

Ján Jakubík

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## ON SOME TYPES OF RADICAL CLASSES

JÁN JAKUBÍK, Košice

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*Abstract.* Let  $m$  be an infinite cardinal. We denote by  $C_m$  the collection of all  $m$ -representable Boolean algebras. Further, let  $C_m^0$  be the collection of all generalized Boolean algebras  $B$  such that for each  $b \in B$ , the interval  $[0, b]$  of  $B$  belongs to  $C_m$ . In this paper we prove that  $C_m^0$  is a radical class of generalized Boolean algebras. Further, we investigate some related questions concerning lattice ordered groups and generalized  $MV$ -algebras.

*Keywords:* Boolean algebra, generalized Boolean algebra,  $m$ -representability, lattice ordered group, generalized  $MV$ -algebra, radical class

*MSC 2010:* 06E05, 06F20, 06D35

### 1. INTRODUCTION

In the present paper we deal with some types of radical classes of generalized Boolean algebras, lattice ordered groups and generalized  $MV$ -algebras.

Let  $m$  be an infinite cardinal. The  $m$ -representability of Boolean algebras was investigated by Chang [1], Scott [16], Pierce [14], Smith [17] and Sikorski [18], [19]; cf. also the monograph Sikorski [20].

The notion of radical class of generalized Boolean algebras was studied by the author [9]; for the analogous notion concerning lattice ordered groups cf. the author [8], Conrad [2], Darnel [4] and Ton [21]; for  $MV$ -algebras cf. the author [10].

The collection of all  $m$ -representable Boolean algebras will be denoted by  $C_m$ . Let  $C_m^0$  be the collection of all generalized Boolean algebras  $B$  such that for each  $b \in B$ , the interval  $[0, b]$  of  $B$  is an element of  $C_m$ .

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In the present paper we prove that  $C_m^0$  is a radical class of generalized Boolean algebras.

For dealing with the  $m$ -representability of Boolean algebras in Theorem 29.3 of [20], several equivalent conditions concerning a Boolean algebra  $\mathfrak{A}$  and the infinite cardinal  $m$  have been considered. Each of these conditions is necessary and sufficient for the  $m$ -representability of  $\mathfrak{A}$ .

Only two from these conditions (namely,  $(r_1)$  and  $(r_2)$ ) are expressed in terms of elements of  $\mathfrak{A}$ ; the other conditions deal with filters of  $\mathfrak{A}$  or with topological notions related to the Stone space of  $\mathfrak{A}$ .

The detailed definitions of  $(r_1)$  and  $(r_2)$  are given in Section 2 below. According to the mentioned result of Sikorski, we have

$$(1) \quad (r_1) \Leftrightarrow (r_2)$$

for each Boolean algebra  $\mathfrak{A}$ .

The conditions  $(r_1)$  and  $(r_2)$  can be applied also for generalized  $MV$ -algebras. We prove that (1) is valid for projectable generalized  $MV$ -algebras. This is a generalization of the mentioned Sikorski's result, since each Boolean algebra is the underlying lattice of some projectable  $MV$ -algebra; cf. Section 6 for a more detailed description of this fact.

We prove that the collection of all generalized  $MV$ -algebras satisfying the condition  $(r_1)$  is a radical class. The analogous assertion concerning  $(r_2)$  fails to be valid.

Some related questions concerning lattice ordered groups are studied.

## 2. PRELIMINARIES

For Boolean algebras we apply the terminology and the notation as in Sikorski [20] with the distinctions that 0 and 1 denote the least and the greatest element of a Boolean algebra, respectively, and that symbols  $\wedge, \vee$  are used for denoting lattice operations.

We recall some definitions which will be frequently used in the sequel. Let  $m$  be an infinite cardinal.

A Boolean algebra  $\mathfrak{A}$  is  *$m$ -representable* if it is isomorphic to a quotient algebra  $\mathfrak{F}/\Delta$ , where  $\mathfrak{F}$  is an  $m$ -field of sets and  $\Delta$  is an  $m$ -ideal of  $\mathfrak{F}$ .

We remark that the classical result proved independently by Loomis [13] and Sikorski [18] can be formulated as follows:

(LS) Each Boolean  $\sigma$ -algebra is  $\sigma$ -representable.

It was remarked in [20] that in the case  $\mathfrak{m} = \aleph_0$ , Theorem 29.3 (dealing with the conditions  $(r_1)$  and  $(r_2)$ ) yields the following generalization of Theorem (LS):

(LS<sub>0</sub>) (Cf. [20], 29.4.) Every Boolean algebra is  $\sigma$ -representable.

The detailed formulations of the conditions  $(r_1)$  and  $(r_2)$  concerning a Boolean algebra  $\mathfrak{A}$  and an infinite cardinal  $\mathfrak{m}$  are as follows:

$(r_1)$  If  $(a_{t,s})_{t \in T, s \in S}$  is an  $\mathfrak{m}$ -indexed set of elements of  $\mathfrak{A}$  (i.e.,  $\text{card } T = \text{card } S = \mathfrak{m}$ ) such that

$$(1) \text{ there exist all joins } \bigvee_{s \in S} a_{t,s} \text{ and the meet } \bigwedge_{t \in T} \bigvee_{s \in S} a_{t,s},$$

and if

$$(2) \bigwedge_{t \in T} \bigvee_{s \in S} a_{t,s} \neq 0,$$

then there exists a function  $\varphi \in S^T$  with

$$(3) \bigwedge_{t \in T'} a_{t, \varphi(t)} \neq 0 \text{ for each finite subset } T' \text{ of } T.$$

