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2-NORMALIZATION OF LATTICES

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Abstract. Let τ be a type of algebras. A valuation of terms of type τ is a function v assigning to each term t of type τ a value $v(t) \geqslant 0$. For $k \geqslant 1$, an identity $s \approx t$ of type τ is said to be k-normal (with respect to valuation v) if either $s = t$ or both s and t have value $\geqslant k$. Taking $k = 1$ with respect to the usual depth valuation of terms gives the well-known property of normality of identities. A variety is called k-normal (with respect to the valuation v) if all its identities are k-normal. For any variety V, there is a least k-normal variety $N_k(V)$ containing V, namely the variety determined by the set of all k-normal identities of V. The concept of k-normalization was introduced by K. Denecke and S. L. Wismath in their paper (Algebra Univers., 50, 2003, pp.107-128) and an algebraic characterization of the elements of $N_k(V)$ in terms of the algebras in V was given in (Algebra Univers., 51, 2004, pp. 395–409). In this paper we study the algebras of the variety $N_2(V)$ where V is the type $(2, 2)$ variety L of lattices and our valuation is the usual depth valuation of terms. We introduce a construction called the 3-*level inflation* of a lattice, and use the order-theoretic properties of lattices to show that the variety $N_2(L)$ is precisely the class of all 3-level inflations of lattices. We also produce a finite equational basis for the variety $N_2(L)$.

Keywords: 2-normal identities, lattices, 2-normalized lattice, 3-level inflation of a lattice

MSC 2010: 08A40, 08B15, 06B20

1. Introduction

Let $\tau = (n_i)_{i \in I}$ be any type of algebras, with an operation symbol f_i of arity n_i for each $i \in I$. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a set of variable symbols, and let $W_\tau(X)$ be the set of all terms of type τ formed using variables from X. We will use the well-known Galois connection Id-Mod between classes of algebras and sets of identities.

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For any class K of algebras of type τ and any set Σ of identities of type τ, we have

$$\operatorname{Mod}\Sigma = \{\text{algebras } \mathscr{A} \text{ of type } \tau\colon\ \mathscr{A} \text{ satisfies all identities in } \Sigma\}, \quad \text{and}$$
$$\operatorname{Id} K = \{\text{identities } s \approx t \text{ of type } \tau\colon\ \text{all algebras in } K \text{ satisfy } s \approx t\}.$$

For each $t \in W_\tau(X)$, we denote by $v(t)$ the depth of t, defined inductively by
(i) $v(t) = 0$, if t is a variable;
(ii) $v(t) = 1 + \max\{v(t_j)\colon\ 1 \leqslant j \leqslant n_i\}$, if t is a composite term $t = f_i(t_1, \dots, t_{n_i})$.

(When a term is portrayed by a tree diagram, with the nodes corresponding to operation symbols in the term and the leaves to variable symbols, the depth of the term t corresponds to the length of the longest path from the root to leaves in the tree diagram for t.) This defines a valuation function v on the set of all terms of type τ (see [6]). Let $k \geqslant 0$ be any natural number. An identity $s \approx t$ of type τ is called *k-normal* (with respect to the depth valuation) if either s and t are identical, or $v(t), v(s) \geqslant k$.

We denote by $N_k(\tau)$ the set of all k-normal identities of type τ. It was proved in [6] that k-normality with respect to the depth of terms is a hereditary property of identities, in the sense that the set $N_k(\tau)$ is closed under the usual five rules of deduction for identities; equivalently, this means that $N_k(\tau)$ is an equational theory. For a variety V of type τ, let $\operatorname{Id} V$ denote the set of all identities of V. Since $\operatorname{Id} V$ is an equational theory, so is $\operatorname{Id}^{N_k} V = N_k(\tau) \cap \operatorname{Id} V$, the set of all k-normal identities satisfied by V. The variety determined by this set, $N_k(V) = \operatorname{Mod}\operatorname{Id}^{N_k} V$, is called the *$k$-normalization of* V. In the special case that $N_k(V) = V$, we say that V is a *k-normal variety*; this occurs when every identity of V is a k-normal identity. Otherwise, V is a proper subvariety of $N_k(V)$, and $N_k(V)$ is the least k-normal variety containing V.

