Czechoslovak Mathematical Journal

Péter T. Nagy; Karl Strambach

Schreier loops

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 3, 759–786

Persistent URL: http://dml.cz/dmlcz/140420

Czechoslovak Mathematical Journal, 58 (133) (2008), 759–786

SCHREIER LOOPS

Péter T. Nagy, Debrecen, and Karl Strambach, Erlangen

(Received July 13, 2006)

Abstract. We study systematically the natural generalization of Schreier's extension theory to obtain proper loops and show that this construction gives a rich family of examples of loops in all traditional common, important loop classes.

Keywords: extension of loops, non-associative extension of groups, weak associativity properties of extensions, central extensions

MSC 2010: 20N05

1. Introduction

The extension theory of groups developed by Schreier has achieved recently a final form of high aesthetic value (cf. [20], [21], [16], Chapter XII, §48-49, pp. 121–131, [24], Chapter 2, §7, pp. 192–200). Any group which is an extension G of a normal subgroup N by a group K is determined by two identities describing the action of K on N and ensuring the associativity of the extension G by the choice of a system of representatives in G for the factor group G/N isomorphic to K.

In [5], Section II.3 Chein discusses the possibilities how to generalize the extension theory of groups to quasigroups and loops. There he documents that there is huge variety of procedures for obtaining classes of extensions with completely different properties. Till now many authors have used these constructions to realize loops in classes investigated by them. In particular, in the last years constructions of loop extensions became a popular method (cf. e.g. [18], Section 17, [7], Section 5, [14], §2, §3, [15], §4, §5, [12], §7, [6].)

In contrast to this, in this paper we investigate thoroughly and systematically a variation of extensions which yields loops as extensions of groups by loops such that

This research was supported by the DAAD-MÖB project 2005/2006 "Loops in Group Theory and Lie Theory".

these extensions can be described by the same type of identities as in the group case. This class of extensions seems to us to be the most natural generalization of Schreier's theory to loops.

Although our extension theory is structurally very close to the group case we show in the second part of this paper that it gives examples of loops in all traditional common, important loop classes. Moreover, this is true already for the central extensions of abelian groups by abelian groups as well as for Scheerer extensions (cf. [18], Section 2), which are for instance needed in the classification of Bol loops. Only for the subclass of semidirect products the Schreier construction restricts essentially the type of extended loops. Although we can construct Schreier semidirect products of groups which are proper right Bol loops, any such semidirect product which satisfies the left inverse property is a group. Hence to obtain proper loops having the inverse property it is necessary to use more general constructions for semidirect products (cf. [3], [4], [1], [2], [25], [11]).

In general, the group generated by the right translations or by the left translations of a proper loop L is a big subgroup of the permutation group of the underlying set of L (cf. [18], Sections 17, 18, 19 and 29). This is the case for the group generated by the left translations of a proper loop L which is a semidirect product of two groups. But the group generated by the right translations of L is a product $N\Sigma$, where Σ is a subgroup of the direct product of K and the automorphism group of the normal subgroup N of $N\Sigma$.

2. Preliminaries

A set L with a binary operation $(x,y) \mapsto x \cdot y$ is called a *loop* if there exists an element $e \in L$ such that $e \cdot x = x \cdot e = x$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a \setminus b$ and $x = b/a$. The right translations $\varrho_a\colon\ y \mapsto y \cdot a\colon\ L \to L$ as well as the left translations $\lambda_a\colon\ y \mapsto a \cdot y\colon\ L \to L$ are bijections for any $a \in L$.

The kernel of a homomorphism $\alpha\colon\ L \to L'$ of a loop L into a loop L' is a *normal subloop* N of L, i.e. a subloop of L such that

$$x \cdot N = N \cdot x, \quad (x \cdot N) \cdot y = x \cdot (N \cdot y) \quad \text{and} \quad x \cdot (y \cdot N) = (x \cdot y) \cdot N$$

holds for all $x, y \in L$.

The *left, right* and *middle nuclei* of L are respectively the subgroups of L which are defined in the following way:

$$N_l = \{u;\ (u \cdot x) \cdot y = u \cdot (x \cdot y),\ x, y \in L\},$$

$$N_r = \{u;\ (x \cdot y) \cdot u = x \cdot (y \cdot u),\ x, y \in L\},$$
$$N_m = \{u;\ (x \cdot u) \cdot y = x \cdot (u \cdot y),\ x, y \in L\}.$$

The intersection $N = N_l \cap N_r \cap N_m$ is called the *nucleus* of L.

The *centre* Z of a loop L is the largest subgroup of the nucleus N of L such that $z \cdot x = x \cdot z$ for all $x \in L, z \in Z$. Any subgroup of Z is a normal subgroup of L, called a *central subgroup* of L.

A loop L possesses the *left inverse property* or the *right inverse property* if there exists a bijection $\iota\colon\ x \mapsto x^{-1}\colon\ L \to L$ such that $x^{-1} \cdot (x \cdot y) = y$ or $(y \cdot x) \cdot x^{-1} = y$ respectively holds for all $x, y \in L$. If the inverse mapping $\iota\colon\ x \mapsto x^{-1}\colon\ L \to L$ is an automorphism of a loop L with the left or right inverse property then L has the *automorphic inverse property*.

A loop L is *left alternative* or *right alternative* if respectively, $x \cdot (x \cdot y) = x^2 \cdot y$ or $(y \cdot x) \cdot x = y \cdot x^2$ for all $x, y \in L$. A loop L is *flexible* if $x \cdot (y \cdot x) = (x \cdot y) \cdot x$ for all $x, y \in L$.

A loop L is a *left* or *right Bol loop* if it satisfies the identity

$$(x \cdot (y \cdot x)) \cdot z = x \cdot (y \cdot (x \cdot z)) \quad \text{or} \quad z \cdot (x \cdot (y \cdot x)) = ((z \cdot x) \cdot y) \cdot x, \ \text{respectively.}$$

A left and right Bol loop is called a *Moufang loop.*

A left Bol loop satisfying one of the following properties is a Moufang loop:

(a) the right inverse property,
(b) the right alternative law,
(c) the flexible law,
(d) the identity $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.

For a proof see e.g. [19], IV.6.9. Theorem.

3. Schreier extensions

Let N be a group, let K be a loop, let $T\colon\ K \to \mathrm{Aut}(N)$ be a function of K into the automorphism group of N with $T(1) = \mathrm{Id}$ and let $f\colon\ K \times K \to N$ be a function with the property $f(1, \tau) = f(\tau, 1) = 1$. A straightforward computation yields

Proposition 3.1. *The multiplication*

$$(\tau, t) \circ (\sigma, s) = (\tau\sigma, f(\tau, \sigma) t^{T(\sigma)} s)$$

on $K \times N$ defines a loop $L(T, f)$ such that

$$(\varrho, r)/(\sigma, s) = (\varrho/\sigma, (f(\varrho/\sigma, \sigma)^{-1} r s^{-1})^{T(\sigma)^{-1}}),$$
$$(\sigma, s) \setminus (\varrho, r) = (\sigma \setminus \varrho, s^{-T(\sigma \setminus \varrho)} f(\sigma, \sigma^{-1}\varrho)^{-1} r).$$

The loop $L(T,f)$ contains $\overline{N} = \{(1,n);\ n \in N\}$ as a normal subgroup and the factor loop is isomorphic to K.

We define the following functions:

$$\Psi\colon\ (\sigma,\tau) \mapsto T(\tau)^{-1}T(\sigma)^{-1}T(\sigma\tau)\iota_{f(\sigma,\tau)}\colon\ K \times K \to \mathrm{Aut}(N),$$

where ι denotes the inner automorphism $\iota_g\colon\ x \mapsto g^{-1}xg\colon\ N \to N$, and

$$\psi\colon\ (\sigma,\tau,\varrho) \mapsto f(\sigma,\tau\varrho)^{-1}f(\sigma\tau,\varrho)f(\sigma,\tau)^{T(\varrho)}f(\tau,\varrho)^{-1}\colon\ K \times K \times K \to N.$$

It is well known (cf. [16], §48) that the loop extension $L(T,f)$ is a group if and only if K is a group and the following two identities hold:

(i) $\Psi(\sigma,\tau) = \mathrm{Id}$,

(ii) $\psi(\sigma,\tau,\varrho) = 1$.

We call a loop $L(T,f)$ a *central extension* of the group N by the loop K if $\overline{N}$ is a central subgroup of $L(T,f)$. The loop $L(T,f)$ is a central extension of N if and only if N is abelian and $T(\sigma) = \mathrm{Id}$ for all $\sigma \in K$. The extension $L(T,f)$ is an abelian group if and only if N and K are abelian groups, $T(\sigma) = \mathrm{Id}$ for all $\sigma \in K$ and $f(\tau,\sigma) = f(\sigma,\tau)$ for all $\sigma,\tau \in K$.

A straightforward computation yields for the loop extension $L(T,f)$ the following

Proposition 3.2.

(i) *The subgroup $\overline{N}$ of the loop $L(T,f)$ is contained in the middle and right nucleus. $\overline{N}$ is contained in the nucleus if and only if $\Psi(\sigma,\tau) = \mathrm{Id}$ for all $\sigma,\tau \in K$.*

(ii) *The loop $L(T,f)$ contains the group $\overline{N}$ in its centre if and only if N is abelian and $T(\sigma) = \mathrm{Id}$ for all $\sigma \in K$.*

(iii) *The loop $L(T,f)$ is abelian if and only if K and N are commutative, $T(\sigma) = \mathrm{Id}$ and $f(\tau,\sigma) = f(\sigma,\tau)$ for all $\tau,\sigma \in K$.*

Proposition 3.3. *In the loop $L(T,f)$ the left inverse of any element coincides with its right inverse, i.e. $x^{-1} = x \setminus 1 = 1/x$, if and only if it is the case in K and the following two identities hold:*

$$\Psi(\sigma,\sigma^{-1}) = \mathrm{Id}, \quad \psi(\sigma,\sigma^{-1},\sigma) = 1.$$

P r o o f. We may assume that the left and right inverses of elements in K coincide. Then the left and right inverses of any element $(\sigma,s) \in L(T,f)$ coincide if and only if the following identity holds:

$$\big(f(\sigma^{-1},\sigma)^{-1}s^{-1}\big)^{T(\sigma)^{-1}} = s^{-T(\sigma^{-1})}f(\sigma,\sigma^{-1})^{-1}.$$

Putting $s = 1$ we obtain

$$f(\sigma^{-1}, \sigma) = f(\sigma, \sigma^{-1})^{T(\sigma)},$$

which gives the second relation in the proposition. Hence we can write

$$(f(\sigma, \sigma^{-1})^{-T(\sigma)} s^{-1})^{T(\sigma)^{-1}} = s^{-T(\sigma^{-1})} f(\sigma, \sigma^{-1})^{-1}$$

or equivalently

$$s^{T(\sigma)^{-1}} f(\sigma, \sigma^{-1}) = f(\sigma, \sigma^{-1}) s^{T(\sigma)^{-1} T(\sigma) T(\sigma^{-1})}.$$

Using the inner automorphism $\iota_{f(\sigma,\sigma^{-1})}$ we have

$$\iota_{f(\sigma,\sigma^{-1})} = T(\sigma)T(\sigma^{-1}), \tag{1}$$

which is equivalent to the first relation in the proposition. □

The conditions of this proposition are in particular satisfied if $T(\sigma)^{-1} = T(\sigma^{-1})$ and if $f(\sigma, \sigma^{-1})$ is contained in the centre of N for all $\sigma \in K$.

