Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 3, 669–681

Persistent URL: `http://dml.cz/dmlcz/140412`

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DIRECTOID GROUPS

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(Received June 21, 2006)

Abstract. We continue the study of directoid groups, directed abelian groups equipped with an extra binary operation which assigns an upper bound to each ordered pair subject to some natural restrictions. The class of all such structures can to some extent be viewed as an equationally defined substitute for the class of (2-torsion-free) directed abelian groups. We explore the relationship between the two associated categories, and some aspects of ideals of directoid groups.

Keywords: directed abelian group, variety

MSC 2010: 06F20, 08B99

1. Introduction

Ježek and Quackenbush [9] introduced *directoids*, not necessarily commutative groupoids which correspond to up-directed sets in the same way as semilattices (commutative semigroups of idempotents) correspond to partially ordered sets in which each pair has a supremum. The authors in [4] began a similar algebraic study of directed groups. We subsequently discovered that the motivating ideas of both [9] and [4] had been anticipated in a slightly earlier paper of Kopytov and Dimitrov [10]. For related ideas see [12], [14], [15]. In this paper we further develop the theory for directed groups.

Thus we consider *directoid groups* (formal definition below), directed abelian groups with a binary operation which assigns an upper bound to each couple and is compatible with the group addition in the way the supremum operation is in lattice-ordered abelian groups. Directoid groups thus represent both an attempt to "equationalize" directed abelian groups and a generalization of abelian l-groups. These two aspects together provide the principal motivation for this study, and various questions involving the three categories—directoid groups, directed abelian groups and abelian l-groups—will be addressed. For instance while for abelian l-groups kernels are precisely convex l-subgroups, the more complicated but analogous role of convex directed subgroups in directoid groups is elucidated, the relationship between directoid group homomorphisms and order homomorphisms is described. In marked contrast to the case of abelian l-groups, the lattice of subvarieties of directoid groups is quite complicated. A non-trivial example of a variety—the class of directoid groups whose one-generator subobjects are l-groups—was given in [10]. In the context of comparisons between our three categories, it is natural to ask whether there are any varieties of directoid groups whose membership is characterized by order alone, rather than involving the directoid operation, analogous to the now well studied *e-varieties* of regular semigroups. There are, but we shall defer the discussion of this question and varieties in general to another paper.

Our notational conventions are consistent with those of [1] and [3]; note, however, that the symbol $\|$ denotes incomparability with respect to a partial order of any kind.

Because of conflicting terminology arising from the independent introduction of ideas, and because we have elected to treat only commutative directoids and only 2-torsion-free abelian groups (in the former case because it seems natural, in the latter by virtual necessity) we shall begin by defining some terms as we shall use them.

We shall call a groupoid D a *directoid* if it satisfies the identities $xx \approx x; xy \approx yx; (xy)x \approx xy; x((xy)z) \approx (xy)z$. If we define $a \leqslant b$ to mean $ab = b$, D becomes an up-directed partially ordered set ($a, b \leqslant ab$). Conversely we can make any up-directed set into a directoid by defining $x \cdot y$ to be y if $x \leqslant y$, x if $y \leqslant x$ and otherwise to be any chosen upper bound of x, y as long as $x \cdot y = y \cdot x$. Of course (except when $\leqslant$ is linear) lots of different directoid structures will correspond to a given order. All of this was established by Kopytov and Dimitrov [10] and Jezek and Quackenbush [9]. What we call a directoid was called a *commutative directoid* in [9], while in [10] sets which are both up—and down-directed were associated with structures carrying two binary operations related by absorption laws.

In [4] we considered directed abelian groups from a similar point of view. Here we recall the definition and give a little more detail of the precise connection between our structures and directed abelian groups.

(An earlier 'algebraic' approach to directed groups, that of Fuchs [2], used the set of all upper bounds of $\{x, y\}$ instead of selecting an upper bound and thereby defining an operation. This approach was also used in a related context by McAlister [13]. On the other hand, choosing elements so as to define an extra *unary* operation is an established technique in the theory of regular semigroups (see Hall [5] for instance) and this is quite analogous to what we are doing here.)

A *directoid group* is an abelian group G with a directoid operation $\cdot$ such that $a + b \cdot c = (a + b) \cdot (a + c)$ for all $a, b, c \in G$.