$(r_2)$  If  $(a_{t,s})_{t \in T, s \in S}$  is an  $\mathfrak{m}$ -indexed set of elements of  $\mathfrak{A}$  such that (1) is valid and if

$$(4) \bigwedge_{t \in T} \bigvee_{s \in S} a_{t,s} = 1,$$

then for each  $a \in \mathfrak{A}$  with  $a \neq 0$  there exists  $\varphi \in S^T$  such that

$$(5) a \wedge \bigwedge_{t \in T'} a_{t, \varphi(t)} \neq 0 \text{ for each finite subset } T' \text{ of } T.$$

**Theorem 2.1** (Cf. [20], 29.3.). *Let  $\mathfrak{A}$  be a Boolean algebra and  $i \in \{1, 2\}$ . Then  $\mathfrak{A}$  is  $\mathfrak{m}$ -representable if and only if the condition  $(r_i)$  is satisfied.*

**Lemma 2.2.** *Assume that  $\mathfrak{A}$  is a Boolean algebra satisfying the condition  $(r_1)$ . Let  $b_1, b_2 \in \mathfrak{A}$ ,  $b_1 \leq b_2$ . Then the interval  $[b_1, b_2]$  of  $\mathfrak{A}$  also satisfies the condition  $(r_1)$ .*

**Proof.** From the definition of  $(r_1)$  we immediately obtain that if  $x \in \mathfrak{A}$ , then the interval  $[0, x]$  of  $\mathfrak{A}$  satisfies the condition  $(r_1)$ . Let  $b_3$  be the complement of  $b_1$  in  $\mathfrak{A}$ . Then the interval  $[b_1, b_2]$  is isomorphic to the interval  $[0, b_2 \wedge b_3]$ . Thus  $[b_1, b_2]$  satisfies  $(r_1)$ .  $\square$

**Lemma 2.3.** *Assume that  $\mathfrak{A}$  is a Boolean algebra which does not satisfy the condition  $(r_1)$ . Then there is  $b \in \mathfrak{A}$  such that*

*(\*) if  $b' \in [0, b]$ ,  $b' > 0$ , then the interval  $[0, b']$  does not satisfy  $(r_1)$ .*

**Proof.** In view of the assumption there exists an  $\mathfrak{m}$ -indexed system  $(a_{t,s})_{s \in T, s \in S}$  of elements of  $\mathfrak{A}$  such that (1) and (2) are valid and, moreover, there does not exist

any function  $\varphi \in S^T$  satisfying (3). Hence for each  $\varphi \in S^T$  there exists a finite subset  $T'$  of  $T$  such that

$$(3') \quad \bigwedge_{t \in T'} a_{t, \varphi(t)} = 0.$$

We put  $b = \bigwedge_{t \in T} \bigvee_{s \in S} a_{t, s}$  and

$$a'_{t, s} = b' \wedge a_{t, s} \quad \text{for each } t \in T, s \in S,$$

where  $b' \in [0, b]$ ,  $b' > 0$ .

Since  $\mathfrak{A}$  is infinitely distributive, for each  $t \in T$  we have

$$\begin{aligned} b' \wedge \left( \bigvee_{s \in S} a_{t, s} \right) &= \bigvee_{s \in S} (b' \wedge a_{t, s}) = \bigvee_{s \in S} a'_{t, s}, \\ b' &= b' \wedge b = b' \wedge \left( \bigwedge_{t \in T} \bigvee_{s \in S} a_{t, s} \right) = \bigwedge_{t \in T} \bigvee_{s \in S} (b' \wedge a_{t, s}) = \bigwedge_{t \in T} \bigvee_{s \in S} a'_{t, s}. \end{aligned}$$

Moreover, according to (3') for each  $\varphi \in S^T$ ,

$$0 = b' \wedge (\bigwedge_{t \in T'} a_{t, \varphi(t)}) = \bigwedge_{t \in T'} (b' \wedge a_{t, \varphi(t)}) = \bigwedge_{t \in T'} \bigvee_{s \in S} a'_{t, \varphi(t)}.$$

Thus the interval  $[0, b']$  does not satisfy the condition (r<sub>1</sub>). □

### 3. GENERALIZED BOOLEAN ALGEBRAS

A lattice  $L$  with the least element 0 is a *generalized Boolean algebra* if for each  $x \in L$ , the interval  $[0, x]$  of  $L$  is a Boolean algebra.

For a generalized Boolean algebra  $L$  we denote by  $\mathcal{I}(L)$  the system of all ideals of  $L$ . This system is partially ordered by the set-theoretical inclusion. Then  $\mathcal{I}(L)$  is a complete lattice.

If  $\{L_i\}_{i \in I}$  is a nonempty system of elements of  $\mathcal{I}(L)$ , then

$$\text{a) } \bigwedge_{i \in I} L_i = \bigcap_{i \in I} L_i$$

and

$$\text{b) } \bigvee_{i \in I} L_i \text{ is the set of all } x \in L \text{ such that there exist } i(1), \dots, i(n) \in I \text{ and } x_{i(1)} \in L_{i(1)}, \dots, x_{i(n)} \in L_{i(n)} \text{ with } x = x_{i(1)} \vee \dots \vee x_{i(n)}.$$

**Definition 3.1.** A nonempty collection  $C$  of generalized Boolean algebras is a *radical class* of generalized Boolean algebras if it satisfies the following conditions:

- (i)  $C$  is closed under isomorphisms;
- (ii) if  $L \in C$  and  $L_1$  is an ideal of  $L$ , then  $L_1 \in C$ ;
- (iii) if  $L$  is a generalized Boolean algebra and  $\{L_i\}_{i \in I} \subseteq \mathcal{S}(L) \cap C$ , then  $\bigvee_{i \in I} L_i$  belongs to  $C$ .