Proposition 3.4. *Let $L(T, f)$ be a loop in which for any element the left and right inverses coincide. The loop $L(T, f)$ satisfies the identity*

$$(1, 1) = [(\tau, t) \circ (\sigma, s)] \circ [(\sigma, s)^{-1} \circ ((\tau, t)^{-1}]$$

if and only if the following identities hold:
(i) $1 = (\tau \cdot \sigma) \cdot (\sigma^{-1} \cdot \tau^{-1})$,
(ii) $\Psi(\sigma, \tau) = \mathrm{Id}$,
(iii) $f(\tau \cdot \sigma, \sigma^{-1} \cdot \tau^{-1}) = f(\tau, \tau^{-1}) f(\sigma, \sigma^{-1})^{T(\tau^{-1})} f(\tau, \sigma)^{-T(\sigma^{-1})T(\tau^{-1})} f(\sigma^{-1}, \tau^{-1})^{-1}$.

Proof. The claim is true if and only if

$$\begin{aligned}(1, 1) &= [(\tau, t) \circ (\sigma, s)] \circ [((\sigma, s) \setminus (1, 1)) \circ ((\tau, t) \setminus (1, 1))] \\ &= ((\tau \cdot \sigma) \cdot (\sigma^{-1} \cdot \tau^{-1}), f(\tau \cdot \sigma, \sigma^{-1} \cdot \tau^{-1}) [f(\tau, \sigma) t^{T(\sigma)} s]^{T(\sigma^{-1}\tau^{-1})} f(\sigma^{-1}, \tau^{-1}) \\ &\quad \cdot [s^{-T(\sigma^{-1})} f(\sigma, \sigma^{-1})^{-1}]^{T(\tau^{-1})} t^{-T(\tau^{-1})} f(\tau, \tau^{-1})^{-1}).\end{aligned}$$

We may assume that the condition (i) is satisfied. Then the assertion of the proposition holds if and only if

$$\begin{aligned}& s^{T(\sigma^{-1} \cdot \tau^{-1})} f(\sigma^{-1}, \tau^{-1}) s^{-T(\sigma^{-1})T(\tau^{-1})} \\ &\quad = t^{-T(\sigma)T(\sigma^{-1} \cdot \tau^{-1})} f(\tau, \sigma)^{-T(\sigma^{-1} \cdot \tau^{-1})} f(\tau \cdot \sigma, \sigma^{-1} \cdot \tau^{-1})^{-1} \\ &\quad\quad \times f(\tau, \tau^{-1}) t^{T(\tau^{-1})} f(\sigma, \sigma^{-1})^{T(\tau^{-1})}.\end{aligned} \tag{2}$$

Since the left hand side of (2) is independent of t and the right hand side of (2) is independent of s we obtain the condition (ii).

Using the identities $T(\sigma^{-1}\cdot\tau^{-1}) = T(\sigma^{-1})T(\tau^{-1})\iota_{f(\sigma^{-1},\tau^{-1})^{-1}}$ and $T(\sigma)T(\sigma^{-1}) = \iota_{f(\sigma,\sigma^{-1})}$ we obtain from (2) an equivalent identity

$$1 = f(\sigma,\sigma^{-1})^{T(\tau^{-1})} f(\tau,\sigma)^{-T(\sigma^{-1})T(\tau^{-1})} f(\sigma^{-1},\tau^{-1})^{-1} f(\tau\cdot\sigma,\sigma^{-1}\cdot\tau^{-1})^{-1} f(\tau,\tau^{-1}),$$

which proves the assertion. □

Proposition 3.5. *Let $L(T,f)$ be a loop in which for any element the left and right inverses coincide. $L(T,f)$ satisfies the automorphic inverse property if and only if the loop K has the automorphic inverse property, the group N is commutative, $T(\sigma) = \mathrm{Id}$ for any $\sigma \in K$ and the following identity holds:*

$$f(\tau\cdot\sigma,\tau^{-1}\sigma^{-1}) = f(\sigma,\sigma^{-1}) f(\tau,\tau^{-1}) f(\tau^{-1},\sigma^{-1})^{-1} f(\tau,\sigma)^{-1}.$$

P r o o f. The loop $L(T,f)$ has the automorphic inverse property if and only if

$$\begin{aligned}(1,1) &= [(\tau,t)\circ(\sigma,s)]\circ[((\tau,t)\setminus(1,1))\circ((\sigma,s)\setminus(1,1))]\\ &= ((\tau\cdot\sigma)\cdot(\tau^{-1}\cdot\sigma^{-1}), f(\tau\cdot\sigma,\tau^{-1}\cdot\sigma^{-1})[f(\tau,\sigma)t^{T(\sigma)}s]^{T(\tau^{-1}\cdot\sigma^{-1})}\\ &\qquad\times f(\tau^{-1},\sigma^{-1})[t^{-T(\tau^{-1})}f(\tau,\tau^{-1})^{-1}]^{T(\sigma^{-1})}s^{-T(\sigma^{-1})}f(\sigma,\sigma^{-1})^{-1}).\end{aligned}$$

This identity is valid if and only if the loop K has the automorphic inverse property, the group N is commutative and the following identity holds:

$$\begin{aligned}(3)\qquad & f(\tau\cdot\sigma,\tau^{-1}\cdot\sigma^{-1})f(\tau,\sigma)^{T(\tau^{-1}\cdot\sigma^{-1})}t^{T(\sigma)T(\tau^{-1}\sigma^{-1})}s^{T(\tau^{-1}\cdot\sigma^{-1})}f(\tau^{-1},\sigma^{-1})\\ &= f(\sigma,\sigma^{-1})s^{T(\sigma^{-1})}f(\tau,\tau^{-1})^{T(\sigma^{-1})}t^{T(\tau^{-1})T(\sigma^{-1})}.\end{aligned}$$

Putting here $\sigma = 1$ we obtain $s^{T(\sigma^{-1})} = s$ for all $\tau \in K$, $s \in N$, and hence $T(\sigma) = \mathrm{Id}$ for all $\sigma \in K$. It follows that $L(T,f)$ has the automorphic inverse property if and only if the identity in the assertion is satisfied. □

Proposition 3.6. *The loop $L(T,f)$ has the right inverse property if and only if K has the right inverse property and the following identities hold:*

$$\Psi(\sigma,\sigma^{-1}) = \mathrm{Id},\ \ \psi(\tau,\sigma,\sigma^{-1}) = 1.$$

P r o o f. $L(T,f)$ has the right inverse property if and only if

$$\begin{aligned}((\tau,t)\circ(\sigma,s))\circ((\sigma,s)\setminus(1,1)) &= (\tau,t)\\ &= ((\tau\cdot\sigma)\cdot\sigma^{-1}, f(\tau\cdot\sigma,\sigma^{-1})[f(\tau,\sigma)t^{T(\sigma)}s]^{T(\sigma^{-1})}s^{-T(\sigma^{-1})}f(\sigma,\sigma^{-1})^{-1})\end{aligned}$$

holds for any $t, s \in N$ and $\sigma, \tau \in K$. This identity is equivalent to the right inverse property in K and to the identity

$$tf(\sigma, \sigma^{-1}) = f(\tau \cdot \sigma, \sigma^{-1}) f(\tau, \sigma)^{T(\sigma^{-1})} t^{T(\sigma) T(\sigma^{-1})}.$$

If the loop $L(T, f)$ has the right inverse property then the left and right inverses of any element coincide and it follows from Proposition 3.3 that $\Psi(\sigma, \sigma^{-1}) = \text{Id}$ for all $\sigma \in K$ and the identity (1) hold. Using this we obtain from the previous identity $f(\sigma, \sigma^{-1}) = f(\tau\sigma, \sigma^{-1}) f(\tau, \sigma)^{T(\sigma^{-1})}$, which shows that the identities in the claim are valid.

The same arguments yield that the conditions in the proposition imply the right inverse property in $L(T, f)$. □

If K satisfies the right inverse property, $f(\sigma, \tau) = 1$ for all $\sigma, \tau \in K$ and $\Psi(\sigma, \tau) \neq \text{Id}$ for some $\sigma, \tau \in K$, then the proper loop $L(T, 1)$ has the right inverse property if and only if for any element the left and right inverses coincide.

Proposition 3.7. *The loop $L(T, f)$ has the left inverse property if and only if the loop K satisfies the left inverse property and the following identities hold:*

$$\Psi(\tau, \sigma) = \text{Id}, \quad \psi(\tau, \tau^{-1}, \sigma) = 1.$$

Proof. The loop $L(T, f)$ has the left inverse property if and only if

$$\begin{aligned}(\sigma, s) &= [(1, 1)/(\tau, t)] \circ [(\tau, t) \circ (\sigma, s)] \\ &= (\tau^{-1}, [tf(\tau^{-1}, \tau)]^{-T(\tau)^{-1}}) \circ (\tau \cdot \sigma, f(\tau, \sigma) t^{T(\sigma)} s) \\ &= (\tau^{-1} \cdot (\tau \cdot \sigma), f(\tau^{-1}, \tau \cdot \sigma)[tf(\tau^{-1}, \tau)]^{-T(\tau)^{-1} T(\tau \cdot \sigma)} f(\tau, \sigma) t^{T(\sigma)} s)\end{aligned}$$

for all $\tau, \sigma \in K$ and $t, s \in N$. This is equivalent to the left inverse property of K and to the identity

$$f(\tau, \sigma) t^{T(\sigma)} f(\tau^{-1}, \tau \cdot \sigma) = [tf(\tau^{-1}, \tau)]^{T(\tau)^{-1} T(\tau \cdot \sigma)}. \tag{4}$$

For $t = 1$ we obtain from this

$$f(\tau, \sigma) f(\tau^{-1}, \tau \cdot \sigma) = f(\tau^{-1}, \tau)^{T(\tau)^{-1} T(\tau \cdot \sigma)}.$$

Since in a loop with the left inverses property the left and right inverse of any element coincide, Proposition 3.3 yields the identity $\psi(\tau, \tau^{-1}, \tau) = 1$ or

$f(\tau^{-1},\tau) = f(\tau,\tau^{-1})^{T(\tau)}$. Hence it follows $f(\tau,\sigma)f(\tau^{-1},\tau\cdot\sigma) = f(\tau,\tau^{-1})^{T(\tau\cdot\sigma)}$. This identity may be written in the form

$$\psi(\tau,\tau^{-1},\tau\cdot\sigma) = f(\tau,\sigma)^{-1}f(\tau,\tau^{-1})^{T(\tau\cdot\sigma)}f(\tau^{-1},\tau\cdot\sigma)^{-1} = 1.$$

Replacing in this identity $\tau\sigma$ by σ we obtain the second relation in the proposition. Using the relations $f(\tau^{-1},\tau)^{-T(\tau)} = f(\tau,\tau^{-1})$ and $f(\tau,\sigma)f(\tau^{-1},\tau\cdot\sigma) = f(\tau,\tau^{-1})^{T(\tau\cdot\sigma)}$ the identity (4) assumes the form $f(\tau,\sigma)t^{T(\sigma)} = t^{T(\tau)^{-1}T(\tau\sigma)}f(\tau,\sigma)$. This yields $T(\tau)T(\sigma) = T(\tau\sigma)\iota_{f(\tau,\sigma)}$, which gives the condition $\Psi(\tau,\sigma) = \mathrm{Id}$ for all $\tau,\sigma\in K$. The same arguments show that the conditions of the proposition imply that $L(T,f)$ has the left inverse property. □

Proposition 3.8. *A loop $L(T,f)$ is left alternative if and only if the loop K is left alternative and the following identities hold:*

$$\Psi(\tau,\sigma) = \mathrm{Id},\quad \psi(\tau,\tau,\sigma) = 1.$$

P r o o f. The loop $L(T,f)$ is left alternative if and only if the identity

$$(\tau\cdot(\tau\cdot\sigma), f(\tau,\tau\cdot\sigma)t^{T(\tau\cdot\sigma)}f(\tau,\sigma)t^{T(\sigma)}s) = (\tau^2\cdot\sigma, f(\tau^2,\sigma)[f(\tau,\tau)t^{T(\tau)}t]^{T(\sigma)}s)$$

holds. This is equivalent to the left alternative law in K and to the identity

$$f(\tau,\tau\cdot\sigma)t^{T(\tau\cdot\sigma)}f(\tau,\sigma) = f(\tau^2,\sigma)f(\tau,\tau)^{T(\sigma)}t^{T(\tau)T(\sigma)}. \tag{5}$$