Theorem 1.1. (i) *Every directoid group is a directed group with respect to the directoid order i.e. $a \leqslant b$ if and only if $a \cdot b = b$.*

(ii) *Directoid groups have no elements of order 2 ([4], [10]).*

(iii) *Conversely, every 2-torsion-free abelian directed group can be made into a directoid group by a directoid operation which defines its order.*

For the proof of (iii) (and at many points throughout the paper) we make use of

Lemma 1.2. *Every 2-torsion-free abelian group G has a subset M such that $G = \{0\} \mathbin{\dot\cup} M \mathbin{\dot\cup} \{-m\colon\ m \in M\}$ (disjoint union). If G is partially ordered, M can be chosen to contain the positive elements.*

Proof. If $G = \{0\}$, let $M = \emptyset$. If not, let

$$\mathscr{F} = \{S\colon\ S \subseteq G \setminus \{0\}; x \in S \Rightarrow -x \notin S\}.$$

Then $\{a\} \in \mathscr{F}$ for every $a \neq 0$ as a can't have order 2. By Zorn's Lemma, $\mathscr{F}$ has a maximal member M. Suppose G has an element $b \neq 0$ such that $b, -b \notin M$. Then $M \cup \{b\} \in \mathscr{F}$, contradicting the maximality of M. Thus $M \cup \{-m\colon\ m \in M\} = G \setminus \{0\}$. For the second assertion, we use the set of positive elements instead of $\{a\}$ to start the Zorn's Lemma argument. $\square$

Proof of 1.1. (i) Let G be a directoid group. Then certainly G is an up-directed set with respect to the order $\leqslant$ defined by the directoid operation. If $a \leqslant b$ then $a \cdot b = b$ so for every c we have $(a + c) \cdot (b + c) = a \cdot b + c = b + c$ so that $a + c \leqslant b + c$.

(ii) If $2a = 0$, then $a + 0 \cdot a = a + 0 \cdot (-a) = a \cdot 0 = 0 \cdot a$ so $a = 0$.

(iii) Let G be a 2-torsion-free directed group, M a subset as described in 1.2 containing all positive elements. We define a binary operation $\cdot$ on G in several steps. If $a \in M$ we let $a \cdot 0 = 0 \cdot a = a$ if $a > 0$ and otherwise we let $a \cdot 0 = 0 \cdot a$ be any chosen upper bound of $\{a, 0\}$. Then we set $(-a) \cdot 0 = 0 \cdot (-a) = -a + a \cdot 0$ for all $a \in M$. (Note that if $-a < 0$, i.e. $a > 0$, then $(-a) \cdot 0 = 0$.) Since $0 \leqslant a \cdot 0$ we have $-a \leqslant (-a) \cdot 0$, and since $a \leqslant a \cdot 0$ we have $0 = -a + a \leqslant -a + a \cdot 0 = 0 \cdot (-a) = (-a) \cdot 0$. Of course we set $0 \cdot 0 = 0$ and now we have defined $g \cdot 0$ ($=0 \cdot g$) for every $g \in G$. Finally, we set $d \cdot c = c + (d - c) \cdot 0$ for all $c, d \in G$.

If $d = c$, then $c \cdot c = c + 0 \cdot 0 = c$. If $d - c \in M$, then $c \cdot d = d + (c - d) \cdot 0 = d + (c - d) + (d - c) \cdot 0 = c + (d - c) \cdot 0 = d \cdot c$, while if $d - c \notin M$ and $d \neq c$, then $c - d \in M$ so $d \cdot c = c \cdot d$. Thus $\cdot$ is idempotent and commutative. Since $(d - c) \cdot 0 \geqslant 0$

we have $c \leqslant c + (d-c) \cdot 0 = d \cdot c$ and similarly $d \leqslant c \cdot d = d \cdot c$ for all $c, d \in G$. If $c < d$ then $(d-c) > 0$ so $d \cdot c = c + (d-c) \cdot 0 = c + (d-c) = d$. We therefore have a commutative binary operation which assigns to each (c,d) an upper bound of $\{c,d\}$ and $c \cdot d = \max\{c,d\}$ if c and d are comparable. Hence G is a directoid with respect to $\cdot$. For every $g, h, l \in G$ we have $(g+h) \cdot (g+l) = (g+l) + (g+h-(g+l)) \cdot 0 = (g+l) + (h-l) \cdot 0 = g + (l + (h-l) \cdot 0) = g + h \cdot l$ so G is a directoid group. $\square$

It will be noted that commutativity of addition is explicitly used in the proof just given. One could modify the definition of directoid group by removing abelianness, but there are non-abelian torsion-free directed groups which do not admit such a structure (see [10], Examples, 6.1).