Let  $m$  be an infinite cardinal and let  $C_m^0$  be as in Section 1.

**Proposition 3.2.**  $C_m^0$  is a radical class of generalized Boolean algebras.

*Proof.* It is obvious that  $C_m^0$  is closed with respect to isomorphisms. If  $L \in C_m^0$  and if  $L_1 \in \mathcal{S}(L)$ , then in view of 2.2 we obtain  $L_1 \in C_m^0$ . Hence the conditions (i) and (ii) from 3.1 are satisfied.

By way of contradiction, assume that the condition (iii) from 3.1 fails to be valid. Hence there exists a generalized Boolean algebra  $L$  and a system  $\{L_i\}_{i \in I} \subseteq \mathcal{S}(L) \cap C_m^0$  such that  $\bigvee_{i \in I} L_i$  does not belong to  $C_m^0$ . Then there exists  $0 < x \in \bigvee_{i \in I} L_i$  with  $[0, x] \notin C_m$ . Thus there is  $0 < b \in [0, x]$  satisfying the condition (\*) from 2.3.

In view of the relation  $b \in \bigvee_{i \in I} L_i$  there are  $i(1), \dots, i(n) \in I$  and  $x_{i(1)} \in L_{i(1)}, \dots, x_{i(n)} \in L_{i(n)}$  such that  $b = x_{i(1)} \vee \dots \vee x_{i(n)}$ . Without loss of generality we can suppose that  $x_{i(1)} > 0, \dots, x_{i(n)} > 0$ .

Since  $x_{i(1)} \in L_{i(1)}$ , by applying 2.2 we obtain that  $[0, x_{i(1)}]$  satisfies  $(r_1)$ . Since (\*) is valid for  $b$ , we arrived at a contradiction.  $\square$

#### 4. LATTICE ORDERED GROUPS

The group operation in a lattice ordered group will be written additively, though we do not assume the commutativity of this operation. Let  $G$  be a lattice ordered group. As usual, we set  $G^+ = \{g \in G: g \geq 0\}$ .

Let  $m$  be an infinite cardinal.

**Definition 4.1.** We say that a lattice ordered group  $G$  satisfies  $(r'_1)$  if it fulfils the same conditions as in  $(r_1)$  with the distinction that instead of  $\mathfrak{A}$  we have now the set  $G^+$ .

Let  $u \in G^+$ . The element  $u$  is a *weak unit* of  $G$  if for each  $g \in G$  with  $g > 0$  the relation  $g \wedge u > 0$  is valid. If for each  $g \in G$  there is a positive integer  $n$  such that  $g \leq nu$ , then  $u$  is a *strong unit* of  $G$ . Every strong unit is a weak unit.

**Definition 4.2.** Assume that  $u$  is a weak unit of a lattice ordered group  $G$ . We say that  $G$  satisfies  $(r'_2)$  if it fulfils the same conditions as in  $(r_2)$  with the distinctions that

- 1) instead of  $\mathfrak{A}$  we have the set  $G^+$ ;
- 2) the condition (4) is replaced by

$$(4') \text{ the element } \bigwedge_{t \in T} \bigvee_{s \in S} a_{t,s} \text{ is a weak unit of } G.$$

**Proposition 4.3.** Assume that  $G$  is a lattice ordered group having a weak unit. Then  $(r'_1) \Rightarrow (r'_2)$ .

*Proof.* Suppose that  $(r'_1)$  is valid. By way of contradiction, assume that  $(r'_2)$  fails to hold. Then there exists an  $m$ -indexed set  $(a_{t,s})_{t \in T, s \in S}$  of elements of  $G^+$  such that  $(4')$  is valid; moreover, there exists  $0 < a \in G$  having the property that for each  $\varphi \in S^T$ ,

$$(5') \ a \wedge \bigwedge_{t \in T'} a_{t, \varphi(t)} = 0 \text{ for some finite subset } T' \text{ of } T.$$

Put  $a_0 = a \wedge u$ . Then  $a_0 > 0$ . Further, for each  $t \in T$  and  $s \in S$  we set  $a'_{t,s} = a_0 \wedge a_{t,s}$ . Consider the  $m$ -indexed set  $(a'_{t,s})_{t \in T, s \in S}$ .

All elements  $a'_{t,s}$  belong to  $G^+$ . In view of the infinite distributivity of  $G$  and according to the condition (1) in  $(r_1)$  we conclude that all the joins  $\bigvee_{s \in S} a'_{t,s}$  and the meet  $\bigwedge_{t \in T} \bigvee_{s \in S} a'_{t,s}$  exist in  $G$ . Put  $u = \bigwedge_{t \in T} \bigvee_{s \in S} a_{t,s}$ . From  $(4')$  we obtain

$$0 < u \wedge a_0 = \bigwedge_{t \in T} \bigvee_{s \in S} (a_{t,s} \wedge a_0) = \bigwedge_{t \in T} \bigvee_{s \in S} a'_{t,s}.$$

Applying  $(r'_1)$  to the  $m$ -indexed system  $(a'_{t,s})_{t \in T, s \in S}$  we conclude that there exists  $\varphi \in S^T$  with

$$(+) \ \bigwedge_{t \in T'} a'_{t, \varphi(t)} \neq 0 \text{ for each finite subset } T' \text{ of } T.$$

