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ASYMPTOTICS OF VARIANCE OF THE LATTICE POINT COUNT

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Abstract. The variance of the number of lattice points inside the dilated bounded set rD with random position in $\mathbb{R}^d$ has asymptotics $\sim r^{d-1}$ if the rotational average of the squared modulus of the Fourier transform of the set is $O(\varrho^{-d-1})$. The asymptotics follow from Wiener's Tauberian theorem.

Keywords: point lattice, Fourier transform, volume, variance

MSC 2010: 62D05, 11H06

1. Introduction

The number of lattice points in a set with random position can be used for estimation of the volume of the set and its variance has been studied for a long time [6], [7]. If $\mathbf{T}$ is a d-periodical point lattice of spatial intensity α in the d-dimensional Euclidean space $\mathbb{R}^d$ then the mean value of

$$(\mathbf{card}((B_d(r)+x)\cap \mathbf{T}) - \alpha\lambda^d(B_d(r)))^2,$$

where λ^d is the Lebesgue measure, i.e. the variance of the lattice point count in the ball $B_d(r)$ of radius r with uniform random position, is

$$C_{\mathbf{T}} H^{d-1}(\partial B_d(r))\Phi(r).$$

Here $C_{\mathbf{T}}$ is a lattice constant, H^{d-1} is the surface measure and the function Φ defined by the above equality fulfills $\lim\limits_{t\to\infty} t^{-1}\int_0^t \Phi(t)\,\mathrm{d}t = 1$, is bounded, and $\limsup\limits_{t\to\infty}\Phi(t)\leqslant 2$ [4], [5]. Hence the variance of the lattice point count in the ball has asymptotics "in the mean" $C_{\mathbf{T}} H^{d-1}(\partial B_d(1)) r^{d-1}$ and is $O(r^{d-1})$, $r\to\infty$.

Let D be a compact body the isotropic covariogram $\overline{\gamma}_D$ (the rotational average of $\gamma_D = I_D \star I_{-D}$) of which is a fractional integral of the Weyl type of a finite signed measure σ on $\mathbb{R}^+$:

$$\overline{\gamma}_D(t) = \int_t^\infty (s-t)^{(d+1)/2}\, \mathrm{d}\sigma(s)$$

for $t \in \mathbb{R}^+$. This means that the fractional derivative of $\overline{\gamma}_D$ of the order $\frac{1}{2}(d+1)$ has bounded variation. Moreover, let the derivative of $\overline{\gamma}_D$ from the right fulfill $\overline{\gamma_D}'^{+}(0) = -(\kappa_{d-1}/\mathrm{d}\kappa_d)H^{d-1}(\partial D)$. Then the asymptotics "in the mean" of the form $C_{\mathbf{T}} H^{d-1}(\partial D) r^{d-1}$ of the variance of the lattice point count inside randomly rotated and shifted set rD follows easilly from Theorem 2.8 in [3].

The variance of the lattice point count inside a randomly rotated and shifted bounded set D dilated by $r > 0$ is $O(r^{d-1})$, $r \to \infty$, if the rotational average of the squared modulus of the Fourier transform of the set D, defined as $\int_{S_{d-1}} |\widehat{I_D}(\varrho\xi)|^2\, \mathrm{d}\xi$, is $O(\varrho^{-d-1})$, $\varrho \to \infty$ [12]; the later property was proved for convex sets and sets with $C^{3/2}$ (see below) boundary [2]. The aim of this paper is to prove that this assumption on the growth of the modulus of the Fourier transform yields also the asymptotics of the form $C_{\mathbf{T}} H^{d-1}(\partial D) r^{d-1}$ for the variance of the lattice point count inside randomly placed bounded full d-dimensional set rD with sufficiently regular boundary (locally finite union of sets of finite reach [9]). The above asymptotics can be thus used for compact bodies with piecewise $\mathbf{C}^2$ smooth boundary and for d-dimensional convex sets in $\mathbb{R}^d$.

