1 \in R_1 \subseteq R_2$.*

(iii) *The mapping $\varphi : v \to \{a \in G;\ av \preceq_S w\}$ is a bijection of S onto $\mathbf{Q}(G, R_1)$ such that $u \preceq_S v$ if and only if $\varphi(v) \subseteq \varphi(u)$.*

(iv) *If S is downwards-regular, then $\varphi(u+v) = \varphi(u) \cap \varphi(v)$ for all $u, v \in S$ such that $u+v \neq o_S$.*

(v) *The mapping $\psi : v \to \{a \in G;\ av \dashv_S w\}$ is a projection of S onto $\mathbf{Q}(G, R_2)$ such that $\ker(\psi) = \pi_S$ and $u \dashv_S v$ if and only if $\psi(v) \subseteq \psi(u)$.*

(vi) *If S is separable, then ψ is a bijection of S onto $\mathbf{Q}(G, R_2)$.*

(vii) *If S is upwards-regular, then $\psi(u+v) = \psi(u) \cap \psi(v)$ for all $u, v \in S$ such that $u+v \neq o_S$.*

Proof. See 2.7, 2.8, 5.1(i), 5.2(iii), 5.5, 5.11, 5.14, 5.12 and 5.15. □

Lemma 5.17. *Let $a, b \in G$, $u = aw$ and $v = bw$. Then:*

(i) *$\varphi(u) \cap \varphi(v) \neq \emptyset$ if and only if $R_1 \cap R_1 a^{-1} b \neq \emptyset$.*

(ii) *$R_1 \cap a^{-1} b R_1 \neq \emptyset$ if and only if there exists $c \in G$ with $cw \preceq_S u$ and $cw \preceq_S v$.*

Proof. (i) We have $\varphi(u) = R_1 a^{-1}$ and $\varphi(v) = R_1 b^{-1}$. The rest is clear.

(ii) If $d = a^{-1}be$, where $d, e \in R_1$, then $ad = c = be$, $cw = adw \preceq_S aw = u$ and $cw = bew \preceq_S bw = v$. Similarly the converse implication. □

Lemma 5.18. *Assume that S is strongly balanced. If $a, b \in G$ are such that $aw + bw = o$, then $R_1 \cap a^{-1} b R_1 \neq \emptyset$.*

Proof. Use 5.17(ii). □

Lemma 5.19. *Let $a, b \in G$, $u = aw$ and $v = bw$. Then:*

(i) *$\psi(u) \cap \psi(v) \neq \emptyset$ if and only if $R_2 \cap R_2 a^{-1} b \neq \emptyset$.*

(ii) *$R_2 \cap a^{-1} b R_2 \neq \emptyset$ if and only if there exists $c \in G$ with $cw \dashv_S u$ and $cw \dashv_S v$.*

Proof. Similar to that of 5.17. □

Lemma 5.20. *Assume that S is balanced. If $a, b \in G$ are such that $aw + bw = o$, then $R_2 \cap a^{-1} b R_2 \neq \emptyset$.*

Proof. Use 5.19(ii). □

Lemma 5.21. *Assume that S is transitive. If $a \in G$ is such that $w + aw \neq o$, then $a \in R_1^{-1}R_1$.*

P r o o f. We have $w + aw = bw$ for some $b \in G$, $aw \preceq_S bw$, $b^{-1}a \in R_1$, $w \preceq_S bw$, $b^{-1} \in R_1$. Consequently, $a \in R_1^{-1}R_1$. □

Lemma 5.22. *Assume that S is strongly balanced. If $a \in G$ is such that $w+aw = o$, then $a \in R_1R_1^{-1}$.*

P r o o f. This follows immediately from 5.18. □

Lemma 5.23. *If S is transitive and strongly balanced, then $G = R_1^{-1}R_1 \cup R_1R_1^{-1}$.*

P r o o f. Combine 5.21 and 5.22. □

6. A few consequences

6.1. Let S be a non-trivial transitive zp-semimodule over a group G such that S is downwards-regular and strongly balanced. By 2.7 and 2.8, S is ideal-simple, $\mathrm{Ann}_S(S) = \{o_S\}$ and $S + S = S$, i.e., S is a zs-semimodule.

Now, choose $w \in S$, $w \neq o_S$, and put $R = R_{1,w} = \{a \in G;\ aw \preceq_S w\}$ and $\varphi = \varphi_w$, where $\varphi_w(v) = \{a \in G;\ av \preceq_S w\}$ for every $v \in S$. According to 5.2(iii), 5.5 and 5.11, R is a subsemigroup of the group G, $1 \in R$ and φ is a bijection of S onto $\mathbf{Q} = \mathbf{Q}(G, R)$ such that $u \preceq_S v$ iff $\varphi(v) \subseteq \varphi(u)$. Moreover, by 5.8 and 5.14, $\varphi(av) = \varphi(v)a^{-1}$, $\varphi(aw) = Ra^{-1}$, $a \in G$, and if $u, v \in S$ are such that $u + v \neq o_S$, then $\varphi(u + v) = \varphi(u) \cap \varphi(v)$.