But according to  $(5')$  we have

$$u \wedge a \wedge \bigwedge_{t \in T'} a_{t, \varphi(t)} = 0 \text{ for some finite } T' \subseteq T,$$

whence

$$\bigwedge_{t \in T'} a'_{t, \varphi(t)} = 0 \text{ for some finite } T' \subseteq T.$$

In view of  $(+)$ , we have arrived at a contradiction. □

For  $X \subseteq G$  we put

$$X^\delta = \{y \in G: |x| \wedge |y| = 0 \text{ for each } x \in X\};$$

$X^\delta$  is a *polar* of  $G$ . The lattice ordered group  $G$  is *projectable* if for each  $x \in G$  we have a direct product decomposition

$$G = \{x\}^{\delta\delta} \times \{x\}^\delta.$$

In such a case,  $x$  is a weak unit of the lattice ordered group  $\{x\}^{\delta\delta}$ .

**Proposition 4.4.** *Assume that  $G$  is a projectable lattice ordered group having a weak unit. Then  $(r'_2) \Rightarrow (r'_1)$ .*

**Proof.** Suppose that  $(r'_2)$  is valid. By way of contradiction, assume that  $(r'_1)$  does not hold.

Thus there exists an  $m$ -indexed set  $(a_{t,s})_{t \in T, s \in S}$  of elements of  $G^+$  such that

- (i) all the joins  $\bigvee_{s \in S} a_{t,s}$  exist in  $G$ ;
- (ii) there is  $v \in G$  such that  $v > 0$  and  $v = \bigwedge_{t \in T} \bigvee_{s \in S} a_{t,s}$ ;
- (iii) for each  $\varphi \in S^T$  and for each finite subset  $T'$  of  $T$  we have

$$\bigwedge_{t \in T'} u_{t, \varphi(t)} = 0.$$

Since  $G$  is projectable, the relation

$$(6) \quad G = \{v\}^{\delta\delta} \times \{v\}^\delta$$

is valid; moreover,  $v$  is a weak unit of the lattice ordered group  $\{v\}^{\delta\delta}$ .

For each  $z \in G$  we denote by  $z(\{v\}^{\delta\delta})$  the component of  $z$  in the direct factor  $\{v\}^{\delta\delta}$  of  $G$ ; the meaning of  $z(\{v\}^\delta)$  is analogous.

In particular, for  $z = v$  we have

$$v(\{v\}^{\delta\delta}) = v, \quad v(\{v\}^\delta) = 0.$$

For each  $t \in T$  and  $x \in S$  we put  $a'_{t,s} = v \wedge a_{t,s}$ . Then, in view of the infinite distributivity of  $G$ , we obtain

$$v = \bigwedge_{t \in T} \bigvee_{s \in S} a'_{t,s}.$$

Since  $0 \leq a'_{t,s} \leq v$ , we get

$$a'_{t,s}(\{v\}^{\delta\delta}) = a'_{t,s}, \quad a'_{t,s}(\{v\}^\delta) = 0.$$



For each  $t \in T$  and  $s \in S$  we denote

$$a''_{t,s} = a'_{t,s} + u(\{v\}^\delta).$$

Then we have

$$a''_{t,s}(\{v\}^{\delta\delta}) = a'_{t,s}, \quad a''_{t,s}(\{v\}^\delta) = u(\{v\}^\delta).$$

This yields that there exist all joins  $\bigvee_{s \in S} a''_{t,s}$  and

$$(7) \quad \bigwedge_{t \in T} \bigvee_{s \in S} a''_{t,s} = v + u(\{v\}^\delta).$$

Let  $u$  be a weak unit in  $G$ ; we infer that  $u(\{v\}^\delta)$  is a weak unit in the direct factor  $\{v\}^\delta$ . Moreover,  $v$  is a weak unit in the direct factor  $\{v\}^{\delta\delta}$ . Therefore in view of (6) and (7),  $\bigwedge_{t \in T} \bigvee_{s \in S} a''_{t,s}$  is a weak unit of the lattice ordered group  $G$ .

Looking at the  $m$ -indexed system  $(a'_{t,s})_{t \in T, s \in S}$  and at the definition of  $(r'_2)$  we conclude that for each  $a \in G$  with  $a > 0$  there exists  $\varphi \in S^T$  such that

$$(8) \quad a \wedge \bigwedge_{t \in T'} a''_{t,\varphi(t)} \neq 0 \quad \text{for each finite subset } T' \text{ of } T.$$

From the definition of  $a'_{t,s}$  and from (iii) we infer that for each  $\varphi \in S^T$  and for each finite subset  $T'$  of  $T$  the relation

$$(9) \quad \bigwedge_{t \in T'} a'_{t,\varphi(t)} = 0$$

is valid.

Since  $a'_{t,s} \in \{v\}^{\delta\delta}$  and  $u(\{v\}^\delta) \in \{v\}^\delta$ , we get

$$a'_{t,s} \wedge u(\{v\}^\delta) = 0,$$

hence

$$a''_{t,s} = a'_{t,s} \vee (u(\{v\}^\delta)).$$

Also,  $v \wedge (u(\{v\}^\delta)) = 0$ .