2. The variance of the lattice point count

Notation 2.1. Let $\mathbf{T}$ be a d-periodic lattice of points in the d-dimensional Euclidean space $\mathbb{R}^d$ defined by a regular matrix $A \in \mathbb{R}^{d\times d}$ as $\mathbf{T}(A) = A\mathbb{Z}^d$, where $\mathbb{Z}^d$ is the set of all points in $\mathbb{R}^d$ with integral coordinates. $\mathbf{T}$ has the fundamental region $F_{\mathbf{T}} = A[0,1)^d$ of volume $\lambda^d(F_{\mathbf{T}}) = \det A$, where λ^d is the Lebesgue measure; hence the spatial intensity of $\mathbf{T}$ is $\alpha = (\det A)^{-1}$. The group dual to the group $\mathbf{T}(A)$ is $\mathbf{T}^* = A^{-1}\mathbb{Z}^d$.

The Fourier transform of a function $f \in \mathbf{L}^1(\mathbb{R}^d)$ is

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \exp(-2\pi \mathrm{i} x\cdot\xi)\, \mathrm{d}x. \tag{2.1}$$

If f is spherically symmetric then $r^{d-1}f(r) \in \mathbf{L}^1(\mathbb{R}^+)$ and the Fourier transform of f can be expressed as the Hankel transform

$$\hat{f}(\varrho) = 2\pi\varrho^{1-d/2} \int_0^\infty r^{d/2} J_{d/2-1}(2\pi\varrho r) f(r)\, \mathrm{d}r, \tag{2.2}$$

where $J_{d/2-1}$ is the Bessel function of the first kind. The inverse Hankel transform is identical to the direct transform.

$\kappa_d = \pi^{d/2}\Gamma(d/2+1)^{-1}$ is the volume of the unit ball $B_d(1)$ in $\mathbb{R}^d$, where Γ is the Euler Gamma function.

I_D is the characteristic function of the set D.

A function $f(x)$ is $O(g(x))$, $x \to \infty$ iff f/g is bounded in a neighbourhood of ∞.

Proposition 2.2. *Let* $\mathbf{T}$ *be a d-periodic lattice of points and let* D *be a bounded measurable set in* $\mathbb{R}^d$. *Then*

$$\int_{F_{\mathbf{T}}} (\mathbf{card}((D+x)\cap \mathbf{T}))\alpha \,\mathrm{d}x = \alpha\lambda^d(D), \tag{2.3}$$

and

$$\int_{F_{\mathbf{T}}} (\mathbf{card}((D+x)\cap \mathbf{T}) - \alpha\lambda^d(D))^2\alpha \,\mathrm{d}x = \sum_{0\neq\xi\in\mathbf{T}^*} |\widehat{I_D}(\xi)|^2, \tag{2.4}$$

where α *is the spatial intensity of* $\mathbf{T}$.

P r o o f. Equation (2.3) can be proved by standard arguments and equation (2.4) follows from the Parseval theorem, see Theorem 2.3 in [3]. □

Definition 2.3. Covariogram of a bounded measurable set D is the function

$$\gamma_D(x) = I_D \star I_{-D}(x) = \int_{\mathbb{R}^d} I_D(y)I_D(y-x)\,\mathrm{d}y.$$

It follows from the properties of the Fourier transform that $\widehat{\gamma_D} = |\widehat{I_D}|^2$ is a non-negative function. The isotropic covariogram is $\overline{\gamma_D}(|x|) = \int_{\mathbf{SO}_d} \gamma_{MD}(x)\,\mathrm{d}M$, where MD is the set D rotated by $M \in \mathbf{SO}_d$ and the integration uses the invariant probabilistic measure on $\mathbf{SO}_d$, the group of rotations in $\mathbb{R}^d$; an equivalent definition is $\overline{\gamma_D}(u) = \int_{S_{d-1}} \gamma_D(ux)\,\mathrm{d}x$. The Hankel transform of the isotropic covariogram is $\widehat{\overline{\gamma_D}}$.

Remark 2.4. As was already discussed in Remark 2.5 in [3], it follows from this definition that γ_D is bounded and, as $\widehat{\gamma_D} \geqslant 0$, the function $\widehat{\gamma_D}$ is integrable in $\mathbb{R}^d$ (see [1] Theorem 9). Further, $\varrho^{d-1}\widehat{\overline{\gamma_D}}(\varrho) \geqslant 0$ is integrable in $\mathbb{R}^+$ by Fubini's theorem. γ_D is then the inverse Fourier transform of $\widehat{\gamma_D}$ ([1] Theorem 8) and $\overline{\gamma_D}$ is the (inverse) Hankel transform (2.2) of $\widehat{\overline{\gamma_D}}(\varrho)$. We have from (2.4) and (2.3) using the variance decomposition lemma [8]

$$\int_{\mathbf{SO}_d}\int_{F_{\mathbf{T}}} (\mathbf{card}((MD+x)\cap \mathbf{T}) - \alpha\lambda^d(D))^2\alpha \,\mathrm{d}x\,\mathrm{d}M = \sum_{0\neq\xi\in\mathbf{T}^*} \widehat{\overline{\gamma_D}}(|\xi|).$$