For $t = 1$ we obtain

$$\psi(\tau,\tau,\sigma) = f(\tau,\tau\cdot\sigma)^{-1}f(\tau^2,\sigma)f(\tau,\tau)^{T(\sigma)}f(\tau,\sigma)^{-1} = 1,$$

which is the second identity in the assertion. Using this condition we obtain from (5) an equivalent identity

$$f(\tau^2,\sigma)f(\tau,\tau)^{T(\sigma)}f(\tau,\sigma)^{-1}t^{T(\tau\cdot\sigma)}f(\tau,\sigma) = f(\tau^2,\sigma)f(\tau,\tau)^{T(\sigma)}t^{T(\tau)T(\sigma)}$$

or

$$\iota_{f(\tau,\sigma)} = T(\tau\cdot\sigma)^{-1}T(\tau)T(\sigma),$$

which yields the first identity in the assertion. □

Proposition 3.9. *The loop $L(T,f)$ is right alternative if and only if K is right alternative and the following identities hold:*

$$\Psi(\sigma,\sigma)=\mathrm{Id},\quad \psi(\tau,\sigma,\sigma)=1.$$

P r o o f. The loop $L(T,f)$ is right alternative if and only if

$$(\tau\cdot\sigma^2, f(\tau,\sigma^2)t^{T(\sigma^2)}f(\sigma,\sigma)s^{T(\sigma)}s)=((\tau\cdot\sigma)\cdot\sigma, f(\tau\cdot\sigma,\sigma)[f(\tau,\sigma)t^{T(\sigma)}s]^{T(\sigma)}s)$$

holds for all $t,s\in N$, $\tau,\sigma\in K$. This is equivalent to the right alternative law in K and to the identity

$$f(\tau,\sigma^2)t^{T(\sigma^2)}f(\sigma,\sigma)=f(\tau\cdot\sigma,\sigma)f(\tau,\sigma)^{T(\sigma)}t^{T(\sigma)^2}. \tag{6}$$

Putting $t=1$ we obtain

$$\psi(\tau,\sigma,\sigma)=f(\tau,\sigma^2)^{-1}f(\tau\cdot\sigma,\sigma)f(\tau,\sigma)^{T(\sigma)}f(\sigma,\sigma)^{-1}=1,$$

which is the second identity in the assertion. Using this condition (6) yields an equivalent identity

$$f(\tau,\sigma^2)t^{T(\sigma^2)}f(\sigma,\sigma)=f(\tau,\sigma^2)f(\sigma,\sigma)t^{T(\sigma)^2}\quad\text{or}\quad \iota_{f(\sigma,\sigma)}=T(\sigma^2)^{-1}T(\sigma)^2,$$

which proves the assertion. □

Proposition 3.10. *The loop $L(T,f)$ is flexible if and only if K is flexible and the following identities hold:*

$$\Psi(\tau,\sigma)=\mathrm{Id},\quad \psi(\tau,\sigma,\tau)=1.$$

P r o o f. The loop $L(T,f)$ is flexible if and only if

$$(\tau\cdot(\sigma\cdot\tau), f(\tau,\sigma\cdot\tau)t^{T(\sigma\cdot\tau)}f(\sigma,\tau)s^{T(\tau)}t=(\tau\cdot(\sigma\cdot\tau), f(\tau\cdot\sigma,\tau)[f(\tau,\sigma)t^{T(\sigma)}s]^{T(\tau)}t)$$

is satisfied for all $t,s\in N$, $\tau,\sigma\in K$. This is equivalent to the flexible law in K and to the identity

$$f(\tau,\sigma\cdot\tau)t^{T(\sigma\cdot\tau)}f(\sigma,\tau)=f(\tau\cdot\sigma,\tau)f(\tau,\sigma)^{T(\tau)}t^{T(\sigma)T(\tau)}. \tag{7}$$

Using the identity $f(\tau,\sigma\cdot\tau)\psi(\tau,\sigma,\tau)f(\sigma,\tau)=f(\tau\sigma,\tau)f(\tau,\sigma)^{T(\tau)}$ we obtain from (7) the relation $t^{T(\sigma\cdot\tau)}f(\sigma,\tau)=\psi(\tau,\sigma,\tau)f(\sigma,\tau)t^{T(\sigma)T(\tau)}$. The value $t=1$ gives $\psi(\tau,\sigma,\tau)=1$. Hence the last identity reduces to $t^{T(\sigma\cdot\tau)}f(\sigma,\tau)=f(\sigma,\tau)t^{T(\sigma)T(\tau)}$, which is equivalent to the first identity. □

Proposition 3.11. *The loop $L(T,f)$ is a left Bol loop if and only if K is a left Bol loop and the following identities hold:*

$$\Psi(\sigma,\tau)=\mathrm{Id},\ \ \psi(\tau,\sigma\cdot\tau,\varrho)f(\sigma,\tau\cdot\varrho)\psi(\sigma,\tau,\varrho)f(\sigma,\tau\cdot\varrho)^{-1}=1.$$

In particular, if K is a group then $\psi(\sigma,\tau,\varrho)$ is contained in the centre of N for all $\sigma,\tau,\varrho\in K$ and $\psi(\tau,\sigma\cdot\tau,\varrho)\psi(\sigma,\tau,\varrho)=1$.

Proof. The loop $L(T,f)$ is a left Bol loop if and only if

$$\begin{aligned}&(\tau\cdot\{\sigma\cdot[\tau\cdot\varrho]\},f(\tau,\sigma\cdot(\tau\cdot\varrho))t^{T(\sigma\cdot(\tau\cdot\varrho))}f(\sigma,\tau\cdot\varrho)s^{T(\tau\cdot\varrho)}f(\tau,\varrho)t^{T(\varrho)}r)\\&\quad=(\{\tau\cdot[\sigma\cdot\tau]\}\cdot\varrho,f(\tau\cdot(\sigma\cdot\tau),\varrho)[f(\tau,\sigma\cdot\tau)t^{T(\sigma\cdot\tau)}f(\sigma,\tau)s^{T(\tau)}t]^{T(\varrho)}r)\end{aligned}$$

is satisfied for all $\sigma,\tau,\varrho\in K$ and $s,t\in N$. This is equivalent to the left Bol property of K and to the identity

$$\tag{8}\begin{aligned}&f(\tau,\sigma\cdot(\tau\cdot\varrho))t^{T(\sigma\cdot(\tau\cdot\varrho))}f(\sigma,\tau\cdot\varrho)s^{T(\tau\cdot\varrho)}f(\tau,\varrho)\\&\quad=f(\tau\cdot(\sigma\cdot\tau),\varrho)f(\tau,\sigma\cdot\tau)^{T(\varrho)}t^{T(\sigma\cdot\tau)T(\varrho)}f(\sigma,\tau)^{T(\varrho)}s^{T(\tau)T(\varrho)}.\end{aligned}$$

A left Bol loop $L(T,f)$ has the left inverse property, hence it follows from Proposition 3.7 that $\Psi(\tau,\varrho)=\mathrm{Id}$, which is the first identity in the claim. Then we have $T(\tau)T(\varrho)=T(\tau\cdot\varrho)\iota_{f(\tau,\varrho)}$ and $T(\sigma\cdot\tau)T(\varrho)=T((\sigma\cdot\tau)\cdot\varrho)\iota_{f(\sigma\cdot\tau,\varrho)}$. Putting these relations into (8) we obtain the identity

$$\begin{aligned}&f(\tau,\sigma\cdot(\tau\cdot\varrho))t^{T(\sigma\cdot(\tau\cdot\varrho))}f(\sigma,\tau\cdot\varrho)f(\tau,\varrho)\\&\quad=f(\tau\cdot(\sigma\cdot\tau),\varrho)f(\tau,\sigma\cdot\tau)^{T(\varrho)}f(\sigma\cdot\tau,\varrho)^{-1}t^{T((\sigma\cdot\tau)\cdot\varrho)}f(\sigma\cdot\tau,\varrho)f(\sigma,\tau)^{T(\varrho)}.\end{aligned}$$

Using again the relation $\Psi=\mathrm{Id}$ we have

$$T(\sigma\cdot(\tau\cdot\varrho))=T(\sigma)T(\tau)T(\varrho))\iota^{-1}_{f(\tau,\varrho)}\iota^{-1}_{f(\sigma,\tau\cdot\varrho)}$$

and

$$T((\sigma\cdot\tau)\cdot\varrho)=T(\sigma)T(\tau)\iota^{-1}_{f(\sigma,\tau)}T(\varrho)\iota^{-1}_{f(\sigma\cdot\tau,\varrho)}.$$

Putting these relations into the previous identity we obtain

$$f(\tau,\sigma\cdot(\tau\cdot\varrho)f(\sigma,\tau\cdot\varrho)f(\tau,\varrho)=f(\tau\cdot(\sigma\cdot\tau),\varrho)f(\tau,\sigma\cdot\tau)^{T(\varrho)}f(\sigma,\tau)^{T(\varrho)}.$$

Since $\psi(\sigma,\tau,\varrho)=f(\sigma,\tau\cdot\varrho)^{-1}f(\sigma\cdot\tau,\varrho)f(\sigma,\tau)^{T(\varrho)}f(\tau,\varrho)^{-1}$ and $\psi(\tau,\sigma\cdot\tau,\varrho)=f(\tau,(\sigma\cdot\tau)\cdot\varrho)^{-1}f(\tau\cdot(\sigma\cdot\tau),\varrho)f(\tau,\sigma\cdot\tau)^{T(\varrho)}f(\sigma\cdot\tau,\varrho)^{-1}$ we get

$$\psi(\tau,\sigma\cdot\tau,\varrho)f(\sigma,\tau\cdot\varrho)\psi(\sigma,\tau,\varrho)f(\sigma,\tau\cdot\varrho)^{-1}=1,$$

which is the second identity in the claim.

Conversely, the conditions of the proposition imply that $L(T,f)$ is a left Bol loop. If K is a group, then it follows from $T(\sigma(\tau\varrho)) = T((\sigma\tau)\varrho)$ that

$$T(\varrho)\iota_{f(\tau,\varrho)^{-1}}\iota_{f(\sigma,\tau\varrho)^{-1}} = \iota_{f(\sigma,\tau)^{-1}}T(\varrho)\iota_{f(\sigma\tau,\varrho)^{-1}}$$

or

$$\begin{aligned} t^{T(\varrho)}f(\tau,\varrho)^{-1}f(\sigma,\tau\varrho)^{-1}f(\sigma\tau,\varrho)f(\sigma,\tau)^{T(\varrho)}t^{-T(\varrho)} \\ = f(\tau,\varrho)^{-1}f(\sigma,\tau\varrho)^{-1}f(\sigma\tau,\varrho)f(\sigma,\tau)^{T(\varrho)} \end{aligned}$$

for all $t \in N$. This means that $f(\tau,\varrho)^{-1}\psi(\sigma,\tau,\varrho)f(\tau,\varrho)$ is contained in the centre of N, which implies that also $\psi(\sigma,\tau,\varrho)$ is contained in the centre of N for all $\sigma,\tau,\varrho \in K$. □

Corollary 3.12. *Let $L(T,f)$ be a left Bol loop. If K is a group and the centre of N is trivial then $L(T,f)$ is a group.*

Corollary 3.13. *A loop $L(T,f)$ is a Moufang loop if and only if K is Moufang,*

$$\Psi(\sigma,\tau) = \mathrm{Id}, \quad \psi(\tau,\sigma\cdot\tau,\varrho)f(\sigma,\tau\cdot\varrho)\psi(\sigma,\tau,\varrho)f(\sigma,\tau\cdot\varrho)^{-1} = 1$$

for all $\sigma,\tau,\varrho \in K$ and one of the following identities holds:

(i) $\psi(\tau,\sigma,\sigma^{-1}) = 1$,

(ii) $\psi(\tau,\sigma,\sigma) = 1$,

(iii) $\psi(\tau,\sigma,\tau) = 1$,

(iv) $f(\tau\cdot\sigma,\sigma^{-1}\cdot\tau^{-1}) = f(\tau,\tau^{-1})f(\sigma,\sigma^{-1})^{T(\tau^{-1})}f(\tau,\sigma)^{-T(\sigma^{-1})T(\tau^{-1})}f(\sigma^{-1},\tau^{-1})^{-1}$.
In particular, if K is a group then $\psi(\sigma,\tau,\varrho)$ is contained in the centre of N for all $\sigma,\tau,\varrho \in K$ and $\psi(\tau,\sigma\cdot\tau,\varrho)\psi(\sigma,\tau,\varrho) = 1$.