A directed group is both up—and down-directed and just as a lattice group has both meet and join, so too in a directoid group we have another binary operation defined by $(a,b) \mapsto -((-a) \cdot (-b))$ which assigns lower bounds. *We shall normally call this operation* $\circ$. Thus $a \circ b = -((-a) \cdot (-b))$ and $a \cdot b = -((-a) \circ (-b))$. We have $a + b = a \cdot b + a \circ b$ ([4], Proposition 2.6 (i)) and this resemblance to the operation defining quasiregularity in ring theory motivates our choice of notation.

2. Ideals

The class of directoid groups is a variety. Moreover, as $0 \cdot 0 = 0$, directoid groups are *multioperator groups* (see, e.g., [9] or [15]). Adopting the usage of multioperator group theory we shall call a kernel of a directoid group homomorphism an *ideal*. It is convenient to recall the characterization of ideals.

Proposition 2.1 ([4], [10]). *Let G be a directoid group, $H \subseteq G$. The following conditions are equivalent.*

(i) *H is a subgroup of G satisfying*

$$x - y, z - w \in H \Rightarrow x \cdot z - y \cdot w \in H.$$

(ii) *H is a subgroup of G satisfying*

$$x \in G, y \in H \Rightarrow (x+y) \cdot 0 - x \cdot 0 \in H.$$

(iii) *H is an ideal of G.*

Ideals are directoid subgroups and are convex.

If H is an ideal and $a, b \in H$, then $a \cdot b = a \cdot b - 0 \cdot 0 \in H$. Everything else can be obtained from the cited papers.

In (abelian) l-groups an l-ideal is the same thing as a convex l-subgroup. However, a convex directoid subgroup need not be an ideal of a directoid group: an example was given by Jakubík [8]. (There are some misprints in the account of that example and Professor Jakubík has indicated to the authors that the following changes need to be made. In 2.2 Theorem (proof) the defining condition for H should be $d = r_1 = 0$, while in the paragraph before the theorem the second case of b_1 should be $(0, r_1 + |r_2|, 0)$). So the ideals of a directoid group are among the directed convex subgroups. Since a directoid group must be 2-torsion-free an ideal H of a directoid group G must be 2-pure, i.e. we must have $2H = H \cap 2G$. These conditions, conversely, guarantee that a subgroup of a 2-torsion-free directed abelian group is an ideal for some directoid group structure with that order.

Theorem 2.2. *Let G be a 2-torsion-free abelian directed group, H a directed convex subgroup of G such that $H \cap 2G = 2H$. Then G has a directoid group structure for which H is an ideal. In fact every directoid group structure on H extends to one on G for which H is an ideal.*

It will be useful to have the following result.

Lemma 2.3 (Notation as in 2.2)**.**
(i) *If $a \in G$ and $a \equiv -a \pmod H$, then $a \in H$.*
(ii) *If $a, b \in G$, $a > 0$ and $b < 0$, then $a \equiv b \pmod H$ if and only if $a, b \in H$.*

P r o o f. (i) simply says that G/H is 2-torsion-free, and this is equivalent to $2H = H \cap 2G$ as G is 2-torsion-free.

(ii) We have $b < 0 < a$, so $0 < -b < a - b$. If $a - b \in H$, then by convexity $-b \in H$ so $b \in H$ and thus $a = a - b + b \in H$. $\square$

P r o o f o f 2.2. The conditions imposed tell us that G/H is a 2-torsion-free abelian directed group with an order induced by that of G. As in 1.2 let $\tilde{M}$ be a set in G/H such that

$$G/H = \{0\} \mathbin{\dot\cup} \tilde{M} \mathbin{\dot\cup} \{-x\colon\ x \in \tilde{M}\}$$

and $\tilde{M}$ contains all the positive elements. Let N be a similar set for H. Then

$$\begin{aligned} G &= \{0\} \mathbin{\dot\cup} (H \setminus \{0\}) \mathbin{\dot\cup} (G \setminus H) \\ &= \{0\} \mathbin{\dot\cup} N \mathbin{\dot\cup} \{-n\colon\ n \in N\} \mathbin{\dot\cup} \{g\colon\ g + H \in \tilde{M}\} \mathbin{\dot\cup} \{g\colon\ -g + H \in \tilde{M}\}. \end{aligned}$$

If $g \in G \setminus H$ and $g > 0$, then $g + H > 0$ so $g + H \in \tilde{M}$. Let $M = N \cup \{g\colon\ g + H \in \tilde{M}\}$. Then $G = \{0\} \mathbin{\dot\cup} M \mathbin{\dot\cup} \{-m\colon\ m \in M\}$ and M contains all positive elements of G. Also, if $m \in M \setminus N$ and $m \equiv m' \pmod H$ then $m' + H = m + H \neq 0$ so $m' \in M$.