Lemma 2.5. *Let $r^2 f(r) \geqslant 0$ be a bounded measurable function on $\mathbb{R}^+$. Then*

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R r^2 f(r)\,\mathrm{d}r = \lim_{h\to 0+}\frac{\pi^{-3/2}}{h}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}\int_0^\infty\Bigl(1-\Gamma\Bigl(\frac{d}{2}\Bigr)\frac{J_{d/2-1}(2\pi hr)}{(\pi hr)^{d/2-1}}\Bigr)f(r)\,\mathrm{d}r$$

whenever at least one of the two limits exists.

Proof. The case $d=1$ was proved in [13] (Theorem 21) using Wiener's Tauberian theorem (see e.g. [11]): Let $\varphi\in\mathbf{L}^\infty(\mathbb{R})$, $K\in\mathbf{L}^1(\mathbb{R})$, $\widehat{K}$ has no root and $K\star\varphi(t)\to a\widehat{K}(0)$ as $t\to\infty$. Then for each $g\in\mathbf{L}^1(\mathbb{R})$ $g\star\varphi(t)\to a\hat{g}(0)$ as $t\to\infty$. What follows is an extension of this proof to higher dimensions.

By the substitution $r=\exp(t)$ and defining

$$R=\exp(\eta)=\frac{1}{h},\quad r^2 f(r)=\varphi(t),$$

we obtain an equivalent formulation of the theorem suitable for a direct application of Wiener's Tauberian theorem

$$\lim_{\eta\to\infty}\int_{-\infty}^\infty K_1(\eta-t)\varphi(t)\,\mathrm{d}t = \lim_{\eta\to\infty}\int_{-\infty}^\infty K_2(\eta-t)\varphi(t)\,\mathrm{d}t$$

whenever at least one of the two limits exists. Here

$$K_1(s)=I_{\{s\mid s>0\}}\exp(-s),\quad K_2(s)=\exp(s)L(\exp(-s)),$$
$$L(u)=\pi^{-3/2}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}\Bigl(1-\Gamma\Bigl(\frac{d}{2}\Bigr)\frac{J_{d/2-1}(2\pi u)}{(\pi u)^{d/2-1}}\Bigr).$$

We can easily see that $\widehat{K_1}(\tau)=\int_0^\infty\exp(-s-2\pi\mathrm{i}s\tau)\,\mathrm{d}s=(1+2\pi\mathrm{i}\tau)^{-1}$ has no real root and $\widehat{K_1}(0)=1$. If moreover $\widehat{K_2}(0)=\int_0^\infty u^{-2}L(u)\,\mathrm{d}u=1$ and the limit on the left hand side exists, then also the limit on the right hand side exists and the equality follows from Wiener's Tauberian theorem. To prove the statement in the opposite direction it remains to show that $\widehat{K_2}(\tau)=\int_0^\infty u^{-2+2\pi\mathrm{i}\tau}L(u)\,\mathrm{d}u$ has no real root.

To complete the proof by establishing the validity of the assumptions concerning the function $\widehat{K_2}$, we will evaluate the integral

$$\int_0^\infty u^{-2+2\pi\mathrm{i}\tau}\Bigl(1-\Gamma\Bigl(\frac{d}{2}\Bigr)\frac{J_{d/2-1}(2\pi u)}{(\pi u)^{d/2-1}}\Bigr)\,\mathrm{d}u.$$