P r o o f. Condition (i) gives the right inverse property (cf. Proposition 3.6), condition (ii) the right alternative law (cf. Proposition 3.9) and condition (iii) the flexible law (cf. Proposition 3.10). Finally, with condition (iv) we have $(x\cdot y)^{-1} = y^{-1}\cdot x^{-1}$ in $L(T,f)$ (cf. Proposition 3.4). □

Proposition 3.14. *The loop $L(T,f)$ is a right Bol loop if and only if K is a right Bol loop and the following identities hold:*

(a) $T(\tau)T(\sigma)T(\tau) = T((\tau\cdot\sigma)\cdot\tau)\iota_{f(\tau\cdot\sigma,\tau)}\iota_{f(\tau,\sigma)^{T(\tau)}}$,

(b) $f((\varrho\cdot\tau)\cdot\sigma,\tau)f(\varrho\cdot\tau,\sigma)^{T(\tau)}f(\varrho,\tau)^{T(\sigma)T(\tau)} = f(\varrho,(\tau\cdot\sigma)\cdot\tau)f(\tau\cdot\sigma,\tau)f(\tau,\sigma)^{T(\tau)}$.

P r o o f. The loop $L(T,f)$ is a right Bol loop if and only if

$$\begin{aligned} &([(\varrho\cdot\tau)\cdot\sigma]\cdot\tau, f((\varrho\cdot\tau)\cdot\sigma,\tau)(f(\varrho\cdot\tau,\sigma)[f(\varrho,\tau)r^{T(\tau)}t]^{T(\sigma)}s)^{T(\tau)}t) \\ &\quad = ([\varrho\cdot[(\tau\cdot\sigma)\cdot\tau], f(\varrho,(\tau\cdot\sigma)\cdot\tau)r^{T((\tau\cdot\sigma)\cdot\tau)}f(\tau\cdot\sigma,\tau)[f(\tau,\sigma)t^{T(\sigma)}s]^{T(\tau)}t) \end{aligned}$$

is satisfied for all $\sigma, \tau, \varrho \in K$ and $r, s, t \in N$. This is equivalent to the right Bol property of K and to the identity

$$f((\varrho \cdot \tau) \cdot \sigma, \tau) f(\varrho \cdot \tau, \sigma)^{T(\tau)} f(\varrho, \tau)^{T(\sigma)T(\tau)} r^{T(\tau)T(\sigma)T(\tau)} = f(\varrho, (\tau \cdot \sigma) \cdot \tau) r^{T((\tau \cdot \sigma) \cdot \tau)} f(\tau \cdot \sigma, \tau) f(\tau, \sigma)^{T(\tau)}. \tag{9}$$

Setting in this identity $\varrho = 1$ we obtain

$$r^{T(\tau)T(\sigma)T(\tau)} = f(\tau, \sigma)^{-T(\tau)} f(\tau \cdot \sigma, \tau)^{-1} r^{T((\tau \cdot \sigma) \cdot \tau)} f(\tau \cdot \sigma, \tau) f(\tau, \sigma)^{T(\tau)}, \tag{10}$$

which gives the claim (a) of the proposition. Using this we obtain from (9) the assertion (b) of the proposition.

Conversely, multiplying the left and right sides of the identities (10) and (b) of the proposition we obtain the identity (9). □

4. Examples

4.1 Central extensions. A loop $L(T, f)$ is the central extension of the abelian group N only if $T(\sigma) = \mathrm{Id}$. Throughout this section we consider the special case that also K is an abelian group. These loops are centrally nilpotent of nilpotency class two, which means that the factor loop L/N is abelian. Some examples of such loops are given in [5] p. 36, [18], Remark 17.6, [7], Section 5, [14], Lemma 3.1, [15], Theorem 4.1, [12], Theorem 7.6, [6], Theorem 2.3.

Let N and K be abelian groups and let $T(\sigma) = \mathrm{Id}$ for all $\sigma \in K$. Then the multiplication in the loop $L(\mathrm{Id}, f)$ is given by

$$(\tau, t) \circ (\sigma, s) = (\tau + \sigma, f(\tau, \sigma) + t + s).$$

In this case we have $\Psi(\sigma, \tau) = \mathrm{Id}$ and

$$\psi(\sigma, \tau, \varrho) = -f(\sigma, \tau + \varrho) + f(\sigma + \tau, \varrho) + f(\sigma, \tau) - f(\tau, \varrho).$$

By Proposition 3.2 a loop $L(\mathrm{Id}, f)$ is commutative if and only if $f(\tau, \sigma) = f(\sigma, \tau)$ for all $\tau, \sigma \in K$.

By Proposition 3.3 in a loop $L(\mathrm{Id}, f)$ the left and right inverses of any element coincide if and only if

$$\psi(\sigma, -\sigma, \sigma) = f(\sigma, -\sigma) - f(-\sigma, \sigma) = 0 \tag{11}$$

for all $\sigma \in K$.

According to Proposition 3.4 a loop $L(\mathrm{Id}, f)$ in which the left and right inverses of any element coincide satisfies the identity

$$(0,0) = [(\tau, t) \circ (\sigma, s)] \circ \left[(\sigma, s)^{-1} \circ ((\tau, t)^{-1}\right] \tag{12}$$

if and only if

$$f(\tau + \sigma, -\tau - \sigma) + f(\tau, \sigma) + f(-\sigma, -\tau) = f(\tau, -\tau) + f(\sigma, -\sigma). \tag{13}$$

By Proposition 3.5 a loop $L(\mathrm{Id}, f)$ in which the left and right inverse of any element coincide possesses the automorphic inverse property if and only if the identity

$$f(\tau + \sigma, -\tau - \sigma) + f(\tau, \sigma) + f(-\tau, -\sigma) = f(\tau, -\tau) + f(\sigma, -\sigma). \tag{14}$$

According to Proposition 3.6 a loop $L(\mathrm{Id}, f)$ has the right inverse property if and only if

$$\psi(\sigma, \tau, -\tau) = f(\sigma + \tau, -\tau) + f(\sigma, \tau) - f(\tau, -\tau) = 0 \tag{15}$$

for all $\sigma, \tau \in K$.

Proposition 3.7 yields that a loop $L(\mathrm{Id}, f)$ has the left inverse property if and only if

$$\psi(\tau, -\tau, \sigma) = -f(\tau, -\tau + \sigma) + f(\tau, -\tau) - f(-\tau, \sigma) = 0 \tag{16}$$

holds for all $\sigma, \tau \in K$.

By Proposition 3.8 a loop $L(\mathrm{Id}, f)$ is left alternative if and only if

$$\psi(\tau, \tau, \sigma) = -f(\tau, \tau + \sigma) + f(2\tau, \sigma) + f(\tau, \tau) - f(\tau, \sigma) = 0. \tag{17}$$

By Proposition 3.9 a loop $L(\mathrm{Id}, f)$ is right alternative if and only if

$$\psi(\tau, \sigma, \sigma) = -f(\tau, 2\sigma) + f(\tau + \sigma, \sigma) + f(\tau, \sigma) - f(\sigma, \sigma) = 0 \tag{18}$$

for all $\sigma, \tau \in K$.

According to Proposition 3.10 a loop $L(\mathrm{Id}, f)$ is flexible if and only if the following identity holds:

$$\psi(\tau, \sigma, \tau) = -f(\tau, \tau + \sigma) + f(\sigma + \tau, \tau) + f(\tau, \sigma) - f(\sigma, \tau) = 0. \tag{19}$$

Proposition 3.11 yields that a loop $L(\mathrm{Id}, f)$ is a left Bol loop if and only if

$$\begin{aligned}(20)\quad \psi(\tau, \sigma+\tau, \varrho) + \psi(\sigma, \tau, \varrho) = -f(\tau, \sigma+\tau+\varrho) + f(\sigma+2\tau, \varrho) + f(\tau, \sigma+\tau)\\ -f(\sigma, \tau+\varrho) + f(\sigma, \tau) - f(\tau, \varrho) = 0.\end{aligned}$$

According to Proposition 3.14 a loop $L(\mathrm{Id}, f)$ is a right Bol loop if and only if the following identity holds:

$$(21)\quad f(\sigma+\tau+\varrho, \tau) + f(\varrho+\tau, \sigma) + f(\varrho, \tau) - f(\varrho, 2\tau+\sigma) - f(\tau+\sigma, \tau) - f(\tau, \sigma) = 0.$$

A loop $L(\mathrm{Id}, f)$ is a group if and only if

$$(22)\qquad \psi(\sigma, \tau, \varrho) = -f(\sigma, \tau+\varrho) + f(\sigma+\tau, \varrho) + f(\sigma, \tau) - f(\tau, \varrho) = 0$$

for all $\sigma, \tau, \varrho \in K$.

4.1.1 Siberian constructions. The following constructions are motivated by the papers [22] and [23].

Let N and K be abelian groups. Let $f\colon K \times K \to N$ be a mapping vanishing on the set $\Sigma = \{(a, 0);\ a \in K\} \cup \{(0, a);\ a \in K\} \cup \{(a, a);\ a \in K\} \cup \{(a, -a);\ a \in K\}$.

The permutation $\varphi_1\colon (K \times K) \setminus \Sigma \to (K \times K) \setminus \Sigma$ given by $\varphi_1(a, b) = (-a, -b)$ generates a group Γ_1 of order 2^{ε}, where $\varepsilon = 0$ if K is an elementary 2-group and $\varepsilon = 1$ if this is not the case. According to identity (14) the loop $L(\mathrm{Id}, f)$ has the *automorphic inverse property* if and only if $f(a, b) = -f(\varphi_1(a, b))$.

The permutation $\varphi_2\colon (K \times K) \setminus \Sigma \to (K \times K) \setminus \Sigma$ given by $\varphi_2(a, b) = (a+b, -b)$ generates a group Γ_2 of order 2. According to identity (15) the loop $L(\mathrm{Id}, f)$ has the *right inverse property* if and only if $f(a, b) = -f(\varphi_2(a, b))$.

Since the permutations φ_1 and φ_2 commute they generate an elementary abelian group Γ_{12} of order $2^{1+\varepsilon}$. The orbits of Γ_{12} in $(K \times K) \setminus \Sigma$ have the form $\{(a, b),$ $\varphi_1(a, b),\ \varphi_2\varphi_1(a, b),\ \varphi_2(a, b)\}$. Let $f(a, b) = f(\varphi_2\varphi_1(a, b)) = n \in N$ and $f(a, b) = -f(\varphi_1(a, b)) = -f(\varphi_2(a, b)) = -n \in N$. Since φ_1 interchanges (a, b) with $\varphi_1(a, b)$ as well as $\varphi_2(a, b)$ with $\varphi_2\varphi_1(a, b)$, the permutation φ_2 interchanges (a, b) with $\varphi_2(a, b)$ as well as $\varphi_1(a, b)$ with $\varphi_2\varphi_1(a, b)$ and $\varphi_2\varphi_1$ interchanges (a, b) with $\varphi_2\varphi_1(a, b)$ as well as $\varphi_1(a, b)$ with $\varphi_2(a, b)$, the function f is invariant under the action of Γ_{12}. This means that the value of f can be chosen for an element of any orbit freely and the function f is uniquely determined by the action of Γ_{12}. The loop $L(\mathrm{Id}, f)$ has the *automorphic inverse property* and the *right inverse property* if and only if the function f is uniquely determined by the action of Γ_{12}.