In view of  $v > 0$  we can take  $a = v$  in (8) and we obtain for each finite  $T' \subseteq T$ ,

$$\begin{aligned} a \wedge \bigwedge_{t \in T'} a''_{t,\varphi(t)} &= v \wedge \bigwedge_{t \in T'} (a'_{t,\varphi(t)} \vee (u(\{v\}^\delta))) \\ &= \bigwedge_{t \in T'} ((v \wedge a'_{t,\varphi(t)}) \vee (v \wedge (u(\{v\}^\delta)))) = \bigwedge_{t \in T'} (v \wedge a'_{t,\varphi(t)}) \\ &= v \wedge \bigwedge_{t \in T'} a'_{t,\varphi(t)} = 0 \end{aligned}$$

for each finite  $T' \subseteq T$  (according to (9)). In view of (8), we have arrived at a contradiction.  $\square$

From 4.3 and 4.4 we obtain

**Theorem 4.5.** *Assume that  $G$  is a projectable lattice ordered group having a weak unit. Then  $(r'_1) \Leftrightarrow (r'_2)$ .*

## 5. THE RADICAL CLASS $C_m^1$

For a lattice ordered group  $G$  we denote by  $c(G)$  the system of all convex  $\ell$ -subgroups of  $G$ ; the system  $c(G)$  is partially ordered by the set-theoretical inclusion. Then  $c(G)$  is a complete lattice. If  $\{G_i\}_{i \in I}$  is a nonempty subset of  $c(G)$ , then  $\bigwedge_{i \in I} G_i = \bigcap_{i \in I} G_i$ . Further,  $\bigvee_{i \in I} G_i$  is the set of all  $g \in G$  such that there exist  $x_1, \dots, x_n \in \bigcup_{i \in I} G_i$  with  $g = x_1 + \dots + x_n$ . If, moreover,  $g > 0$ , then without loss of generality we can suppose that  $x_1 > 0, \dots, x_n > 0$ .

**Definition 5.1.** A nonempty collection  $X$  of lattice ordered groups is a radical class if it satisfies the following conditions:

- (i)  $X$  is closed with respect to isomorphisms;
- (ii) if  $G_1 \in X$  and  $G_2 \in c(X)$ , then  $G_2 \in X$ ;
- (iii) if  $G$  is a lattice ordered group and  $\emptyset \neq \{G_i\}_{i \in I} \subseteq c(G) \cap X$ , then  $\bigvee_{i \in I} G_i \in X$ .

Let  $m$  be an infinite cardinal. We denote by  $C_m^1$  the class of all lattice ordered groups which satisfy the condition  $(r'_1)$ .

The following assertion is easy to verify; the proof will be omitted.

**Lemma 5.2.** *Let  $G$  be a lattice ordered group and let  $G_1 \in c(G)$ .*

- (i) *Let  $\{x_i\}_{i \in I} \subseteq G_1$  and  $p, q \in G_1$ . If the relation  $p = \bigwedge_{i \in I} x_i$  is valid in  $G_1$ , then this relation holds also in  $G$ . Similarly, if  $q = \bigvee_{i \in I} x_i$  is valid in  $G_1$ , then the same holds in  $G$ .*
- (ii) *Let  $\{y_i\}_{i \in I} \subseteq G_1^+, y \in G$ . If the relation  $\bigwedge_{i \in I} y_i = y$  is valid in  $G$ , then  $y \in G_1^+$  and the mentioned relation holds also in  $G_1$ .*

**Lemma 5.3.** *Let  $G$  and  $G_1$  be as in 5.2. If  $G$  satisfies the condition  $(r'_1)$ , then  $G_1$  satisfies the condition as well.*

*Proof.* This is a consequence of the definition of  $(r'_1)$  and of 5.2. □

**Definition 5.4.** Let  $G$  be a lattice ordered group and  $g \in G^+$ . The interval  $[0, g]$  of  $G$  is said to be regular with respect to  $(r'_1)$  if the assertion of  $(r'_1)$  is valid whenever  $(a_{t,s})_{t \in T, s \in S}$  is an  $m$ -indexed set of elements of the interval  $[0, g]$ .

**Lemma 5.5.** *Let  $G$  be a lattice ordered group which fails to satisfy the condition  $(r'_1)$ . Then there is  $0 < b \in G^+$  such that*

*(\*) if  $0 < b' \in [0, b]$ , then the interval  $[0, b']$  is not regular with respect to  $(r'_1)$ .*

*Proof.* It suffices to apply analogous steps as in the proof of 2.3. □

Let  $\mathfrak{m}$  be an infinite cardinal. We denote by  $C_{\mathfrak{m}}^1$  the class of all lattice ordered groups which satisfy the condition  $(r'_1)$ .

**Proposition 5.6.**  $C_{\mathfrak{m}}^1$  is a radical class of lattice ordered groups.

*Proof.* Consider the conditions (i), (ii) and (iii) from 5.1. The validity of (i) is obvious. The validity of (ii) follows from 5.3.

By way of contradiction, suppose that (iii) fails to be valid. Then there exists a nonempty system  $\{G_i\}_{i \in I} \subseteq c(G) \cap C_{\mathfrak{m}}^1$  such that the lattice ordered group  $\bigvee_{i \in I} G_i$  does not satisfy the condition  $(r'_1)$ . Then according to 5.5 there exists  $0 < b \in \bigvee_{i \in I} G_i$  such that the condition (\*) from 5.5 is satisfied.

The element  $b$  can be expressed in the form  $b = x_1 + \dots + x_n$ , where  $x_1, \dots, x_n$  are elements of the set  $\bigcup_{i \in I} G_i^+$ . Therefore all intervals  $[0, x_1], \dots, [0, x_n]$  are regular with respect to  $(r'_1)$ . Without loss of generality we can suppose that  $x_1 > 0$ . Then in view of (\*), this interval fails to be regular with respect to  $(r'_1)$ ; we have arrived at a contradiction. □

Let us denote by  $C_{\mathfrak{m}}^2$  the class of all lattice ordered groups which satisfy the condition  $(r'_2)$ .