Using $\int t^{-\nu}J_{\nu+1}(t)\,\mathrm{d}t=-t^{-\nu}J_\nu(t)$ we get

$$2\pi\Gamma\Bigl(\frac{d}{2}\Bigr)\int_0^\infty u^{-2+2\pi\mathrm{i}\tau}\int_0^u(\pi s)^{1-d/2}J_{d/2}(2\pi s)\,\mathrm{d}s\,\mathrm{d}u$$

and integration by parts gives

$$\frac{2\pi\Gamma(\frac{d}{2})}{1-2\pi i\tau}\Bigg(-\Bigg[u^{-1+2\pi i\tau}\int_0^u \frac{J_{d/2}(2\pi s)}{(\pi s)^{d/2-1}}\,\mathrm{d}s\Bigg]_0^\infty + \pi^{1-2\pi i\tau}\int_0^\infty \frac{J_{d/2}(2\pi u)}{(\pi u)^{d/2-2\pi i\tau}}\,\mathrm{d}u\Bigg).$$

The first term is zero, as the limits at 0 and infinity are zero by the l'Hospital formula and asymptotic properties of the Bessel function. From the formula $\int_0^\infty t^a J_\nu(2t)\,\mathrm{d}t = \frac{1}{2}\Gamma(\frac{\nu+a+1}{2})\Gamma(\frac{\nu-a+1}{2})^{-1}$, valid if $\operatorname{Re} a < \frac{1}{2}$, $\operatorname{Re} a + \nu > -1$, it follows that

$$\widehat{K_2}(\tau) = \frac{\pi^{-2\pi i\tau-1/2}}{1-2\pi i\tau}\,\frac{\Gamma(\frac{d+1}{2})\Gamma(\frac{1}{2}+\pi i\tau)}{\Gamma(\frac{d+1}{2}-\pi i\tau)}.$$

Now it is easy to see that $\widehat{K_2}(0) = 1$ and $\widehat{K_2}$ has no real root because the Gamma function has none and because all poles of the Gamma function are negative.

Remark 2.6. If D is a bounded full-dimensional locally finite union of sets of finite reach, then the derivative of the covariance from the right is $\overline{\gamma_D}'^+(0) = -(\kappa_{d-1}/\mathrm{d}\kappa_d)H^{d-1}(\partial D)$ [9].

Theorem 2.7. *Let* $\mathbf{T}$ *be a* d*-periodic lattice of points,* $r \in \mathbb{R}^+$*,* D *a bounded measurable set such that a finite* $\overline{\gamma_D}'^+(0)$ *exists and* Φ *the function on* $\mathbb{R}^+$ *defined by the equation*

$$\int_{\mathbf{SO}_d}\int_{F_{\mathbf{T}}} (\mathbf{card}((rMD+x)\cap\mathbf{T}) - \alpha\lambda^d(D))^2\alpha\,\mathrm{d}\lambda^d(x)\,\mathrm{d}M \\ = \Bigg(\sum_{0\neq\xi\in\mathbf{T}^*}|\xi|^{-d-1}\Bigg)\Phi(r)r^{d-1}.$$

Further, let $\varrho^{d+1}\widehat{\overline{\gamma_D}}(\varrho)$ *be bounded on* $\mathbb{R}^+$*. Then* Φ *is bounded and*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \Phi(t)\,\mathrm{d}t = 1. \tag{2.5}$$

If moreover $\lim\limits_{\varrho\to\infty}\varrho^{d+1}\widehat{\overline{\gamma_D}}(\varrho)$ *exists then*

$$\lim_{t\to\infty}\Phi(t) = 1. \tag{2.6}$$

P r o o f. From Remark 2.4 we have

$$\int_{\mathbf{SO}_d}\int_{F_{\mathbf{T}}} (\mathbf{card}((rMD+x)\cap\mathbf{T}) - \alpha\lambda^d(D))^2\alpha\,\mathrm{d}\lambda^d(x)\,\mathrm{d}M = \sum_{0\neq\xi\in\mathbf{T}^*}\widehat{\overline{\gamma_D}}(r|\xi|)r^{2d}.$$

We shall prove first that the auxiliary function Ψ defined by the equation

$$-\overline{\gamma_D}'^{+}(0)\Psi(t) = 2\pi^2\kappa_{d-1}t^{d+1}\widehat{\overline{\gamma_D}}(t)$$

has the property (2.5) and (2.6). It is easy to see that the function

$$\Phi(t) = \frac{\sum\limits_{0\neq\xi\in\mathbf{T}^*} |\xi|^{-d-1}\Psi(t|\xi|)}{\sum\limits_{0\neq\xi\in\mathbf{T}^*} |\xi|^{-d-1}}$$

is bounded and has then the same property (2.5) or (2.6) as the auxiliary function Ψ.