The permutation $\varphi_3\colon (K \times K) \setminus \Sigma \to (K \times K) \setminus \Sigma$ given by $\varphi_3(a, b) = (-a, a+b)$ generates a group Γ_3 of order 2. According to identity (16) the loop $L(\mathrm{Id}, f)$ has the *left inverse property* if and only if $f(a, b) = -f(\varphi_3(a, b))$.

Since the permutations φ_1 and φ_3 commute they generate an elementary abelian group Γ_{13} of order $2^{1+\varepsilon}$. The orbits of Γ_{13} in $(K \times K) \setminus \Sigma$ have the form $\{(a,b), \varphi_1(a,b), \varphi_3\varphi_1(a,b), \varphi_3(a,b)\}$. Let $f(a,b) = f(\varphi_3\varphi_1(a,b)) = n \in N$ and $f(a,b) = -f(\varphi_1(a,b)) = -f(\varphi_3(a,b)) = -n \in N$. Since φ_1 interchanges (a,b) with $\varphi_1(a,b)$ as well as $\varphi_3(a,b)$ with $\varphi_3\varphi_1(a,b)$, the permutation φ_3 interchanges (a,b) with $\varphi_3(a,b)$ as well as $\varphi_1(a,b)$ with $\varphi_3\varphi_1(a,b)$ and $\varphi_3\varphi_1$ interchanges (a,b) with $\varphi_3\varphi_1(a,b)$ as well as $\varphi_1(a,b)$ with $\varphi_3(a,b)$, the value of the function f can be chosen for an element of any orbit freely and then f is uniquely determined by the action of Γ_{13}. The loop $L(\mathrm{Id}, f)$ has the *automorphic inverse property* and the *left inverse property* if and only if the function f is uniquely determined by the action of Γ_{13}.

The permutations φ_2 and φ_3 generate a group Γ_{23} isomorphic to the symmetric group S_3 of order 6. The orbits of Γ_{23} in $(K \times K) \setminus \Sigma$ have the form $\{(a,b), \varphi_2(a,b), \varphi_3\varphi_2(a,b), \varphi_2\varphi_3\varphi_2(a,b), \varphi_2\varphi_3(a,b), \varphi_3(a,b)\}$. Let $f(a,b) = f(\varphi_3\varphi_2(a,b)) = f(\varphi_2\varphi_3(a,b)) = n \in N$ and $f(a,b) = -f(\varphi_2(a,b)) = -f(\varphi_3(a,b)) = -f(\varphi_2\varphi_3\varphi_2(a,b)) = -n \in N$. Since φ_2 and φ_3 interchange odd products of involutions with even products of involutions the value of the function f can be chosen for an element of any orbit freely and then f is uniquely determined by the action of Γ_{23}. The loop $L(\mathrm{Id}, f)$ has the *left inverse property* if and only if the function f is uniquely determined by the action of Γ_{23}.

A loop $L(\mathrm{Id}, f)$ having the *inverse property* satisfies the *automorphic inverse property* if and only if it is commutative. In this case any orbit of $(a,b) \in (K \times K) \setminus \Sigma$ under the group Γ_{123} generated by the permutations φ_1, φ_2 and φ_3 contains the element (b,a). If the exponent of K is different from 2 then Γ_{123} is the direct product of S_3 and the group of order 2. If K is an elementary abelian 2-group then Γ_{123} is the symmetric group S_3. In any case the point $(0,0)$ is a fixed point of Γ_{123}.

Now we assume that K is an elementary abelian 2-group of order at least 8. In this case if $(a,b) \neq (0,0)$ then the Γ_{123}-orbit of (a,b) consists of the elements

$\{(a,b), (a+b,a), (b,a+b)\}$ and the Γ_{123}-orbit of (b,a) consists of the elements $\{(b,a), (a+b,b), (a,a+b)\}$. Hence the mapping $(a,b) \mapsto (b,a)$ induces an involution on the orbit space $(K \times K)/\Gamma_{123}$.

Let N be also an elementary abelian 2-group and let $f\colon K \times K \to N$ be a mapping which is constant on the orbits of Γ_{123}. Since $(\tau, \tau+\sigma)$ and (τ, σ) are on the same Γ_{123}-orbit and $f(\tau,\tau) = 0$ the identity

$$f(\tau, \tau+\sigma) + f(\tau, \sigma) = f(\tau, \tau) \tag{23}$$

holds. Hence the identities (15), (16), (17) and (18)) yield that the loop $L(\mathrm{Id}, f)$ has the *inverse property* and the *alternative property.* Moreover, it follows from the identity (14) that the loop $L(\mathrm{Id}, f)$ has the automorphic inverse property, i.e. $L(\mathrm{Id}, f)$ is *commutative.*

If $L(\mathrm{Id}, f)$ satisfies a Bol condition then $L(\mathrm{Id}, f)$ is a Moufang loop of exponent 2 and hence it is a group (cf. [3], Proposition 2, p. 35).

Since the group K is the union of elementary abelian 2-groups of order 4 we consider the values of the function f on an elementary abelian 2-group $A = \{0, \tau, \sigma, \tau+\sigma\}$ of K. It follows from (23) that $f(\tau, \sigma) = f(\sigma, \tau) = f(\tau, \tau + \sigma) = f(\tau + \sigma, \tau) = f(\sigma, \tau+\sigma) = f(\tau+\sigma, \sigma) = u_A \in N$. The value $u_A \in N$ can be chosen independently for any elementary abelian subgroup A of K which has order 4 since the intersection of such two different subgroups has order $\leqslant 2$. Since the order of K is at least 8 it contains 4 such different elementary abelian subgroups. According to (22) the loop $L(\mathrm{Id}, f)$ is associative only if $f(\alpha + \beta, \gamma) + f(\alpha, \beta) + f(\alpha, \beta + \gamma) + f(\beta, \gamma) = u_{A_4} + u_{A_1} + u_{A_2} + u_{A_3} = 0$, where $A_1 = \{0, \alpha, \beta, \alpha+\beta\}$, $A_2 = \{0, \alpha, \beta+\gamma, \alpha+\beta+\gamma\}$, $A_3 = \{0, \beta, \gamma, \beta+\gamma\}$ and $A_4 = \{0, \alpha+\beta, \gamma, \alpha+\beta+\gamma\}$. Choosing the values u_{A_i} such that $u_{A_4} + u_{A_1} + u_{A_2} + u_{A_3} \neq 0$ we obtain a proper abelian loop having the inverse property and the alternative property.

Summarizing this discussion we obtain

Proposition 4.1. *There are commutative loops $L(\mathrm{Id}, f)$ which are extensions of an elementary abelian group by an elementary abelian group such that the following assertions hold:*

(i) *$L(\mathrm{Id}, f)$ possesses the inverse and the alternative property but it is not a Moufang loop,*

(ii) *any element $\neq (0, 0)$ of $L(\mathrm{Id}, f)$ has order 2,*

(iii) *any pair of elements of $L(\mathrm{Id}, f)$ is contained in a subgroup of $L(\mathrm{Id}, f)$.*

At the end of this subsection we investigate the flexible law.

Proposition 4.2. *Let $L(\mathrm{Id}, f)$ be a central extension of a finite abelian group N of odd order by an elementary abelian 2-group K. The loop $L(\mathrm{Id}, f)$ is flexible if and only if it is abelian.*

P r o o f. It follows from identity (19) that $L(\mathrm{Id}, f)$ is flexible if and only if the function $g(\tau, \sigma) = f(\tau, \sigma) - f(\sigma, \tau)$ satisfies the identity $g(\tau, \sigma) = g(\tau, \tau + \sigma)$. Since $g(\tau, \sigma) = -g(\sigma, \tau)$ for all $\tau, \sigma \in K$, this identity yields

$$g(\tau, \sigma) = -g(\tau + \sigma, \tau) = -g(\tau + \sigma, \sigma) = g(\sigma, \tau + \sigma) = g(\sigma, \tau) = -g(\tau, \sigma).$$

Since N has odd order, $g(\tau, \sigma) \neq -g(\tau, \sigma)$ if $g(\tau, \sigma) \neq 0$. Hence $f(\tau, \sigma) = f(\sigma, \tau)$ for all $\tau, \sigma \in K$. □

Remark. Let N be the group of order 2 and let K be the elementary abelian group of order 4 which is a 2-dimensional vector space over $GF(2)$. Le $\{e_1, e_2\}$ be a basis of K. Then the loop $L(\mathrm{Id}, f)$ defined on $K \times N$ by the multiplication

$$\Big(\sum_{i=1}^{2} \xi_i e_i, x\Big)\Big(\sum_{i=1}^{2} \eta_i e_i, y\Big) = \Big(\sum_{i=1}^{2}(\xi_i + \eta_i) e_i, x + y + f\Big(\sum_{i=1}^{2} \xi_i e_i, \sum_{i=1}^{2} \eta_i e_i\Big)\Big)$$

with $\xi_i, \eta_i \in GF(2)$, where $f\colon K \times K \to N$ is given by $f(e_1, e_2) = f(e_1, e_1 + e_2) = f(e_2, e_1 + e_2) = 1$ and $f = 0$ otherwise, is flexible but neither commutative nor satisfying the left or right alternative law.

P r o o f. A straightforward computation shows that the non-symmetric function f defined in the assertion satisfies the identity (19). Putting $\tau = e_2$ and $\sigma = e_2$ into the identity (17) we get the contradiction

$$f(e_2, e_1) + f(e_2, e_1 + e_2) = 1 \neq 0.$$

A similar contradiction can be obtained putting $\tau = e_1 + e_2$ and $\sigma = e_2$ into the identity (18). □

4.1.2 Abelian loops $L(\mathrm{Id}, f)$**.** A loop $L(\mathrm{Id}, f)$ is commutative if and only if $f(\tau, \sigma) = f(\sigma, \tau)$ for all $\tau, \sigma \in K$. According to the identity (19) any loop $L(\mathrm{Id}, f)$ is flexible.

Putting $\sigma = \tau = \alpha$ into the identity (13) we see that the loop $L(\mathrm{Id}, f)$ does not satisfy the identity (12) if we can find an element $\alpha \in K$ such that $f(2\alpha, -2\alpha) \neq 2f(\alpha, -\alpha) - f(\alpha, \alpha) - f(-\alpha, -\alpha)$. If the group N is not an elementary abelian 2-group then there are functions f satisfying this condition for an element $\alpha \in K$ with $2\alpha \neq \alpha, -\alpha$.

Putting $\sigma = \tau = \beta$ into the identity (15) we obtain that the loop $L(\mathrm{Id}, f)$ does not possess the inverse property if we can find an element $\beta \in K$ such that $f(2\beta, -\beta) \neq f(\beta, -\beta) - f(\beta, \beta)$. If the group N is not an elementary abelian 2-group then there are functions f satisfying this condition for an element $\beta \in K$ with $2\beta \neq \beta, -\beta$. Putting $-\sigma = \tau = \beta$ into the identity (17) we see that this loop $L(\mathrm{Id}, f)$ does not possess the alternative property either.

From this discussion it follows that if N contains two cyclic subgroups $\langle\alpha\rangle$ and $\langle\beta\rangle$ of order $\geqslant 3$ such that $\langle\alpha\rangle \cap \langle\beta\rangle = \{0\}$, then there exist loops $L(\mathrm{Id}, f)$ having neither the automorphic inverse property, nor the inverse property, nor the alternative law.

Proposition 4.3. *If in the group N the mapping $x \mapsto 2x\colon N \to N$ is an automorphism, then the commutative loop $L(\mathrm{Id}, f)$ has the automorphic inverse property if and only if the function $f(\tau, \sigma)$ has the form*

$$f(\tau,\sigma) = \frac{1}{2}\left(u(\tau,\sigma) + w(\tau) + w(\sigma) - w(\tau+\sigma)\right), \tag{24}$$

where the functions $u\colon K \times K \to N$ and $w\colon K \to N$ satisfy $u(-\tau,-\sigma) = -u(\tau,\sigma)$ and $w(-\tau) = w(\tau)$, respectively.