$C_{\mathfrak{m}}^2$  fails to be a radical class since it does not fulfil the condition (iii) from 5.1. Example: Let  $R$  be the set of all reals with the operation  $+$  and with the natural linear order. Let  $I$  be an infinite set of indices and for each  $i \in I$  let  $R_i = R$ ; put  $G = \prod_{i \in I} R_i$ . For  $i_0 \in I$  let  $\overline{R}_{i_0}$  be the set of all  $g \in G$  such that  $g_i = 0$  whenever  $i \in I$  and  $i \neq i_0$ . Then  $\overline{R}_{i_0} \in c(G)$  and  $\overline{R}_{i_0} \in C_{\mathfrak{m}}^2$  for any infinite cardinal  $\mathfrak{m}$ . But  $\bigvee_{i \in I} \overline{R}_i$  has no weak unit, whence it does not belong to  $C_{\mathfrak{m}}^2$ . Therefore the condition (iii) from 5.1 does not hold for  $C_{\mathfrak{m}}^2$ .

Let  $C_{\mathfrak{m}}^{02}$  be the class of all lattice ordered groups  $G$  such that for each  $0 < g \in G$ , the convex  $\ell$ -subgroup  $G_g$  of  $G$  generated by the element  $g$  belongs to  $C_{\mathfrak{m}}^2$ . Without a proof, we present the following result

(\*)  $C_{\mathfrak{m}}^{02}$  is a radical class of lattice ordered groups.

## 6. GENERALIZED $MV$ -ALGEBRAS

The notion of generalized  $MV$ -algebra was introduced independently by Georgescu and Iorgulescu [6], [7] and by Rachůnek [15] (in [6] and [7], the term ‘pseudo  $MV$ -algebra’ was used).

An  $MV$ -algebra  $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$  is an algebraic system of type  $(2,1,1,0,0)$  such the axioms (A1)–(A8) from [15] are satisfied.

Let  $G$  be a lattice ordered group with a strong unit  $u$ . The pair  $(G, u)$  is called a unital lattice ordered group. If no misunderstanding can occur we speak about  $G$  instead of  $(G, u)$ .

Given  $(G, u)$ , we put  $A = [0, u]$  and for  $x, y \in A$  we define

$$x \oplus y = (x + y) \wedge u, \quad x^\neg = u - x, \quad x^\sim = -x + u, \quad 1 = u.$$

Then  $\mathcal{A} = (A; \oplus, \neg, \sim, 0, u)$  is a generalized  $MV$ -algebra; we set  $\mathcal{A} = \Gamma(G, u)$ .

Dvurečenskij [5] proved that for each  $MV$ -algebra  $\mathcal{A}$  there exists a unital lattice ordered group  $(G, u)$  such that the relation

$$(1) \quad \mathcal{A} = \Gamma(G, u)$$

is valid.

Below, when speaking about a generalized  $MV$ -algebra  $\mathcal{A}$  we always assume that (1) holds.

We consider the partial order  $\leq$  on the set  $A$  which is induced by the partial order on  $G$ .

Now we can apply for a generalized  $MV$ -algebra  $\mathcal{A}$  the conditions  $(r_1)$  and  $(r_2)$  from Section 2 with the distinction that we take  $\mathcal{A}$  instead of the Boolean algebra  $\mathfrak{A}$ .

Thus we obviously have

**Lemma 6.1.** *A generalized  $MV$ -algebra  $\mathcal{A}$  satisfies  $(r_1)$  if and only if the interval  $[0, u]$  of  $G$  is regular with respect to  $(r'_1)$ .*

**Lemma 6.2.** *Let  $(G, u)$  be a unital lattice ordered group. The following conditions are equivalent:*

- (i)  $G$  satisfies  $(r'_1)$ ;
- (ii) The interval  $[0, u]$  of  $G$  is regular with respect to  $(r'_1)$ .

**Proof.** The validity of the implication (i)  $\Rightarrow$  (ii) is obvious. Assume that (ii) holds. By way of contradiction, suppose that (i) fails to be valid.

In view of 5.5, there is  $0 < b \in G$  such that, whenever  $b' \in G$ ,  $0 < b' \leq b$ , then the interval  $[0, b']$  is not regular with respect to  $(r'_1)$ . Put  $b' = b \wedge u$ . We have  $0 < b'$ , hence the mentioned assertion for  $b'$  holds. Therefore (ii) is not valid, which is a contradiction. □

**Proposition 6.3.** *Let  $\mathcal{A}$  and  $(G, u)$  be as above. Then the following conditions are equivalent:*

- (i)  $G$  satisfies  $(r'_1)$ ;
- (ii)  $\mathcal{A}$  satisfies  $(r_1)$ .

*Proof.* This is a consequence of 6.1 and 6.2. □

**Proposition 6.4.** *Let  $\mathcal{A}$  be a generalized MV-algebra. Then  $(r_1) \Rightarrow (r_2)$ .*

*Proof.* It suffices to apply the same idea as in the proof of 6.4; we also use the fact that  $u$  is the greatest element of  $\mathcal{A}$ . □

As above, let  $A$  be the underlying set of  $\mathcal{A}$ . For  $a \in A$  we put  $A_a = [0, a]$  and for each  $x_1, x_2 \in A_1$  we set

$$x_1 \oplus_a x_2 = (x_1 + x_2) \wedge a \quad x_1^{-a} = a - x_1, \quad x_1^{\sim a} = -x_1 + a, \quad 1_a = a.$$

Then the algebraic structure  $\mathcal{A}_a = (A_a, \oplus_a, \neg^a, \sim^a, 0, 1_a)$  is a generalized MV-algebra. It will be called an interval subalgebra of  $\mathcal{A}$ .