Lemma 6.1.1. *The condition* (f1) *(see* 4.1*) is satisfied.*

P r o o f. Let $a \in G$ be such that $R \cap Ra = \emptyset$. It follows from 5.18 that $a^{-1}w + w \neq o$, and hence $a^{-1}w + w = b^{-1}w$ for some $b \in G$. Now, $Rb = \varphi(b^{-1}w) = \varphi(a^{-1}w + w) = \varphi(a^{-1}w) \cap \varphi(w) = Ra \cap R$. □

The condition (f1) is true, and so we get the groupoid $\mathbf{Q} = \mathbf{Q}(+)$ due to 4.1.

Lemma 6.1.2. *φ is an isomorphism of $S(+)$ onto $\mathbf{Q}(+)$.*

P r o o f. Since φ is a bijection, we have to show that φ is a homomorphism of the additive structures. For, let $u, v \in S$. We have $\varphi(o_S) = \emptyset$, and hence $\varphi(u + v) = \emptyset = \varphi(u) + \varphi(v)$ provided that either $u = o$ or $v = o$. Now, assume $u \neq o \neq v$. Then $u = aw$ and $v = bw$, $a, b \in G$.

First, let $u+v \neq o$. If $R \cap a^{-1}bR \neq \emptyset$, then $cw \preceq_S u$ and $cw \preceq_S v$ for some $c \in G$ by 5.17(ii) and it follows that $u+v=o$, a contradiction. Thus $R \cap a^{-1}bR = \emptyset$, $Ra^{-1} + Rb^{-1} = Ra^{-1} \cap Rb^{-1} \neq \emptyset$ in $\mathbf{Q}(+)$ and we get $\varphi(u+v) = \varphi(u) \cap \varphi(v) = Ra^{-1} \cap Rb^{-1} = Ra^{-1} + Rb^{-1} = \varphi(u) + \varphi(v)$.

Next, let $u+v=o$. Then $R \cap a^{-1}bR \neq \emptyset$ by 5.18, and therefore $\varphi(u+v) = \varphi(o) = \emptyset = Ra^{-1} + Rb^{-1} = \varphi(u) + \varphi(v)$, too. □

Lemma 6.1.3. *The condition* (f2) *is satisfied.*

P r o o f. By 6.1.2, $S(+)$ is isomorphic to $\mathbf{Q}(+)$. Consequently, $\mathbf{Q}(+)$ is a semigroup and (f2) follows by 4.1.3. □

Lemma 6.1.4. $\mathbf{Q}$ *is a non-trivial transitive zs-semimodule and* $\varphi: S \to \mathbf{Q}$ *is an isomorphism of the semimodules.*

P r o o f. See 4.1, 6.1.2 and 6.1.3. □

Lemma 6.1.5. *The conditions* (f3) *and* (f4) *are satisfied.*

P r o o f. By 6.1.4, $\mathbf{Q}$ ($\cong S$) is a non-trivial zs-semimodule. Now, (f3) follows from 4.1.4 and (f4) is clear from 4.1.6 and 5.11(ii). □

Theorem 6.1.6. *The conditions* (f1), (f2), (f3) *and* (f4) *are satisfied (see* 4.1*) and the semimodules* S *and* $\mathbf{Q}(G,R)$ *are isomorphic.*

P r o o f. See 6.1.2, …, 6.1.5. □

6.2. Let S be a non-trivial transitive zp-semimodule over a group G such that S is upwards-regular and balanced. By 2.7 and 2.8, S is ideal-simple, $\mathrm{Ann}_S(S) = \{o_S\}$ and $S+S=S$, i.e., S is a zs-semimodule.

Now, choose $w \in S$, $w \neq o_S$, and put $R = R_{2,w} = \{a \in G;\ aw \dashv_S w\}$ and $\psi = \psi_w$, where $\psi_w(v) = \{a \in G;\ av \dashv_S w\}$ for every $v \in S$. According to 5.2(iii), 5.5 and 5.12, R is a subsemigroup of the group G, $1 \in R$ and ψ is a projection of S onto $\mathbf{Q} = \mathbf{Q}(G,R)$ such that $\ker(\psi) = \pi_S$ and $u \dashv_S v$ iff $\psi(v) \subseteq \psi(u)$. Moreover, by 5.9 and 5.15, $\psi(av) = \psi(v)a^{-1}$, $\psi(aw) = Ra^{-1}$, $a \in G$, and if $u, v \in S$ are such that $u+v \neq o_S$, then $\psi(u+v) = \psi(u) \cap \psi(v)$.

Lemma 6.2.1. *The condition* (f1) *(see 4.1) is satisfied.*

P r o o f. Similar to that of 6.1.1 (use 5.20). □

The condition (f1) is true, and so we get the groupoid $\mathbf{Q} = \mathbf{Q}(+)$ due to 4.1.