Suppose we are given a binary operation $\cdot$ which makes H a directoid group. We seek an extension of $\cdot$ to G and by the proof of 1.1(iii) we only need to define $m \cdot 0$ for all $m \in M$ (as $n \cdot 0$ is already defined for $n \in N$).

Let a be in $M \setminus N$. If $a + H$ contains a positive element, we may assume $a > 0$. Then we set $a \cdot 0 = a$ and for $h \in H$ let

$$(a+h) \cdot 0 = \begin{cases} a+h & \text{if} \quad a+h > 0, \\ a + h \cdot 0 & \text{otherwise.} \end{cases}$$

Now for $h, k \in H$ we have

$$\begin{aligned} (1) \quad ((a+h)+k) \cdot 0 - (a+h) \cdot 0 &= a+h+k-(a+h) \text{ or } a+h+k-(a+h\cdot 0) \\ &\text{or } a+(h+k)\cdot 0-(a+h) \text{ or } a+(h+k)\cdot 0-(a+h\cdot 0) = k \\ &\text{or } h+k-h\cdot 0 \text{ or } (h+k)\cdot 0-h \text{ or } (h+k)\cdot 0-h\cdot 0 \in H. \end{aligned}$$

If every element of $a + H$ is incomparable with 0 (a still being in $M \setminus N$) we can let $a \cdot 0$ be any suitable upper bound of a and 0 and then let

$$(a+h) \cdot 0 = (a \cdot 0) + (h \cdot 0)$$

for each $h \in H$. Then for $h, k \in H$ we have

$$(2) \quad ((a+h)+k)\cdot 0-(a+h)\cdot 0 = a\cdot 0+(h+k)\cdot 0-(a\cdot 0+h\cdot 0) = (h+k)\cdot 0-h\cdot 0 \in H.$$

We can now extend our directoid operation to G by the procedure in the proof of 1.1(iii). We have also verified 2.1(ii) for members of $M \setminus N$.

Returning to a positive $a \in M \setminus N$, if $h, k \in H$ we have (using (1) at the appropriate point)

$$\begin{aligned} (3) \quad (-a+h+k)\cdot 0-(-a+h)\cdot 0 &= (-(a-h-k))\cdot 0-(-(a-h))\cdot 0 \\ &= (a-h-k)\cdot 0-(a-h-k)-(a-h)\cdot 0+(a-h) \\ &= (a-h-k)\cdot 0-(a-h)\cdot 0+k \in H. \end{aligned}$$

If $a \in M \setminus N$ and $a + H$ contains no positive elements (so that by 2.3(ii) all its elements are incomparable with 0) then for $h, k \in H$ we have

(4)

$$\begin{aligned} (-a+h+k)\cdot 0-(-a+h)\cdot 0 &= (-(a-h-k))\cdot 0-(-(a-h))\cdot 0 \\ &= (a-h-k)\cdot 0-(a-h-k)-(a-h)\cdot 0+(a-h) \\ &= (-h-k)\cdot 0+k-(-h)\cdot 0 \in H. \end{aligned}$$

(We have used (2).) Now (1)–(4) give us 2.1(ii) for all elements of non-zero cosets, i.e. for all elements of $G \setminus H$. But of course this condition is trivial for elements of H, so the proof is complete. $\square$

In abelian l-groups we have *transitivity of normality*: l-ideals of l-ideals are l-ideals. This cannot be generalized to directoid groups, however. In our account of an example demonstrating this (and elsewhere) the following result will be useful.

Proposition 2.4. *Let G be a 2-torsion-free abelian directed group, $g \in G$ and $g \parallel 0$. If $g \leqslant b$ and $0 \leqslant b$, then G has a directoid group structure for which $g \cdot 0 = b$.*

Proof. Let M be a subset of G as in 1.2 with $G = \{0\} \mathbin{\dot\cup} M \mathbin{\dot\cup} \{-m \colon m \in M\}$. Then $g \in M$ or $-g \in M$ and with a minor adjustment we can assume $g \in M$. As in the proof of 1.1(iii) we can make $g \cdot 0 = b$. $\square$

Now for the example.