Let $\varrho^{d+1}\widehat{\overline{\gamma_D}}(\varrho)$ be bounded measurable on $\mathbb{R}^+$. From Lemma 2.5, Proposition and Remark 2.4 it follows that

$$\begin{aligned}
&\lim_{R\to\infty}\frac{1}{R}\int_0^R 2\pi^2\kappa_{d-1}\varrho^{d+1}\widehat{\overline{\gamma_D}}(\varrho)\,\mathrm{d}\varrho\\
&\quad= \lim_{h\to 0+}\frac{1}{h}\int_0^\infty (\,\mathrm{d}\kappa_d - 2\pi(h\varrho)^{1-d/2}J_{d/2-1}(2\pi h\varrho))\varrho^{d-1}\widehat{\overline{\gamma_D}}(\varrho)\,\mathrm{d}\varrho\\
&\quad= \lim_{h\to 0+}\frac{1}{h}(\overline{\gamma_D}(0) - \overline{\gamma_D}(h)) = -\overline{\gamma_D}'^{+}(0)
\end{aligned}$$

since the derivative exists by assumption, which proves (2.5). If moreover

$$\lim_{\varrho\to\infty} 2\pi^2\kappa_{d-1}\varrho^{d+1}\widehat{\overline{\gamma_D}}(\varrho)$$

exists then it must be equal to

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R 2\pi^2\kappa_{d-1}\varrho^{d+1}\widehat{\overline{\gamma_D}}(\varrho)\,\mathrm{d}\varrho,$$

which implies (2.6). □

Corollary 2.8. *From Theorem* 2.7 *and Remark* 2.6 *it follows that if D is a bounded full-dimensional locally finite union of sets of finite reach such that $\varrho^{d+1}\widehat{\overline{\gamma_K}}(\varrho)$ is bounded (or has a limit in $+\infty$), then*

$$\begin{aligned}
(2.7)\qquad &\int_{\mathbf{SO_d}}\int_{F_{\mathbf{T}}} (\mathbf{card}((rMD+x)\cap\mathbf{T}) - \alpha\lambda^{\mathbf{d}}(\mathbf{D}))^{\mathbf{2}}\alpha\,\mathrm{d}\lambda^{\mathbf{d}}(\mathbf{x})\,\mathrm{d}\mathbf{M}\\
&\quad= C_{\mathbf{T}}H^{d-1}(\partial D)\Phi(r)r^{d-1},
\end{aligned}$$

where

$$C_{\mathbf{T}} = \frac{1}{2\pi^2\,\mathrm{d}\kappa_d}\sum_{0\neq n\in\mathbb{Z}^d}|A^{-1}n|^{-d-1}$$

is a lattice constant and Φ fulfills (2.5) *(or has limit* 1*).*

3. Discussion

The assumption of Theorem 2.7 and Corollary 2.8, that $\widehat{\overline{\gamma_D}}(\varrho)$ is $O(\varrho^{-d-1})$, $\varrho \to \infty$, hold for convex sets and sets with $C^{3/2}$ boundary (the boundary of the set can be decomposed into finitely many neighbourhoods such that given any pair of points P, Q in the neighbourhood, $|(P-Q)n(Q)| \leqslant c|P-Q|^{3/2}$, where $n(Q)$ is a unit normal to the set in Q) [2]. The asymptotics $\sim r^{d-1}$ of the variance of the lattice point count in the mean value (2.7) thus holds for sets with piecewise $\mathbf{C^2}$ smooth boundary and convex sets in $\mathbb{R}^d$.

As was already said in the introduction, Theorem 2.7 gives results similar to Theorem 2.8 in [3] for a compact body D with smooth isotropic covariogram $\overline{\gamma}_D$. Slightly different situation is studied in [3]: locally finite periodic measure is studied instead of the point lattice and the size of the body is fixed while the scale s of the lattice tends to zero and the counting measure is multiplied by the factor s^d; consequently, the asymptotics $\sim s^{-d-1}$ are obtained there. As in [3] we may generalize the results of the present paper to locally finite periodic measures. In [3], lattice constants $C_{\mathbf{T}}$ in Equation 2.7 were calculated for some important point lattices.

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