Let  $a$  be as above and let  $b \in A$  such that  $a \wedge b = 0, a \vee b = 1$ . For each  $x \in A$  we put  $\varphi(x) = (x \wedge a, x \wedge b)$ . Then  $\varphi$  is an isomorphism of  $\mathcal{A}$  onto the direct product  $\mathcal{A}_a \times \mathcal{A}_b$  (cf. [11]). In this situation we will write  $\mathcal{A} = \mathcal{A}_a \times \mathcal{A}_b$ .

For  $\emptyset \neq X \subseteq A$  we put

$$X^{\delta_1} = \{y \in A : y \wedge x = 0 \text{ for each } x \in X\}.$$

The generalized MV-algebra  $\mathcal{A}$  is *projectable* if for each  $x \in A$ , the relation

$$\mathcal{A} = \{x\}^{\delta_1 \delta_1} \times \{x\}^{\delta_1}$$

is valid (meaning that  $\{x\}^{\delta_1 \delta_1}$  has a greatest element  $a$ ,  $\{x\}^{\delta_1}$  has a greatest element  $b$  and  $\mathcal{A} = \mathcal{A}_a \times \mathcal{A}_b$ ).

It is easy to verify that there exists a one-to-one correspondence between polars of  $\mathcal{A}$  and polars of  $G$ . From this and from Theorem 6.4 in [11] we can deduce by a simple argument

**Lemma 6.5.** *Let  $\mathcal{A}$  and  $(G, u)$  be as above. Then  $\mathcal{A}$  is projectable if and only if  $G$  is projectable.*

**Lemma 6.6.** *Let us apply the notation as above. The following conditions are equivalent:*

- (i)  $G$  satisfies  $(r'_2)$ ;
- (ii)  $\mathcal{A}$  satisfies  $(r_2)$ .

*Proof.* The case  $G = \{0\}$  being trivial we assume that  $G \neq \{0\}$ .

a) The validity of the implication (i)  $\Rightarrow$  (ii) is obvious (since  $u = 1$  is a strong unit of  $G$ ).

b) Assume that (ii) is valid. By way of contradiction, suppose that (i) fails to hold. Hence there is an  $\mathbf{m}$ -indexed set  $(a_{t,s})_{t \in T, s \in S}$  of elements of  $G^+$  such that

$\alpha$ ) the element

$$\bigwedge_{t \in T} \bigvee_{s \in S} a_{t,s} = v$$

exists in  $G$ ;

$\beta$ )  $v$  is a weak unit in  $G$ ;

$\gamma$ ) there exists  $0 < a \in G$  such that for each  $\varphi \in S^T$ ,

$$a \wedge \bigwedge_{t \in T'} a_{t,\varphi(t)} = 0 \quad \text{for each finite subset } T' \text{ of } T.$$

We denote by  $G_1$  the convex  $\ell$ -subgroup of  $G$  generated by the element  $v$ . We have  $v > 0$ , thus  $G_1 \neq \{0\}$ . For each  $t \in T$  and  $x \in S$  we put  $a'_{t,s} = a_{t,s} \wedge v$ . Then  $(a'_{t,s})_{t \in T, s \in S}$  is an  $\mathbf{m}$ -indexed set of elements of the interval  $[0, v]$  such that

$$\alpha_1) \bigwedge_{t \in T} \bigvee_{s \in S} a'_{t,s} = v,$$

$\beta_1$ )  $v$  is the greatest element of the generalized  $MV$ -algebra  $\Gamma(G, v) = \mathcal{A}_1$ .

The underlying set  $A_1$  of  $\mathcal{A}_1$  is an interval of the underlying lattice of  $\mathcal{A}$ . From this and from the fact that  $\mathcal{A}$  satisfies  $(r'_2)$  we conclude that  $\mathcal{A}_1$  satisfies  $(r'_2)$  as well.

Let  $0 < a \in A$ . Then  $0 < a \wedge v$  and  $a \wedge v \in G_1$ . Put  $a \wedge v = a_1$ . Since  $G_1$  satisfies  $(r'_2)$ , there exists  $\varphi \in S^T$  such that

$$(1) \quad a_1 \wedge \bigwedge_{t \in T'} a'_{t,\varphi(t)} > 0 \quad \text{for each finite subset } T' \text{ of } T.$$

Let the element  $a$  be as in  $\gamma$ ). We have

$$a \wedge \bigwedge_{t \in T'} a_{t,\varphi(t)} \geq a_1 \wedge \bigwedge_{t \in T'} a'_{t,\varphi(t)} > 0,$$

which contradicts  $\gamma_1$ ). Therefore (i) must hold. □

**Proposition 6.7.** *Let  $\mathcal{A}$  be a generalized MV-algebra. Suppose that  $\mathcal{A}$  is projectable. Then the relation  $(r_1) \Leftrightarrow (r_2)$  holds for  $\mathcal{A}$ .*

**Proof.** In view of 6.5,  $G$  is projectable. Hence according to 4.5 we have  $(r'_1) \Leftrightarrow (r'_2)$ . Further, 6.3 yields  $(r_1) \Leftrightarrow (r'_1)$ . In view of 6.6,  $(r'_2) \Leftrightarrow (r_2)$ . Hence  $(r_1) \Leftrightarrow (r_2)$ .  $\square$

Specker groups were investigated by Conrad and Darnel [3] and by the author [12]. For each Boolean algebra  $B$  there exists a Specker group  $G_0$  such that  $G_0$  is a lattice ordered group having the property that there exists an ideal  $B^1$  of the underlying lattice of  $G_0^+$  with  $B \simeq B^1$ .