Example 2.5. Consider $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ with the standard componentwise order, $H = \mathbb{Z} \times \mathbb{Z} \times 0$, $K = \mathbb{Z} \times 0 \times 0$. Let M be a subset as in 1.2. Then $(-1,0,1) \parallel (0,0,0)$ and as in 2.4 we can assume that $(-1,0,1) \in M$ and define a directoid group structure on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ by setting

$$(-1,0,1) \cdot (0,0,0) = (1,1,1)$$

and

$$(a,b,c) \cdot (0,0,0) = (a,b,c) \vee (0,0,0) \text{ for all other } (a,b,c) \in M.$$

Then $(1,0,-1) \cdot (0,0,0) = (-1,0,1) \cdot (0,0,0) + (1,0,-1) = (1,1,1) + (1,0,-1) = (2,1,0)$ and otherwise $(-a,-b,-c) \cdot (0,0,0) = (a,b,c) \cdot (0,0,0) + (-a,-b,-c) = (a,b,c) \vee (0,0,0) - (a,b,c) = (-a,-b,-c) \vee (0,0,0)$. (Thus $x \cdot 0 = x \vee 0$ if $x \neq \pm(-1,0,1)$.) For every $u, v \in \mathbb{Z}$ we have

$$\begin{aligned}&[(-1,0,1) + (u,v,0)] \cdot (0,0,0) - (-1,0,1) \cdot (0,0,0)\\ &\qquad = (u-1,v,1) \cdot (0,0,0) - (-1,0,1) \cdot (0,0,0) = (*,*,1) - (1,1,1) \in H\end{aligned}$$

and

$$\begin{aligned}&[(1,0,-1) + (u,v,0)] \cdot (0,0,0) - (1,0,-1) \cdot (0,0,0)\\ &\qquad = (u+1,v,-1) \cdot (0,0,0) - (1,0,-1) \cdot (0,0,0) = (*,*,0) - (2,1,0) \in H.\end{aligned}$$

Here the asterisks indicate irrelevant values. Taking any other (r,s,t), we get

$$\begin{aligned}&[(r,s,t) + (u,v,0)] \cdot (0,0,0) - (r,s,t) \cdot (0,0,0)\\ &\qquad = (r+u,s+v,t) \cdot (0,0,0) - (r,s,t) \cdot (0,0,0)\\ &\qquad = (r+u,s+v,t) \vee (0,0,0) - (r,s,t) \vee (0,0,0) \in H\end{aligned}$$

unless $(r+u, s+v, t) = (-1, 0, 1)$ or $(1, 0, -1)$. In the former case the difference is still in H as both terms have third component 1. On the other hand if $(r+u, s+v, t) = (1, 0, -1)$, then

$$(r+u, s+v, t) \cdot (0,0,0) = (2,1,0)$$

and

$$(r, s, t) \cdot (0,0,0) = (r, s, -1) \cdot (0,0,0) = (*, *, 0)$$

so again the difference is in H. By 2.1(ii) H is an ideal of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Now H is an l-group and K a direct factor, whence an l-ideal and so an ideal of H *qua* directoid group (e.g. because l-groups form a variety of directoid groups; in any case, direct factors of multioperator groups are ideals). However,

$$\begin{aligned}&[(0,0,1) + (-1,0,0)] \cdot (0,0,0) - (0,0,1) \cdot (0,0,0)\\ &\qquad = (-1,0,1) \cdot (0,0,0) - (0,0,1) \cdot (0,0,0) = (1,1,1) - (0,0,1) = (1,1,0) \notin K,\end{aligned}$$

so K is not an ideal of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Of course if we use the product-of-l-groups structure of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, then K *is* an ideal. More generally, it will be fairly clear that the freedom of choice which we have in setting up a directoid group structure, while allowing us to "make into an ideal" a given directed convex subgroup, can make it just as easy to disqualify the subgroup. Nevertheless, in the context of trying to make directoid groups an "equational substitute" for directed groups it is natural to seek out ideals which work as such in a directed group regardless of the directoid operation considered. The (ideal defining the) minimum lattice congruence, for example, might be one to look at. It turns out, however, that such "absolute ideals" are quite uncommon.

Proposition 2.6. *Let G be a 2-torsion-free abelian directed group, H a proper subgroup of G. The following conditions are equivalent.*

(i) *H is a directoid subgroup for every directoid group structure on G.*

(ii) *H is linearly ordered.*

Proof. $\neg$(ii) $\Rightarrow \neg$(i): Let a be in H with $a \parallel 0$. Let b be an element of G with $a, 0 \leqslant b$. By 2.4 G has a directoid group operation $\cdot$ for which $a \cdot 0 = b$. Were H to be a directoid subgroup, we would have $b \in H$. But for every positive $c \in G$ we also have $a, 0 \leqslant b + c$ so similarly $b + c \in H$, whence $c \in H$. Thus we would have $G = H$, as G is generated by its positive elements. The converse is clear. $\square$

Theorem 2.7. *Let G be a 2-torsion-free abelian directed group, H a subgroup such that*

(i) $\{0\} \subset H \subset G$ *(proper inclusions) and*

(ii) H *is an ideal in every directoid group on G.*

Then G is linearly ordered. Conversely every convex subgroup of a linearly ordered 2-torsion-free abelian group is an ideal in the unique directoid group on that group.