The variety $N_k(V)$ is defined equationally, by means of the set of all k-normal identities of V. An algebraic characterization of the algebras in $N_k(V)$ was given by Denecke and Wismath in [5], using the concept of a k-choice algebra. They showed that any algebra in $N_k(V)$ is a homomorphic image of a k-choice algebra constructed from an algebra in V.

In this paper we characterize the algebras in the variety $N_k(V)$ in one special case, when $k = 2$ and V is the type (2,2) variety L of lattices. It is well-known that lattices are two-sided objects: as well as being algebras of type (2,2) with operations $\vee$ and $\wedge$, they are sets with a partial order relation $\leqslant$ in which any two elements have a (unique) least upper bound and a (unique) greatest lower bound. We shall refer to any algebra in $N_2(L)$ as a *2-normalized lattice*. After some background on $N_2(L)$ in Section 2, we introduce in Section 3 a construction called the *3-level inflation* of a lattice and show that any 3-level inflation of a lattice is in $N_2(L)$. Then in Section 4 we use the order-theoretic nature of lattices to show conversely that any algebra in $N_2(L)$ is a 3-level inflation of some lattice. Our 3-level inflation construction is a slightly simpler version of the k-choice construction of [5], and we obtain a stronger result: that the variety $N_2(L)$ is precisely the class of all 3-level inflations of lattices. Finally, in Section 5 we give a finite equational basis for the variety $N_2(L)$.

2. The variety $N_2(L)$

The variety $N_2(L)$ is a type (2,2) variety, with two binary operations which we shall denote by $\wedge$ and $\vee$. By definition, $N_2(L)$ is the equational class determined by the set of all 2-normal identities satisfied by the variety L of lattices. The variety $N_2(L)$ is closely related to two other type (2,2) varieties constructed from L. The variety $E(L)$ is called the *externalization of* L, and it is the variety determined by all *externally compatible identities* of L. An identity $s \approx t$ (of any type τ) is said to be *externally compatible*, if either s and t are the same variable, or $s = f_j(s_1, \dots, s_{n_j})$ and $t = f_j(t_1, \dots, t_{n_j})$ for some terms $s_1, \dots, s_{n_j}, t_1, \dots, t_{n_j}$ and some index $j \in I$. Externally compatible identities were defined by J. Płonka [9] and studied by Chromik in [3] and Graczyńska in [8]. A characterization of the algebras in $E(L)$ was given in [1]. The variety $N(L)$ is the usual *normalization of* L, the variety determined by the set of all normal identities of L. An identity $s \approx t$ (of any type τ) is said to be *normal*, if either s and t are identical, or $v(t), v(s) \geqslant 1$.

The collection of all varieties of type (2,2) forms a lattice under the inclusion ordering. We observe from the definitions that in this ordering $L \leqslant N(L) \leqslant N_2(L)$. However, $L \neq N(L)$, since L satisfies the non-normal idempotent identity $x \vee x \approx x$. Also $N(L) \neq N_2(L)$ since $N(L)$ satisfies $x \vee x \approx x \wedge x$, but $N_2(L)$ does not. All external identities are normal identities, implying $N(L) \leqslant E(L)$ but $N(L) \neq E(L)$ since $N(L)$ satisfies $x \vee x \approx x \wedge x$ but $E(L)$ does not. We know also that $N_2(L)$ is not a subvariety of $E(L)$, since $E(L)$ satisfies $x \vee x \approx (x \vee x) \vee (x \vee x)$, but $N_2(L)$ does not. Finally, $E(L)$ is not a subvariety of $N_2(L)$ either, since $N_2(L)$ satisfies $(x \vee (x \vee x)) \wedge (x \vee (x \vee x)) \approx x \vee (x \vee x)$, but $E(L)$ does not. Fig. 1 shows the inclusion relationships

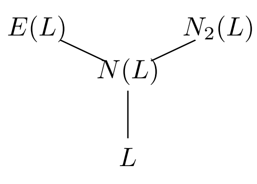


Figure 1.

of the varieties L, $N(L)$, $E(L)$ and $N_2(L)$. The variety $N(L)$ covers L (see [7]), but

$N_2(L)$ does not cover $N(L)$ since the variety $N_2(L) \cap \mathrm{Comm}$, where Comm is the commutative variety of type (2,2), is strictly in between $N(L)$ and $N_2(L)$.