P r o o f. The loop $L(\mathrm{Id}, f)$ has the automorphic inverse property if and only if

$$f(\tau,\sigma) + f(-\sigma,-\tau) = f(\tau,-\tau) + f(\sigma,-\sigma) - f(\tau+\sigma,-\tau-\sigma). \tag{25}$$

If the function $f(\tau,\sigma)$ satisfies the identity (25) then we denote $w(\tau) = f(\tau,-\tau)$ and $u(\tau,\sigma) = f(\tau,\sigma) - f(-\tau,-\sigma)$. One has $w(-\tau) = w(\tau)$ and $u(-\tau,-\sigma) = -u(\tau,\sigma)$ and hence $f(\tau,\sigma)$ has the form (24). Conversely, for any functions u and w with $u(-\tau,-\sigma) = -u(\tau,\sigma)$ and $w(-\tau) = w(\tau)$ the function determined by (24) satisfies the identity (25). □

Corollary 4.4. *There exist commutative loops $L(\mathrm{Id}, f)$ having the automorphic inverse property but neither the inverse property nor the alternative property.*

P r o o f. Let $L(\mathrm{Id}, f)$ be a commutative loop such that $f(\tau,\sigma)$ has the form (24). The loop $L(\mathrm{Id}, f)$ has neither the inverse property nor the alternative property if there exists $\beta \in K$ such that $f(2\beta,-\beta) \neq f(\beta,-\beta) - f(\beta,\beta)$, which is equivalent to the condition $u(2\beta,-\beta) \neq -u(\beta,\beta)$. Clearly there exist functions satisfying the identity $u(-\tau,-\sigma) = -u(\tau,\sigma)$ and for a suitable $\beta \in K$ also $u(2\beta,-\beta) \neq -u(\beta,\beta)$. □

Let F be a field of characteristic $p > 0$ such that $|F| \geqslant p^2$. The loop $L(\mathrm{Id}, f)$ defined on $F \times F$ by

$$f(\sigma,\tau) = (\sigma-\tau)(\sigma^p\tau^{p^2} - \sigma^{p^2}\tau^p)$$

is an abelian loop. It follows from the identity (14) that $L(\mathrm{Id}, f)$ has the automorphic inverse property. Using the identity (15) we see that $L(\mathrm{Id}, f)$ has the inverse property if and only if $p = 3$, in which case it is a commutative Moufang loop (cf. [17], Theorem 3.4).

4.1.3 Bol and Moufang loops. Let N be a finite abelian group of even order and let K be the direct product $K = C_m \times C_n$ of the cyclic group C_m of order m and C_n of order n, where m, n are even. Choose in N elements r, s, t, z, w such that

$r^m = r^n = s^2 = t^2 = 1$ and $s \neq 1$ or $t \neq 1$. Then the multiplication defined on $K \times N = C_m \times C_n \times N$ by

$$(\alpha_1, \alpha_2, a) \cdot (\beta_1, \beta_2, b) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, a + b + f(\alpha_1, \alpha_2, \beta_1, \beta_2))$$

with $f(\alpha_1, \alpha_2, \beta_1, \beta_2) = r^{\beta_1\alpha_2} s^{\alpha_1\beta_1\beta_2} t^{\beta_1\alpha_2\beta_2} z^p w^q$, where

$$p = \begin{cases} 0 & \text{for} \quad \alpha_1 + \beta_1 < m, \\ 1 & \text{for} \quad \alpha_1 + \beta_1 \geqslant m, \end{cases} \qquad q = \begin{cases} 0 & \text{for} \quad \alpha_2 + \beta_2 < n, \\ 1 & \text{for} \quad \alpha_2 + \beta_2 \geqslant n \end{cases}$$

if we identify C_m and C_n respectively with the integers $0, \dots, m-1$ and $0, \dots, n-1$, yields a loop $L(\mathrm{Id}, f)$. This loop is according to Theorem 2.3 and Lemma 6.1 in [6] a right Bol loop but it is not Moufang.

Let N be the group of order 2 and let K be the elementary abelian group of order 8. We choose a basis $\{e_1, e_2, e_3\}$ of K which is a 3-dimensional vector space over $GF(2)$. Then the multiplication defined on $K \times N$ by

$$\Big(\sum_{i=1}^{3} \xi_i e_i, x\Big)\Big(\sum_{i=1}^{3} \eta_i e_i, y\Big) = \Big(\sum_{i=1}^{3}(\xi_i + \eta_i)e_i, x + y + f\Big(\sum_{i=1}^{3} \xi_i e_i, \sum_{i=1}^{3} \eta_i e_i\Big)\Big)$$

with $\xi_i, \eta_i \in GF(2)$, where the function $f\colon K \times K \to N$ is determined by the relations

$$f(e_i, c) = 0 \quad \text{for all} \quad c \in K \quad \text{and} \quad i = 1, 2, 3,$$
$$f(e_i + e_j, c) = \begin{cases} 1 & \text{if} \quad c = \xi_i e_i + \xi_j e_j + e_k \quad \text{and} \quad \{i, j, k\} = \{1, 2, 3\}, \\ 0 & \text{otherwise}, \end{cases}$$
$$f(e_1 + e_2 + e_3, c) = \begin{cases} 1 & \text{if} \quad c = e_i \quad (i = 1, 2, 3) \quad \text{or} \quad c = e_1 + e_2 + e_3, \\ 0 & \text{otherwise}, \end{cases}$$

gives a loop $L(\mathrm{Id}, f)$. According to [12], Theorem 7.6 this loop is a proper non-commutative Moufang loop of order 16.

4.1.4 Loops $L(\mathrm{Id}, f)$ with polynomial f. Let N and K be vector spaces over a commutative field $\mathbb{F}$. We consider trilinear mappings p, $r\colon K \times K \times K \to N$ such that p is symmetric in the first two variables and r is symmetric in the last two variables. The loop $L(p, r)$ is defined on $K \oplus N$ by

$$(\tau, t) \circ (\sigma, s) = (\tau + \sigma, t + s + p(\tau, \tau, \sigma) + r(\tau, \sigma, \sigma)).$$

In this extension $T(\sigma) = \mathrm{Id}$ and $f(\tau, \sigma) = p(\tau, \tau, \sigma) + r(\tau, \sigma, \sigma)$ for all $\tau, \sigma \in K$. In this case we have

$$\psi(\sigma, \tau, \varrho) = 2p(\sigma, \tau, \varrho) - 2r(\sigma, \tau, \varrho).$$

If the characteristic of the field $\mathbb{F}$ is 2 then $L(p,r)$ is a group, hence we assume that the characteristic $\neq 2$.

By equation (11), in the loop $L(p,r)$ the left and right inverses of any element coincide if and only if

$$p(\sigma,\sigma,\sigma) = r(\sigma,\sigma,\sigma) \tag{26}$$

for all $\sigma \in K$. According to equation (15) the loop $L(p,r)$ has the right inverse property if and only if

$$p(\sigma,\tau,\tau) = r(\sigma,\tau,\tau) \tag{27}$$

for all $\sigma, \tau \in K$.

The relation (16) yields that the loop $L(p,r)$ has the left inverse property if and only if

$$\psi(\sigma,\tau,\varrho) = -f(\sigma,\tau+\varrho) + f(\sigma+\tau,\varrho) + f(\sigma,\tau) - f(\tau,\varrho). \tag{28}$$

$$p(\tau,\tau,\sigma) = r(\tau,\tau,\sigma) \tag{29}$$

hold for all $\sigma \in K$.

The identies (17) and (18) show that a loop $L(p,r)$ has the left or right inverse property if and only if it is respectively left or right alternative.

Let $K = \mathbb{F} \oplus \mathbb{F}$ and $N = \mathbb{F}$. We consider the trilinear form

$$\begin{aligned} p(\sigma,\tau,\varrho) = {} & a_{11}x_1y_1z_1 + a_{12}x_1y_2z_1 + a_{12}x_2y_1z_1 + a_{22}x_2y_2z_1 \\ & + b_{11}x_1y_1z_2 + b_{12}x_1y_2z_2 + b_{12}x_2y_1z_2 + b_{22}x_2y_2z_2, \end{aligned}$$

where $\sigma = (x_1,x_2), \tau = (y_1,y_2), \varrho = (z_1,z_2)$ and $a_{ij}, b_{ij} \in \mathbb{F}$. According to identity (11), in the loop $L(p,r)$ the left and right inverses of any element coincide if and only if the trilinear form $r(\sigma,\tau,\varrho)$ has the form

$$\begin{aligned} r(\sigma,\tau,\varrho) = {} & a_{11}x_1y_1z_1 + c_{12}x_1y_1z_2 + c_{12}x_1y_2z_1 + (a_{22} + 2(b_{12} - d_{12}))\, x_1y_2z_2 \\ & + (b_{11} + 2(a_{12} - c_{12}))\, x_2y_1z_1 + d_{12}x_2y_1z_2 + d_{12}x_2y_2z_1 + b_{22}x_2y_2z_2 \end{aligned}$$

where c_{12}, d_{12} are arbitrary elements of $\mathbb{F}$. The identity (15) yields that $L(p,r)$ has the right inverse property if and only if $r(\sigma,\tau,\varrho)$ has the form

$$\begin{aligned} r(\sigma,\tau,\varrho) = {} & a_{11}x_1y_1z_1 + \tfrac{1}{2}(a_{12}+b_{11})x_1y_1z_2 + \tfrac{1}{2}(a_{12}+b_{11})x_1y_2z_1 + b_{12}x_1y_2z_2 \\ & + a_{12}x_2y_1z_1 + \tfrac{1}{2}(a_{22}+b_{12})x_2y_1z_2 + \tfrac{1}{2}(a_{22}+b_{12})x_2y_2z_1 + b_{22}x_2y_2z_2. \end{aligned}$$

According to identity (16) the loop $L(p,r)$ has the left inverse property if and only if the trilinear form $r(\sigma,\tau,\varrho)$ can be expressed by

$$\begin{aligned} r(\sigma,\tau,\varrho) &= a_{11}x_1y_1z_1 + b_{11}x_1y_1z_2 + b_{11}x_1y_2z_1 + (2b_{12}-a_{22})x_1y_2z_2 \\ &\quad + (2a_{12}-b_{11})x_2y_1z_1 + a_{22}x_2y_1z_2 + a_{22}x_2y_2z_1 + b_{22}x_2y_2z_2. \end{aligned}$$

It follows that the loop $L(p,r)$ have the inverse property if and only if the trilinear forms $p(\sigma,\tau,\varrho)$ and $r(\sigma,\tau,\varrho)$ have the form

$$\begin{aligned} p(\sigma,\tau,\varrho) &= a_{11}x_1y_1z_1 + a_{12}x_1y_2z_1 + a_{12}x_2y_1z_1 + a_{22}x_2y_2z_1 \\ &\quad + a_{12}x_1y_1z_2 + a_{22}x_1y_2z_2 + a_{22}x_2y_1z_2 + b_{22}x_2y_2z_2 \end{aligned}$$

and

$$\begin{aligned} r(\sigma,\tau,\varrho) &= a_{11}x_1y_1z_1 + a_{12}x_1y_1z_2 + a_{12}x_1y_2z_1 + a_{22}x_1y_2z_2 \\ &\quad + a_{12}x_2y_1z_1 + a_{22}x_2y_1z_2 + a_{22}x_2y_2z_1 + b_{22}x_2y_2z_2. \end{aligned}$$

In this case the trilinear forms $p(\sigma,\tau,\varrho)$ and $r(\sigma,\tau,\varrho)$ coincide and they are totally symmetric and hence $L(p,r)$ is an abelian group.