Proof. Let G and H be as described. First suppose that there are elements $g \in G \setminus H$, $h \in H \setminus \{0\}$ with $(g+h) \parallel 0$. Let $*$ be a directoid group operation on G. Then by 2.1(ii),

$$(g+h) * 0 - g * 0 \in H.$$

Now take any $a \in G$ with $a > 0$. Then $0, g+h \leqslant (g+h) * 0 < a + (g+h) * 0$ and we can now (by 2.4) define a new directoid group on G with an operation $\sharp$ for which

$$(g+h) \sharp 0 = a + (g+h) * 0.$$

Now $g \neq (g+h)$ and if $-g$ were $g+h$ we would have $2g = -h \in H$ so (as H is an ideal) $g \in H$-a contradiction. Hence we can put both g and $g+h$ in a set M as in 1.2, and this allows us to define $\sharp$ so that $g \sharp 0 = g * 0$ if $g \parallel 0$. But if $g \geqslant 0$ or $g \leqslant 0$, then $g \sharp 0 = g = g * 0$ or $g \sharp 0 = 0 = g * 0$ respectively. We now have

$$(g+h) \sharp 0 - g \sharp 0 = a + (g+h) * 0 - g * 0$$

and since H is an ideal with respect to both $\sharp$ and $*$ it follows that $a \in H$. Since a is anything > 0 this means that $G = H$. From the resultant contradiction we conclude that

$$\text{for every } g \in G \setminus H \text{ and every } h \in H \setminus \{0\}, \text{ either } g+h > 0 \text{ or } g+h < 0.$$

Now 2.6 tells us already that H is linearly ordered. If $g \in G \setminus H$ and $h \in H \setminus \{0\}$ then $g - h \in G \setminus H$ and $g = (g-h) + h$, so by the above argument $g > 0$ or $g < 0$. Thus G is linearly ordered.

Conversely, if G is a linearly ordered group, H a convex subgroup and the unique directoid group operation on G is called $\cdot$, then for $g \in G$, $h \in H$ we have

$$(g+h) \cdot 0 - g \cdot 0 = \begin{cases} g+h-g = h & \text{if } g+h,\ g \geqslant 0, \\ g+h-0 = g+h & \text{if } g+h \geqslant 0,\ g \leqslant 0, \\ 0-g = -g & \text{if } g+h \leqslant 0,\ g \geqslant 0, \\ 0-0 = 0 & \text{if } g+h,\ g \leqslant 0. \end{cases}$$

But if $g \leqslant 0 \leqslant g+h$, then $0 \leqslant -g \leqslant h$ so $-g \in H$ and thus $g+h \in H$, while if $g+h \leqslant 0 \leqslant g$ then $h \leqslant -g \leqslant 0$ so $-g \in H$. Thus H is an ideal. $\square$

Corollary 2.8. *The minimum l-group congruences on all directoid groups on G coincide if and only if G is linearly ordered (so that there is only one directoid group and the congruence is zero).*

We end this section by examining the relationship between the categories of (2-torsion-free abelian) directed groups and directoid groups, the morphisms in the former case being the order-preserving group homomorphisms and in the latter the directoid group homomorphisms. We precede the characterization of the directoid group homomorphisms among the order-preserving ones with two simple but useful results. The first is analogous to 2.2 but much more straightforward.

Proposition 2.9. *Let H be a directed subgroup of a 2-torsion-free abelian directed group. Then every directoid group structure on H extends to one on G making H a directoid subgroup.*

Proof. Let $G = \{0\} \mathbin{\dot\cup} M \mathbin{\dot\cup} \{-m\colon\ m \in M\}$. Then

$$H = \{0\} \mathbin{\dot\cup} (M \cap H) \mathbin{\dot\cup} \{-m\colon\ m \in M \cap H\}.$$

All we need to do is assign appropriate values for $m \cdot 0$, $m \in M$, taking care that $m \cdot 0$ is defined by the existing directoid structure on H whenever $m \in M \cap H$. $\square$

If G is a partially ordered group, $f\colon\ G \to H$ a group homomorphism with convex kernel, then we can make H a partially ordered group by defining $h_1 \leqslant h_2$ if and only if there exist $g_1, g_2 \in G$ such that $g_1 \leqslant g_2$, $f(g_1) = h_1$ and $f(g_2) = h_2$. As there does not seem to be a standard name for it, we shall call this induced order on H the *quotient order defined by f*.