3. The 3-level inflation construction

Let L be the type (2,2) variety of lattices, with $N_2(L)$ its 2-normalization. In this section we introduce a construction called the 3-level inflation construction, which we use to produce an algebra in $N_2(L)$ from any lattice in L. Since we will be talking about algebras of type (2,2) from three different varieties, we shall use different operation symbols for the two binary operations to distinguish them, with the symbols $\vee$ and $\wedge$ now used for lattices.

Our 3-level inflation construction is a generalization of the usual inflation construction which is well-known in universal algebra. Given a base algebra $\mathscr{A}$, an inflation of $\mathscr{A}$ is formed by adding disjoint sets of new elements to the base set A, one set C_a (containing a) for each element a of A. The union of these new sets then forms the base set of a new algebra, in which operations are performed by the rule that any element in the set C_a always acts like a. For more information on the inflation construction, see [4].

Now let $\mathscr{A} = (A; \vee, \wedge)$ be a lattice. As in the usual inflation process, we inflate the set A by adding to each $a \in A$ a set C_a containing a, such that for $a \neq b \in A$ the sets C_a and C_b are disjoint. Let $A^* = \bigcup\{C_a \colon a \in A\}$. For each element $a_1 \in A^*$ there is a unique element $\bar{a}_1 \in A$ such that $a_1 \in C_{\bar{a}_1}$. For each $a \in A$ we will refer to C_a as the class of a. These classes form a partition of A^* which induces an equivalence relation θ on A^*. A mapping $\psi\colon p(A^*) \to A^*$ satisfying $\psi(C_a) \in C_a$ for all $a \in A$ will be called a *θ-choice function*. Unlike the usual inflation, for each $a \in A$ we partition C_a into three subclasses or levels, C_a^j, for $j = 0, 1, 2$, such that $|C_a^2| \geqslant 1$, but C_a^0 and C_a^1 are possibly empty. Thus, $C_a = \bigcup\{C_a^j \colon j = 0, 1, 2\}$.

Our new algebra $\mathscr{A}^*$ will have the inflated set A^* as its universe, with binary operations $\vee_\theta, \wedge_\theta$ defined as follows:

Definition 3.1. Let $\mathscr{A} = (A; \vee, \wedge)$ be a lattice with A^* and θ as above. Let φ be a θ-choice function such that $\varphi(C_{\bar{a}}) \in C_{\bar{a}}^2$ for any $a \in A^*$. We define two operations $\vee_\theta$ and $\wedge_\theta$ on A^* by setting, for any $a_1, a_2 \in A^*$,

$$a_1 \vee_\theta a_2 = \begin{cases} \text{any element of } C^1_{\bar{a}_1 \vee \bar{a}_2} \cup C^2_{\bar{a}_1 \vee \bar{a}_2} & \text{if } a_1 \in C^0_{\bar{a}_1} \text{ and } a_2 \in C^0_{\bar{a}_2}, \\ \varphi(C_{\bar{a}_1 \vee \bar{a}_2}) & \text{otherwise;} \end{cases}$$

$$a_1 \wedge_\theta a_2 = \begin{cases} \text{any element of } C^1_{\bar{a}_1 \wedge \bar{a}_2} \cup C^2_{\bar{a}_1 \wedge \bar{a}_2} & \text{if } a_1 \in C^0_{\bar{a}_1} \text{ and } a_2 \in C^0_{\bar{a}_2}, \\ \varphi(C_{\bar{a}_1 \wedge \bar{a}_2}) & \text{otherwise.} \end{cases}$$

The algebra $\mathscr{A}^* = (A^*; \vee_\theta, \wedge_\theta) = \mathrm{Inf}_3(\mathscr{A}, \theta)$ will be called a 3-*level inflation* of $\mathscr{A}$.

The key observation about our new algebra $\mathscr{A}^*$ is the following fact. Any element of A^* that is an output of $\vee_\theta$ or $\wedge_\theta$ will be at level 1 or level 2. Hence, any element that is the output of a term of depth 2 or more had to be determined by φ and so must be at level 2.

We let L^* be the class of all algebras $\mathscr{A}^* = \mathrm{Inf}_3(\mathscr{A}, \theta)$ formed from some lattice $\mathscr{A} \in L$. Our goal now is to show that $L^* \subseteq N_2(L)$, that is, that any algebra constructed in this way from a lattice is in $N_2(L)$. Our proof will use the following lemma.