Let $K = F^n$ and $N = F$, where F is a field of characteristic $\neq 2$. A loop $L(p,r)$ is commutative if and only if $p(\mathbf{x},\mathbf{x},\mathbf{y}) + r(\mathbf{x},\mathbf{y},\mathbf{y}) = p(\mathbf{y},\mathbf{y},\mathbf{x}) + r(\mathbf{y},\mathbf{x},\mathbf{x})$ or equivalently $r(\mathbf{x},\mathbf{y},\mathbf{y}) = p(\mathbf{y},\mathbf{y},\mathbf{x})$ for all $\mathbf{x},\mathbf{y} \in K$. According to identity (15) this loop $L(p)$ has the inverse property if and only if

$$-p(\mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y},\mathbf{y}) + p(\mathbf{y},\mathbf{y},\mathbf{x}+\mathbf{y}) + p(\mathbf{x},\mathbf{x},\mathbf{y}) + p(\mathbf{y},\mathbf{y},\mathbf{x}) + p(\mathbf{y},\mathbf{y},\mathbf{y}) - p(\mathbf{y},\mathbf{y},\mathbf{y}) = 0,$$

which is equivalent to

$$p(\mathbf{x},\mathbf{x},\mathbf{y}) = p(\mathbf{y},\mathbf{x},\mathbf{x})$$

for all $\mathbf{x},\mathbf{y} \in K$.

Lemma 4.5. *A loop $L(p)$ with the inverse property is a commutative Moufang loop.*

P r o o f. $L(p)$ is a commutative Moufang loop if and only if it satisfies the Bol condition (21). This means

$$\begin{aligned} &p(\mathbf{x}+\mathbf{y}+\mathbf{z},\mathbf{x}+\mathbf{y}+\mathbf{z},\mathbf{y}) + p(\mathbf{y},\mathbf{y},\mathbf{x}+\mathbf{y}+\mathbf{z}) + p(\mathbf{y}+\mathbf{z},\mathbf{y}+\mathbf{z},\mathbf{x}) + p(\mathbf{x},\mathbf{x},\mathbf{y}+\mathbf{z}) \\ &+ p(\mathbf{y},\mathbf{y},\mathbf{z}) + p(\mathbf{z},\mathbf{z},\mathbf{y}) + p(\mathbf{x}+2\mathbf{y},\mathbf{x}+2\mathbf{y},\mathbf{z}) - p(\mathbf{z},\mathbf{z},\mathbf{x}+2\mathbf{y}) - p(\mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y},\mathbf{y}) \\ &- p(\mathbf{y},\mathbf{y},\mathbf{x}+\mathbf{y}) - p(\mathbf{x},\mathbf{x},\mathbf{y}) - p(\mathbf{y},\mathbf{y},\mathbf{x}) = 0, \end{aligned}$$

or equivalently

$$p(\mathbf{x},\mathbf{z},\mathbf{y})+p(\mathbf{y},\mathbf{z},\mathbf{y})+p(\mathbf{y},\mathbf{z},\mathbf{x})-p(\mathbf{y},\mathbf{y},\mathbf{z})-2p(\mathbf{y},\mathbf{x},\mathbf{z})=0$$

for all $\mathbf{x},\mathbf{y},\mathbf{z}\in K$. Since a Moufang loop $L(p)$ has the inverse property we have $p(\mathbf{y},\mathbf{z},\mathbf{y})=p(\mathbf{z},\mathbf{y},\mathbf{y})=p(\mathbf{y},\mathbf{y},\mathbf{z})$. Moreover, polarization of this identity yields

$$p(\mathbf{x},\mathbf{z},\mathbf{y})+p(\mathbf{y},\mathbf{z},\mathbf{x})-2p(\mathbf{y},\mathbf{x},\mathbf{z})=0 \tag{30}$$

which proves the claim. □

Let $p(\mathbf{x},\mathbf{y},\mathbf{z})=\sum\limits_{i,j,k=1}^{n}p_{ijk}x^iy^jz^k$, where $\mathbf{x}=(x^i),\mathbf{y}=(y^i),\mathbf{z}=(z^i)$ and $p_{ijk}=p_{jik}\in F$. We investigate now the the identity (30) which characterizes commutative Moufang loops $L(p)$. The trilinear form $p(\mathbf{x},\mathbf{y},\mathbf{z})=\sum\limits_{i,j,k=1}^{n}p_{ijk}x^iy^jz^k$ satisfies the identity (30) if and only if

$$2p_{ijk}=p_{kij}+p_{kji}. \tag{31}$$

If $i=j=k$ then the equation (31) is true. If two of the indices coincide we get that $p_{iik}=p_{kii}$ for any $i\neq k$. If all three indices i,j,k are different then we have the system of equations

$$\begin{aligned}2p_{ijk}&=p_{kij}+p_{kji}=p_{kij}+p_{jki},\\2p_{jki}&=p_{ijk}+p_{ikj}=p_{ijk}+p_{kij},\\2p_{kij}&=p_{jki}+p_{jik}=p_{jki}+p_{jik}.\end{aligned}$$

If the characteristic of F is different from 3 then the solutions are given by the condition $3p_{ijk}=3p_{jki}=3p_{kij}$. Hence in this case the trilinear form $p(\mathbf{x},\mathbf{y},\mathbf{z})$ is symmetric in all three variables and $L(p)$ is a group. If the characteristic of F is 3, then this system is equivalent to the equation $p_{ijk}+p_{kij}+p_{jki}=0$. Choosing at least for one triple (i,j,k) of different indices two of p_{ijk},p_{kij},p_{jki} distinctly we obtain a trilinear form $p(\mathbf{x},\mathbf{y},\mathbf{z})$ which is not symmetric in all three variables. Hence for $n\geqslant 3$ we obtain a great variety of commutative Moufang loops $L(p)$.

4.2 Differentiable Schreier loops. The loops $L(\mathrm{Id},f)$ with polynomial f over the field of real or complex numbers have analytic operations. But there exist many other examples of differentiable Schreier loops which are central extensions of the additive group $\mathbb{R}$ by $\mathbb{R}$. The importance of the following examples is based on the fact that the group generated by the left translations of them is a 3-dimensional non-nilpotent Lie group with one-dimensional centre [18], Section 23.2, p. 298. Let $L(\mathrm{Id}, f)$ with $f(\sigma,\tau) = g(\sigma)(1-\mathrm{e}^{\tau})$, where g is a differentiable real function. The multiplication of the loop $L(\mathrm{Id}, f)$ is given by

$$(\tau,t)\circ(\sigma,s) = (\tau+\sigma, t+s+g(\tau)(1-\mathrm{e}^{\sigma}))\,, \quad \tau,\sigma,t,s\in\mathbb{R}.$$

According to equation (11), in a loop $L(\mathrm{Id}, f)$ the left and right inverses of any element coincide if and only if $g(\sigma)\mathrm{e}^{-\sigma} = -g(-\sigma)$.

Hence in a loop $L(\mathrm{Id}, f)$ the left and right inverses of any element coincide if and only if there exists a differentiable function $h(\sigma)$ defined for $\sigma \geqslant 0$ satisfying $h(0) = 0$ such that $g(\sigma) = h(\sigma)$ for $\sigma \geqslant 0$ and $g(\sigma) = -h(-\sigma)\mathrm{e}^{-\sigma}$ for $\sigma \leqslant 0$.

Using the identity (16) we see that a loop $L(\mathrm{Id}, f)$ in which the left and right inverses of any element coincide already has the left inverse property.

According to equation (15) a loop $L(\mathrm{Id}, f)$ has the right inverse property if and only if

$$g(\sigma+\tau)(1-\mathrm{e}^{-\tau}) + g(\sigma)(1-\mathrm{e}^{\tau} + g(\tau)(1-\mathrm{e}^{-\tau}) = 0,$$

which yields

$$g(\sigma+\tau) - g(\sigma)\mathrm{e}^{\tau} = g(\tau) \tag{32}$$

for all $\tau,\sigma\in\mathbb{R}$. Deriving this identity with respect to σ we obtain $g'(\tau) = g'(0)\mathrm{e}^{\tau}$ from which it follows that $g(\tau) = \alpha + \gamma\mathrm{e}^{\tau}$ with a constant α and $\gamma = g'(0)$. Putting this into the equation (32) we obtain $\alpha = -\gamma$ and $g(\tau) = \alpha(1-\mathrm{e}^{\tau})$. But in this case the loop $L(\mathrm{Id}, f)$ is the two-dimensional vector group $\mathbb{R}^2$ (cf. Theorem 23.7 (ii) in [18]).

Using the identities (18) and (19) an analogous computation shows that a loop $L(\mathrm{Id}, f)$ is also the two-dimensional vector group $\mathbb{R}^2$ if $L(\mathrm{Id}, f)$ is flexible or right alternative.

According to equation (17) a loop $L(\mathrm{Id}, f)$ is left alternative if and only if

$$g(\tau)\left(\mathrm{e}^{\tau+\sigma} - \mathrm{e}^{\tau} + \mathrm{e}^{\sigma} - 1\right) = g(2\tau)\left(\mathrm{e}^{\tau} - 1\right)$$

or

$$g(2\tau) = g(\tau)\left(\mathrm{e}^{\tau} + 1\right). \tag{33}$$

Clearly, the function $g(\tau) = \alpha\,(1-\mathrm{e}^{\tau})$ which corresponds to the 2-dimensional vector group satisfies this identity. If $g(\tau)$ is a real analytic function than it has a power series expansion $g(\tau) = \sum_{i=1}^{\infty} a_i \tau^i$ since $g(0) = 0$. According to equation (33) we have

$$\sum_{i=1}^{\infty} a_i 2^i \tau^i = \left(\sum_{j=1}^{\infty} a_j \tau^j\right)\left(2 + \sum_{k=1}^{\infty} \frac{1}{k!}\tau^k\right) = \sum_{j=1}^{\infty} a_j \tau^j + \sum_{l=2}^{\infty}\left(\sum_{j=1}^{l-1} \frac{a_j}{(l-j)!}\right)\tau^l.$$

If we compare the coefficients of these power series we obtain

$$a_l = \frac{1}{2(2^{l-1}-1)} \sum_{j=1}^{l-1} \frac{a_j}{(l-j)!}$$

for $l \geqslant 2$. Hence the function $g(\tau) = \sum_{i=1}^{\infty} a_i \tau^i$ is uniquely determined by the coefficient a_1. The unique solution is $g(\tau) = -a_1 (1 - \mathrm{e}^{\tau})$, which means that $L(\mathrm{Id}, f)$ is a group.

4.3 Scheerer extensions. Let K be a proper loop having G as the group generated by its left translations, let H be the stabilizer of the identity of K in G and let $\sigma\colon G/H \to G$ be the section such that $\Sigma = \sigma(G/H)$ is the set of left translations of K. The multiplication

$$(x, y) \mapsto x \star y = \pi(xy)\colon \ \Sigma \times \Sigma \to \Sigma,$$

where $\pi\colon G \to \Sigma$ is the mapping assigning to $g \in G$ the unique element of Σ contained in the coset gH, defines a loop $(\Sigma, \star)$ wich is isomorphic to K (cf. [18], p. 18).

If ϱ is a homomorphism from H into the centre $Z(N)$ of N then the loop $L(\mathrm{Id}, f)$ defined by

$$(\alpha, a) \circ (\beta, b) = \left(\alpha \star \beta, a\varrho\left((\alpha\beta)^{-1}(\alpha \star \beta)\right) b\right) = \left(\alpha \star \beta, \varrho\left((\alpha\beta)^{-1}\sigma(\alpha\beta H)\right) ab\right)$$

is a Schreier loop with $f(\alpha, \beta) = \varrho\left((\alpha\beta)^{-1}(\alpha \star \beta)\right)$, which is a Scheerer extension of the group N by the loop K (cf. [18], Proposition 2.4, p. 44). The loop $L(\mathrm{Id}, f)$ is a central extension of N by K if and only if N is abelian.

According to Propositions 2.7 and 2.8 ([18], pp. 46-47), the loop $L(\mathrm{Id}, f)$ belongs to the class of right alternative loops, left Bol loops, Moufang loops or to the class of loops with left inverse property if and only if K belongs to the same class of loops.