Proposition 2.10. *Let $f\colon\ G_1 \to G_2$ be a surjective homomorphism of directoid groups. Then the order of G_2 is the quotient order defined by f.*

Proof. If $c \leqslant d$, $f(r) = c$ and $f(s) = d$, then $f(r \cdot s) = f(r) \cdot f(s) = c \cdot d = d$ and $r \leqslant r \cdot s$, so we can take $a = r$ and $b = r \cdot s$. Conversely, if $f(a) = c, f(b) = d$ and $a \leqslant b$, then $a \cdot b = b$ so

$$d = f(b) = f(a \cdot b) = f(a) \cdot f(b) = c \cdot d$$

and thus $c \leqslant d$. $\square$

Theorem 2.11. *Let G_1, G_2 be 2-torsion-free abelian directed groups, $f\colon G_1 \to G_2$ an order homomorphism. Then G_1 and G_2 carry directoid group structures for which f is a directoid group homomorphism if and only if the restriction to* $\mathrm{Im}(f)$ *of the order of G_2 is the quotient order defined by f and* $\mathrm{Ker}(f)$ *is directed.*

Proof. Let f be a directoid group homomorphism for directoid groups on G_1, G_2. Then $\mathrm{Ker}(f)$ is an ideal and so is directed. The other required property of f is given by 2.10 (as $\mathrm{Im}(f)$, of course, is a directed subgroup of G_2). Conversely, if f satisfies the stated conditions, then $\mathrm{Ker}(f)$ is both convex and directed and so by 2.2 there is a directoid group structure on G_1 for which $\mathrm{Ker}(f)$ is an ideal. We denote its directoid operation by $\cdot$. Let $\leqslant$ denote the given orders on both G_1 and G_2. If $a, b \in G_1$ let $t \in G$ be such that $a, b \leqslant t$. Then $f(a), f(b) \leqslant f(t)$ so $\mathrm{Im}(f)$ is directed. For each $c, d \in \mathrm{Im}(f)$ set $c \cdot d = f(r \cdot s)$ for any $r, s \in G_1$ for which $f(r) = c$ and $f(s) = d$. If $f(r) = f(r') = c$ and $f(s) = f(s') = d$, then $r - r', s - s' \in \mathrm{Ker}(f)$ so (as $\mathrm{Ker}(f)$ is known to be an ideal) $r \cdot s - r' \cdot s' \in \mathrm{Ker}(f)$ and thus $\cdot$ is well-defined. But if $a, b \in G_1$ then $f(a) \cdot f(b) = f(a \cdot b)$. From this it follows that $\mathrm{Im}(f)$ is a directoid group and f induces a directoid group homomorphism $G_1 \to \mathrm{Im}(f)$. By 2.9 we can extend the directoid operation of $\mathrm{Im}(f)$ to G_2 and $f\colon G_1 \to G_2$ then becomes a directoid group homomorphism. $\square$

3. Examples

We now present a gallery of examples of directoid groups, which will be used to illustrate concepts and results from earlier sections.

Example 3.1. Let G be a 2-torsion-free abelian group which is directed with respect to an order $\leqslant$ and let r, s be relatively prime positive integers. We define a new order $\preceq$ on G as follows.

$$a \preceq b \text{ if and only if } b = a + rg + sh,\ g, h \in G,\ g \geqslant 0,\ h \geqslant 0.$$

We denote the positive cones of G with respect to $\leqslant$, $\preceq$ by $G^+(\leqslant)$, $G^+(\preceq)$ respectively. Note that $G^+(\preceq) = \{rg + sh\colon g, h \in G^+(\leqslant)\}$. Let m, k be integers such that $mr + ks = 1$. If $a \in G^+(\leqslant)$, then $a = rma + ska \in G^+(\preceq) - G^+(\preceq)$. If $c \in G$ we have $c = a - d$ for some $a, d \in G^+(\leqslant)$ whence $c \in G^+(\preceq) - G^+(\preceq)$. Hence G is directed with respect to $\preceq$. We consider a special case: $G = \mathbb{Z}, (r, s) = (n, n+1)$ for some $n \in \mathbb{Z}^+$. We shall call the resulting directed group $\mathbb{Z}^{(n)}$. For every $a, b \in \mathbb{Z}$ with $a \geqslant b$ we have $a - b \geqslant 0$ so $n(a - b) \succeq 0$, i.e. $na \succeq nb$. Hence $\preceq$ is linear on $n\mathbb{Z}$ so regardless of the directoid group operation we put on $\mathbb{Z}^{(n)}$, $n\mathbb{Z}^{(n)}$ is a linearly ordered directoid subgroup.