Lemma 3.2. *Let $\mathscr{A}^* = (A^*; \vee_\theta, \wedge_\theta)$ be a 3-level inflation of a lattice $\mathscr{A}$. For any term t of arity n and any $a_1, \ldots, a_n \in A^*$, $t^{\mathscr{A}^*}(a_1, \ldots, a_n)$ is in the θ-class of $t^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n)$, which is in A.*

P r o o f. Let t be any term of arity n and let $a_1, \ldots, a_n$ be any elements of A^*. We will give a proof by induction on the complexity of t. First, if $t = x_j$ for some $j \geqslant 1$, then

$$t^{\mathscr{A}^*}(a_1, \ldots, a_n) = a_j \quad \text{and} \quad \overline{t^{\mathscr{A}^*}(a_1, \ldots, a_n)} = \bar{a}_j = t^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n).$$

Therefore, both a_j, $\bar{a}_j$ are in the same θ-class, $C_{\bar{a}_j}$. We note also that for terms of depth 1, the definition of $\vee_\theta$ guarantees that $x \vee_\theta y$ and $\bar{x} \vee \bar{y}$ are both in $C_{\bar{x} \vee \bar{y}}$ and hence in the same θ-class, and similarly for $\wedge_\theta$.

Inductively, let $t = f(t_1, t_2)$ be a compound term, and suppose without loss of generality that $f = \vee$. So $t = t_1 \vee t_2 = \bigvee(t_1, t_2)$. Hence,

$$t^{\mathscr{A}^*}(a_1, \ldots, a_n) = \bigvee_\theta (t_1^{\mathscr{A}^*}(a_1, \ldots, a_n), t_2^{\mathscr{A}^*}(a_1, \ldots, a_n)).$$

By definition of $\vee_\theta$, we have

$$\bigvee_\theta (t_1^{\mathscr{A}^*}(a_1, \ldots, a_n), t_2^{\mathscr{A}^*}(a_1, \ldots, a_n)) \in C_{\overline{t_1^{\mathscr{A}^*}(a_1, \ldots, a_n)} \vee \overline{t_2^{\mathscr{A}^*}(a_1, \ldots, a_n)}}.$$

By induction,

$$\overline{t_1^{\mathscr{A}^*}(a_1, \ldots, a_n)} = t_1^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n) \quad \text{and} \quad \overline{t_2^{\mathscr{A}^*}(a_1, \ldots, a_n)} = t_2^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n).$$

Therefore, $t^{\mathscr{A}^*}(a_1, \ldots, a_n) \in C_{t_1^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n) \vee t_2^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n)}$. Now,

$$\bigvee (t_1^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n), t_2^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n)) = t^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n).$$

Therefore, $t^{\mathscr{A}^*}(a_1, \ldots, a_n) \in C_{t^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n)}$ and thus $t^{\mathscr{A}^*}(a_1, \ldots, a_n)$ is in the θ-class of $t^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n)$, which is in A. $\square$

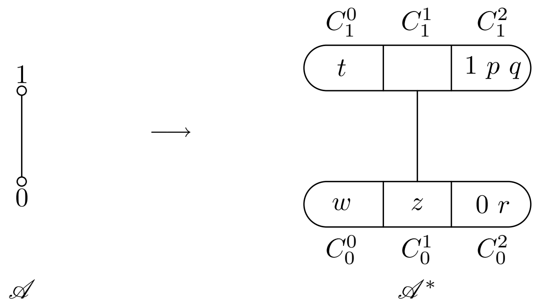
Theorem 3.3. *Any algebra $\mathscr{A}^*$ constructed as a 3-level inflation of a lattice $\mathscr{A}$ is in $N_2(L)$. Consequently, $L^* \subseteq N_2(L)$.*