The Schreier loops which are Scheerer extensions play an inportant role in the classification of differentiable Bol loops and of loops L for wich the stabilizer of the identity in the group generated by the left translations consists of automorphisms of L (cf. [8], Theorem 6. p. 446, [9], Theorem 14. p. 406 and [10], Theorem 6. p. 74).

4.4 Semidirect products. In this section we consider loops $L(T,1)$ which generalize the semidirect products in the theory of groups. Since we are interested in proper loops $L(T,1)$ we will always assume that $T(\sigma) \neq Id$ for some $\sigma \in K$. Multiplication in $L(T,1)$ on $K \times N$ is given by

$$(\tau, t) \cdot (\sigma, s) = (\tau\sigma, t^{T(\sigma)} s).$$

By Proposition 3.3, in a loop $L(T,1)$ the left and the right inverses of any element coincide if and only if it is the case in the loop K and $T(\sigma^{-1}) = T(\sigma)^{-1}$ for any $\sigma \in K$ holds.

Proposition 3.6 yields that a loop $L(T,1)$ in which for any element the left inverse and the right inverse coincide has the right inverse property if K has the right inverse property.

According to Propositions 3.5, 3.7, 3.8 and 3.10 a proper loop $L(T,1)$ which is not the direct product $K \times N$ has neither the automorphic inverse property nor the left inverse property nor the left alternative law as well as the flexible law.

By Proposition 3.9 a loop $L(T,1)$ is right alternative if and only if K is right alternative and $T(\sigma^2) = T(\sigma)^2$ for any $\sigma \in K$.

According to Proposition 3.14 a loop $L(T,1)$ is a right Bol loop if and only if K is a right Bol loop and $T(\tau)T(\sigma)T(\tau) = T((\tau \cdot \sigma) \cdot \tau)$ for all $\tau, \sigma \in K$.

4.4.1 Right Bol loops and right alternative loops. Let N be an abelian group, let α be an involutory automorphism of N and let K be an elementary abelian 2-group of cardinality > 2. Then the semidirect product $L_\alpha = L(T,1)$ defined on $K \times N$ by a non-constant function $T\colon K \to \langle \alpha \rangle$ with $T(1) = \mathrm{Id}$ is a proper right Bol loop if and only if there exists no subgroup N of index 2 in K such that the restrictions of T satisfy $T|_M = Id$ and $T|_{Mx} = \alpha$ for $x \notin M$.

This is clear since the identity $T(\tau)T(\sigma)T(\tau) = T(\tau\sigma\tau)$ is satisfied (cf. Proposition 3.14) and $T\colon K \to \langle \alpha \rangle$ is a homomorphism if and only if $M = \mathrm{Ker}\,(T)$.

Let N be an abelian group and let α, β be two non-commuting involutory automorphisms of N. Let K be an elementary abelian 2-group. We divide the set $N \setminus \{1\}$ into two non-empty disjoint subsets M_1 and M_2 and define

$$T(\sigma) = \begin{cases} \alpha & \text{if} \quad \sigma \in M_1, \\ \beta & \text{if} \quad \sigma \in M_2, \\ Id & \text{if} \quad \sigma = 1. \end{cases}$$

The loop $L_{\alpha,\beta} = L(T,1)$ with this function T is right alternative since $T(\sigma^2) = T(1) = Id = T(\sigma)^2$ for any $\sigma \in K$. However, $L(T,1)$ does not satisfy the right Bol identity because for $\sigma \in M_1$, $\tau \in M_2$ one has $T(\tau)T(\sigma)T(\tau) = \beta\alpha\beta \neq \alpha = T(\tau\sigma\tau) = T(\sigma)$.

4.4.2 Groups generated by translations. Let the semidirect product $L(T,1)$ be an extension of a group N by a group K. The right translation $\varrho_{(\sigma,s)}$ of $L(T,1)$ is the map

$$\varrho_{(\sigma,s)}\colon\ (\tau,t) \mapsto (\tau\sigma, t^{T(\sigma)}s)\colon\ L(T,1) \to L(T,1)$$

and its inverse is the map

$$\varrho^{-1}_{(\sigma,s)}\colon\ (\tau,t) \mapsto (\tau\sigma^{-1}, (ts^{-1})^{T(\sigma)^{-1}}).$$

Since $\varrho_{(\sigma,s)}\varrho_{(1,v)}\varrho^{-1}_{(\sigma,s)}\colon\ (\tau,t) \mapsto (\tau, ts^{-1}v^{T(\sigma)}s)$ holds one has $\varrho_{(\sigma,s)}\varrho_{(1,v)}\varrho^{-1}_{(\sigma,s)} = \varrho_{(1,s^{-1}v^{T(\sigma)}s)}$. Hence the group G_ϱ generated by the right translations of $L(T,1)$ contains a normal subgroup $N_\varrho = \{\varrho_{(1,v)};\ v \in N\}$ isomorphic to N. Because of $\varrho_{(\sigma,s)} = \varrho_{(1,s)}\varrho_{(\sigma,1)}$ and $N_\varrho = \{\varrho_{(1,s)};\ s \in N\}$ one has $G_\varrho = N_\varrho\Sigma$, where Σ is the group generated by the set $\{\varrho_{(\sigma,1)};\ \sigma \in K\}$. If Θ denotes the group of automorphisms of N generated by the set $\{T(\sigma);\ \sigma \in K\}$, then Σ is the subgroup of the direct product $K \times \Theta$.

This relatively simple structure of the group G_ϱ generated by the right translations of a loop $L(T,1)$ allows to determine G_ϱ more precisely. So the group G_ϱ generated by the right translations of the right Bol loop L_α defined in the previous section is the semidirect product $N_\varrho \rtimes (K \times \langle\alpha\rangle) = N_\varrho \rtimes \Theta$. This group contains a subgroup of index 2 which is isomorphic to the direct product $N \times K$.

The left translation $\lambda_{(\sigma,s)}$ of $L(T,1)$ is the map

$$\lambda_{(\sigma,s)}\colon\ (\tau,t) \mapsto (\sigma\tau, s^{T(\tau)}t)\colon\ L(T,1) \to L(T,1)$$

and its inverse is the map

$$\lambda^{-1}_{(\sigma,s)}\colon\ (\tau,t) \mapsto (\sigma^{-1}\tau, s^{-T(\sigma^{-1}\tau)}t).$$

The set $N_\lambda = \{\varrho_{(1,v)};\ v \in N\}$ is a subgroup of the group G_λ generated by the left translations of $L(T,1)$. One has

$$\omega(\sigma,s,v) = \lambda_{(\sigma,s)}\lambda_{(1,v)}\lambda^{-1}_{(\sigma,s)}\colon\ (\tau,t) \mapsto (\tau, (svs^{-1})^{T(\sigma^{-1}\tau)}t).$$

The mapping $\omega(\sigma,s,v)$ is equal to the element $\lambda_{(1,w)}$ of N_λ for a suitable $w \in N$ if and only if the relation $(\tau, (svs^{-1})^{T(\sigma^{-1}\tau)}t) = (\tau, w^{T(\tau)}t)$ and $s, v \in N$ for all $(\tau,t) \in L(T,1)$ holds. This is equivalent to $(svs^{-1})^{T(\sigma^{-1}\tau)} = w^{T(\tau)}$ or to

$$(svs^{-1})^{T(\sigma^{-1}\tau)} = (svs^{-1})^{T(\sigma^{-1})T(\tau)} \tag{34}$$

for any $\tau \in K$. The group N_λ is a normal subgroup of the group G_λ generated by the left translations if and only if the relation (34) holds true for all $\sigma, \tau \in K$ and $s, v \in N$. But this is the case if and only if the mapping T is a homomorphism, which means that $L(T, 1)$ is a group.

References

[1] *G. Birkenmeier, B. Davis, K. Reeves and S. Xiao*: Is a semidirect product of groups necessarily a group? Proc. Amer. Math. Soc. *118* (1993), 689–692.

[2] *G. Birkenmeier and S. Xiao*: Loops which are semidirect products of groups. Comm. in Algebra *23* (1995), 81–95.

[3] *O. Chein*: Moufang loops of small order I. Trans. Amer. Math. Soc. *188* (1974), 31–51.

[4] *O. Chein*: Moufang loops of small order. Mem. Amer. Math. Soc. *13* (1978), 31–51.

[5] *O. Chein*: Examples and methods of construction, Chapter II in Quasigroups and Loops: Theory and Applications (O. Chein, H. O. Pflugfelder, J. D. H. Smith, eds.). Heldermann Verlag, Berlin, 1990, pp. 27–93.

[6] *O. Chein and E. G. Goodaire*: A new construction of Bol loops of order $8k$. Journal of Algebra *287* (2005), 103–122.

[7] *P. Csörgő and A. Drápal*: Left conjugacy closed loops of nilpotency class two. Results Math. *47* (2005), 242–265.

[8] *A. Figula*: 3-dimensional Bol loops as sections in non-solvable Lie groups. Forum Math. *17* (2005), 431–460.

[9] *A. Figula*: 3-dimensional loops on non-solvable reductive spaces. Adv. Geom. *5* (2005), 391–420.

[10] *A. Figula*: 3-dimensional Bol loops corresponding to solvable Lie triple systems. Publ. Math. Debrecen *70* (2007), 59–101.

[11] *A. Figula and K. Strambach*: Loops which are semidirect products of groups. Acta Math. Hungar. *114* (2007), 247–266.

[12] *M. K. Kinyon and K. Kunen*: The structure of extra loops. Quasigroups Related Systems *12* (2004), 39–60.

[13] *M. K. Kinyon, K. Kunen and J. D. Phillips*: A generalization of Moufang and Steiner loops. Algebra Universalis *48* (2002), 81–101.

[14] *M. K. Kinyon, J. D. Phillips and P. Vojtěchovský*: C-loops; extensions and constructions. J. Algebra Appl. *6* (2007), 1–20.

[15] *M. K. Kinyon, J. D. Phillips and P. Vojtěchovský*: When is the commutant of a Bol loop a subloop. Trans. Amer. Math. Soc. *360* (2008), 2393–2408.

[16] *A. G. Kurosh*: The Theory of Groups, Vol. 2, transl. from Russian by K. A. Hirsch. Chelsea Publishing Co., New York, 1956.

[17] *G. P. Nagy*: Algebraic commutative Moufang loops. Forum Math. *15* (2003), 37–62.

[18] *P. T. Nagy and K. Strambach*: Loops in Group Theory and Lie Theory, Expositions in Mathematics 35. Walter de Gruyter, Berlin-New York, 2002.

[19] *H. O. Pflugfelder*: Quasigroups and Loops: An Introduction. Heldermann Verlag, Berlin, 1990.

[20] *O. Schreier*: Über die Erweiterung von Gruppen I. Monatshefte f. Math. *34* (1926), 165–180.

[21] *O. Schreier*: Über die Erweiterung von Gruppen II. Abh. Math. Sem. Univ. Hamburg *4* (1926), 321–346.

[22] *N. M. Suvorov and N. I. Krjuckov*: Examples of certain quasigroups and loops that permit only the discrete topologization. Sib. Mat. Zh. *17* (1976), 471–473. (In Russian.)

[23] *N. M. Suvorov*: A commutative IP-loop that admits only discrete topologization. Sib. Mat. Zh. *32* (1991), 193. (In Russian.)
[24] *M. Suzuki*: Group Theory, Vol. 1. Grundlehren der Mathematischen Wissenschaften 10, Springer Verlag, Berlin-New York, 1982.
[25] *E. Winterroth*: Right Bol loops with a finite dimensional group of multiplications. Publ. Math. Debrecen *59* (2001), 161–173.

Authors' addresses: Péter T. Nagy, Institute of Mathematics, University of Debrecen, P.O.B. 12, 4010 Debrecen, Hungary, e-mail: `nagypeti@math.klte.hu`; Karl Strambach, Mathematisches Institut der, Universität Erlangen-Nürnberg, Bismarckstr. $1\frac{1}{2}$, 910 54 Erlangen, Germany, e-mail: `strambach@mi.uni-erlangen.de`.