The identity homomorphism $\mathbb{Z}^{(n)} \to \mathbb{Z}$ (where $\mathbb{Z}$ has the standard order) is order-preserving but not a directoid group homomorphism (for any operations on $\mathbb{Z}^{(n)}, \mathbb{Z}$) as the order on $\mathbb{Z}$ is not the quotient order (even though the kernel is trivially directed (2.11).

Example 3.2. Let $\mathbb{Z}^0$ denote the integers with the discrete order (i.e. equality) and let $\mathbb{Z} * \mathbb{Z}^0$ denote the lexicographic product, where $\mathbb{Z}$ (as opposed to $\mathbb{Z}^0$) carries the standard order. Thus $(m,k) \leqslant (m',k')$ means $m < m'$ or $m = m'$ and $k = k'$. We can make $\mathbb{Z} * \mathbb{Z}^0$ into a directoid group by setting $(0,a) \cdot (0,0) = (a,0)$ and $(0,-a) \cdot (0,0) = (a,-a)$ for all $a > 0$. (If $(m,k) \parallel (0,0)$ then $m = 0$.)

The natural homomorphism $\mathbb{Z} * \mathbb{Z}^0 \to \mathbb{Z}$ preserves order, and the natural order on $\mathbb{Z}$ is the quotient order. However the kernel $\mathbb{Z}^0$ is not directed, so the map can't be made a directoid group homomorphism by 2.11.

Example 3.3. Let $\mathbb{Z}_n$ be the group of integers modulo n, $\mathbb{Z}_n^0$ this group with the discrete order, $H^{(n)}$ the lexicographic product $\mathbb{Z} * \mathbb{Z}_n^0$ This is directed.

As in the previous example, the natural map $\mathbb{Z} * \mathbb{Z}_n^0$ can't be made a directoid group map by 2.11.

Example 3.4 (Generalized Jaffard group)**.** For a positive integer n, let

$$J_n = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \colon\ a \equiv b \pmod{n}\}.$$

If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $(a-c) - (b-d) = (a-b) - (c-d)$ so $a - c \equiv b - d \pmod{n}$ and thus $(a-c, b-d) \in J_n$ whenever $(a,b),(c,d) \in J_n$. Thus J_n is a subgroup of $\mathbb{Z} \times \mathbb{Z}$. It is partially ordered by the product order on $\mathbb{Z} \times \mathbb{Z}$. If $(a,b),(c,d) \in J_n$, then $a, c \leqslant |a| + |c| \leqslant n(|a| + |c|)$ and $b, d \leqslant |b| + |d| \leqslant n(|b| + |d|)$. Hence

$$(a,b),(c,d) \leqslant (n(|a| + |c|\,, |b| + |d|)) \in J_n$$

and J_n is directed. (The case $n = 2$ is an example of Jaffard [7] which is treated as a directoid group in Example 2.10 of [4].)

Let $f\colon J_n \to \mathbb{Z}$ be given by $f(a,b) = a$. If $r, s \in \mathbb{Z}$ and $r \leqslant S$, then e.g. $r = f(r,r), s = f(s,s)$ with $(r,r) \leqslant (s,s)$ so $\mathbb{Z}$ has the quotient order. We have $\mathrm{Ker}(f) = \{(a,b)\colon\ a \equiv b \pmod{n}\}$ and $a = 0\} = \{(0,b)\colon\ n \mid b\}$, and this is directed. If $(0,nc) \leqslant (x,y) \leqslant (0,nd)$, then $0 \leqslant x \leqslant 0$ so $x = 0$ whence (as $(x,y) \in J_n$) $n|y$. This shows that $\mathrm{Ker}(f)$ is convex, so by 2.11 f can be made a directoid group homomorphism and $\mathrm{Ker}(f)$ will then be an ideal. Note that the latter can also be deduced from 2.2; we just point out the proof of 2-purity. If $(0,nc) = 2(a,b) = (2a,2b)$, then $a = 0$, so $n \mid b$. Let $b = nb'$. Then $(0,nc) = 2(0,b) = 2(0,nb')$.

If G is one of the groups in Examples 3.1–3.4, then nG is linearly ordered and consequently G satisfies identities such as

$$(nx \cdot ny) \cdot nz \approx nx \cdot (ny \cdot nz).$$

This observation is important for the study of varieties of directoid groups, which we shall pursue elsewhere.

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