We denote by  $u$  the greatest element of  $B^1$ . Further, let  $G$  be the convex  $\ell$ -subgroup of  $G_0$  generated by the element  $u$ . Put  $\mathcal{A} = \Gamma(G, u)$ . Then without loss of generality we can assume that  $B$  is the underlying lattice of  $\mathcal{A}$ .

Each Specker group is projectable and abelian; this yields that  $G$  is projectable as well. The condition  $(r_1)$  for the Boolean algebra  $B$  is, in fact, identical to the condition  $(r_1)$  for the MV-algebra  $\mathcal{A}$ ; the situation for  $(r_2)$  is analogous. Therefore the equivalence  $(r_1) \Leftrightarrow (r_2)$  for Boolean algebras (cf. Section 1) is a consequence of 6.7.

We remark that in the proof of the implication  $(r_2) \Rightarrow (r_1)$  for Boolean algebras in the proof of 19.3 in [20], Stone's theorem and arguments of topological character were applied. In our method, all the proofs are purely algebraic.

## 7. RADICAL CLASSES OF GENERALIZED MV-ALGEBRAS

The collection of all MV-algebras or of all generalized MV-algebras will be denoted by  $\mathcal{M}$  or by  $\mathcal{M}_g$ , respectively.

The notion of radical class of MV-algebras was introduced in [10]. If we modify the definition from [10] in such a way that instead of elements of  $\mathcal{M}$ , the elements of  $\mathcal{M}_g$  are taken into account, then we obtain the definition of a *radical class of generalized MV-algebras*.

For the sake of completeness, we present the definition in detail.

Let  $X$  be a nonempty subclass of  $\mathcal{M}_g$  such that the following conditions are satisfied:

- (i)  $X$  is closed with respect to isomorphisms;
- (ii) if  $\mathcal{A} \in X$ , then each interval subalgebra of  $\mathcal{A}$  belongs to  $X$ ;
- (iii) if  $\mathcal{A} \in \mathcal{M}_g$  and if  $a_1, \dots, a_n \in A$  such that all interval subalgebras  $\mathcal{A}_{a_1}, \dots, \mathcal{A}_{a_n}$  of  $\mathcal{A}$  belong to  $X$ , then the interval subalgebra  $\mathcal{A}_a$  also belongs to  $X$ , where  $a = a_1 \vee \dots \vee a_n$ .

Under these assumptions,  $X$  is called a radical class of generalized  $MV$ -algebras.

We denote by  $\mathcal{R}_\ell$  and  $\mathcal{R}_g$  the collection of all radical classes of lattice ordered groups or the collection of all radical classes of generalized  $MV$ -algebras, respectively.

Let  $X \in \mathcal{R}_\ell$ . We denote by  $\varphi_1(X)$  the class of all generalized  $MV$ -algebras  $\mathcal{A}$  such that there exists a unital lattice ordered group  $(G, u)$  with  $(G, u) \in X$ ,  $\Gamma(G, u) = \mathcal{A}$ .

Further, let  $Y \in \mathcal{R}_g$ . The symbol  $\varphi_2(Y)$  will denote the class of all lattice ordered groups  $G$  such that for each  $0 \leq u \in G$  we have  $\Gamma(G_u, u) \in Y$ , where  $G_u$  is the convex  $\ell$ -subgroup of  $G$  generated by the element  $u$ .

**Lemma 7.1.** *For each  $X \in \mathcal{R}_\ell$  and  $Y \in \mathcal{R}_g$  we have  $\varphi_1(X) \in \mathcal{R}_g$  and  $\varphi_2(Y) \in \mathcal{R}_\ell$ . The mapping  $\varphi_1$  is a bijection of  $\mathcal{R}_\ell$  onto  $\mathcal{R}_g$ ; further,  $\varphi_2$  is a bijection of  $\mathcal{R}_g$  onto  $\mathcal{R}_\ell$  and  $\varphi_2 = \varphi_1^{-1}$ .*

**Proof.** It suffices to apply the same argument as in [10] by investigating the relation between radical classes of abelian lattice ordered groups and radical classes of  $MV$ -algebras. □

Let  $m$  be an infinite cardinal and  $i \in \{1, 2\}$ . We denote by  $D_m^i$  the class of all generalized  $MV$ -algebras which satisfy the condition  $(r_1)$ .

**Proposition 7.2.**  *$D_m^1$  is a radical class of generalized  $MV$ -algebras.*

**Proof.** This is a consequence of 5.6 and 7.1. □

On the other hand,  $D_m^2$  fails to be a radical class. We verify this as follows. Put  $D_m^2 = Y$  and let  $\varphi_2(Y)$  be as above. Further, let  $C_m^2$  be as in Section 5. Then in view of 6.6 we have  $\varphi_2(Y) = C_m^2$ . If  $Y \in \mathcal{R}_g$ , then in view of 7.1 we would have  $C_m^2 \in \mathcal{R}_\ell$ . But in Section 5 we observed that  $C_m^2$  fails to be a radical class; hence we have arrived at a contradiction.

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*Author's address*: J á n J a k u b í k, Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia, e-mail: [kstefan@saske.sk](mailto:kstefan@saske.sk).