Lemma 6.2.2. *ψ is a homomorphism of $S(+)$ onto $\mathbf{Q}(+)$.*

P r o o f. We have to show that ψ is a homomorphism of the additive structures. For, let $u, v \in S$. We have $\psi(o_S) = \emptyset$, and hence $\psi(u+v) = \emptyset = \psi(u)+\psi(v)$ provided that either $u = o$ or $v = o$. Now, assume that $u \neq o \neq v$. Then $u = aw$ and $v = bw$, $a, b \in G$.

First, let $u + v \neq o$. If $R \cap a^{-1}bR \neq \emptyset$, then $cw \dashv_S u$ and $cw \dashv_S v$ for some $c \in G$ by 5.19(ii). Consequently, $\mathrm{Ann}_S(cw) \subseteq \mathrm{Ann}_S(u) \cap \mathrm{Ann}_S(v)$, $cw \in \mathrm{Ann}_S(cw)$ implies $cw + u = o$, $u \in \mathrm{Ann}_S(cw)$ and, finally, $u \in \mathrm{Ann}_S(v)$, $u + v = o$, a contradiction. Thus $R \cap a^{-1}bR = \emptyset$, $Ra^{-1} + Rb^{-1} = Ra^{-1} \cap Rb^{-1} \neq \emptyset$ in $\mathbf{Q}(+)$ and we get $\psi(u + v) = \psi(u) \cap \psi(v) = Ra^{-1} \cap Rb^{-1} = Ra^{-1} + Rb^{-1} = \psi(u) + \psi(v)$.

Next, let $u+v = o$. Then $R \cap a^{-1}bR \neq \emptyset$ by 5.20, and therefore $\psi(u+v) = \psi(o) = \emptyset = Ra^{-1} + Rb^{-1} = \psi(u) + \psi(v)$, too. □

Lemma 6.2.3. *The condition* (f2) *is satisfied.*

P r o o f. By 6.2.2, $\mathbf{Q}(+)$ is a homomorphic image of $S(+)$. Consequently, $\mathbf{Q}(+)$ is a semigroup and (f2) follows by 4.1.3. □

Lemma 6.2.4. *$\mathbf{Q}$ is a non-trivial transitive zs-semimodule and $\psi : S \to \mathbf{Q}$ is a projective homomorphism of the semimodules.*

P r o o f. We have $\pi_S \neq S \times S$, and hence $\mathbf{Q}$ is non-trivial. The rest is clear from 4.1, 6.2.2 and 6.2.3. □

Lemma 6.2.5. *The conditions* (f3) *and* (f5) *are satisfied.*

P r o o f. By 6.2.4, $\mathbf{Q}$ is a non-trivial zs-semimodule and (f3) follows from 4.1.4. Now, consider the condition (f5). According to 4.1.9 and 4.1.5(v), it suffices to show that $Rb \subseteq Ra$ whenever $a, b \in G$ are such that $Ra \dashv_{\mathbf{Q}} Rb$. We have $Ra = \psi(u)$ and $Rb = \psi(v)$, $u = a^{-1}w$, $v = b^{-1}w$. If $z \in \mathrm{Ann}_S(u)$, then $\psi(z) \in \mathrm{Ann}_{\mathbf{Q}}(Ra)$, and so $\psi(z) \in \mathrm{Ann}_{\mathbf{Q}}(Rb)$ and $\psi(z + v) = \psi(z) + \psi(v) = \emptyset$ $(= o_{\mathbf{Q}})$. Thus $(z + v, o_S) \in \pi_S$, $z + v \in \mathrm{Ann}_S(S) = \{o_S\}$, $z + v = o_S$ and $z \in \mathrm{Ann}_S(v)$. It follows that $u \dashv_S v$ and $Rb = \psi(v) \subseteq \psi(u) = Ra$ by 5.12. □

Theorem 6.2.6. *The conditions* (f1), (f2), (f3) *and* (f5) *are satisfied and there exists a projection of the semimodule S onto the semimodule $\mathbf{Q}(G, R)$. This projection is an isomorphism if and only if S is separable.*

P r o o f. See 6.2.1, …, 6.2.5. □

References

[1] *R. El Bashir and T. Kepka*: Commutative semigroups with few invariant congruences. Semigroup Forum *64* (2002), 453–471.

[2] *V. Flaška and T. Kepka*: Commutative zeropotent semigroups. Acta Univ. Carolinae *47/1* (2006), 3–14.

[3] *J. Ježek*: Simple semilattices with two commuting automorphisms. Algebra Univ. *15* (1982), 162-175.

[4] *J. Ježek and T. Kepka*: Medial groupoids. Rozpravy ČSAV, vol. 93, 1983, pp. 93.

[5] *M. Maróti*: Semilattices with a group of automorphisms. Algebra Univ. *38* (1997), 238–265.

[6] *S. H. McCleary*: o-Primitive ordered permutation groups. Pacific J. Math. *40* (1972), 349–372.

[7] *S. H. McCleary*: o-Primitive ordered permutation groups (II). Pacific J. Math. *49* (1973), 431–445.

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