P r o o f. Let $\mathscr{A}^* = \mathrm{Inf}_3(\mathscr{A}, \theta)$ for some lattice $\mathscr{A}$. We will show that $\mathscr{A}^* \in N_2(L)$ by showing that it satisfies any 2-normal identity $s \approx t$ of L. By Lemma 3.2 we have that $s^{\mathscr{A}^*}(a_1, \ldots, a_n)\, \theta\, s^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n)$ and $t^{\mathscr{A}^*}(a_1, \ldots, a_n)\, \theta\, t^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n)$. Since L satisfies $s \approx t$ and all the elements $\bar{a}_1, \ldots, \bar{a}_n$ are in A, and $\mathscr{A} \in L$, we have $s^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n) = t^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n)$. Therefore, $s^{\mathscr{A}^*}(a_1, \ldots, a_n)\, \theta\, t^{\mathscr{A}^*}(a_1, \ldots, a_n)$. That is, $s^{\mathscr{A}^*}(a_1, \ldots, a_n)$ and $t^{\mathscr{A}^*}(a_1, \ldots, a_n)$ are in the same θ-class; specifically, $s^{\mathscr{A}^*}(a_1, \ldots, a_n)$ and $t^{\mathscr{A}^*}(a_1, \ldots, a_n)$ are both in $C_{s^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n)}$.

Moreover, we know that $v(s), v(t) \geqslant 2$, so by the comment following Definition 3.1, $s^{\mathscr{A}^*}(a_1, \ldots, a_n) = \varphi(C_{s^{\mathscr{A}}(\bar{a}_1, \ldots, \bar{a}_n)}) = t^{\mathscr{A}^*}(a_1, \ldots, a_n)$. Thus $s^{\mathscr{A}^*}(a_1, \ldots, a_n) = t^{\mathscr{A}^*}(a_1, \ldots, a_n)$. This shows that $\mathscr{A}^*$ satisfies $s \approx t$, as required. $\square$

For any $\mathscr{A} \in L$, if no new elements are added in the 3-level inflation of $\mathscr{A}$ to $\mathscr{A}^*$, then $\mathscr{A}^*$ is just $\mathscr{A}$ again. This means that we have $L \subseteq L^* \subseteq N_2(L)$. If sufficiently many new elements are added in an inflation of $\mathscr{A}$, then it is possible for the new algebra $\mathscr{A}^*$ to break the non-normal identities of L but keep the normal identities of L, and to put $\mathscr{A}^*$ in the variety $N(L)$ determined by all normal identities of L. If enough new elements are added in the 3-level inflation, then it is possible to break the non-2-normal identities of L, but keep all the 2-normal identities of L, and so have $\mathscr{A}^* \in N_2(L) - N(L)$. In the following example, sufficiently many elements were added to the original lattice $\mathscr{A}$ to form a new algebra $\mathscr{A}^*$ which is in $N_2(L)$ but not in $N(L)$.

Example 3.4. Let $\mathscr{A} = (\{1, 0\}, \vee, \wedge)$ be a two-element lattice. Let $C_0^0 = \{w\}$, $C_0^1 = \{z\}$, $C_0^2 = \{0, r\}$, $C_1^0 = \{t\}$, $C_1^1 = \emptyset$ and $C_1^2 = \{1, p, q\}$. Let $C_0 = C_0^0 \cup C_0^1 \cup C_0^2$ and $C_1 = C_1^0 \cup C_1^1 \cup C_1^2$. Let $\mathscr{A}^* = (C_0 \cup C_1; \vee_\theta, \wedge_\theta)$ be the algebra constructed as in Definition 3.1. Let $\varphi(C_0) = r$ and $\varphi(C_1) = p$. Since $p \in C_1^2$ and $z \in C_0^1$, we have $p \vee_\theta z = \varphi(C_{\overline{p} \vee \overline{z}}) = \varphi(C_1) = p$. Since $t \in C_1^0$ and $w \in C_0^0$, we can select any element of $C^1_{\overline{t} \vee \overline{w}} \cup C^2_{\overline{t} \vee \overline{w}}$ $(= C_1^1 \cup C_1^2)$ for $t \vee_\theta w$. In this example we set $t \vee_\theta w = 1$. We also set $w \vee_\theta t = p$, $t \wedge_\theta w = r$ and $w \wedge_\theta t = z$. Note that having $w \vee_\theta t \neq t \vee_\theta w$ and $w \wedge_\theta t \neq t \wedge_\theta w$ breaks commutativity and hence normality, so our constructed algebra $\mathscr{A}$ is in $N_2(L)$ but not in $N(L)$. Fig. 2 shows the algebras $\mathscr{A}$ and $\mathscr{A}^*$, along with the Cayley tables for the operations $\vee_\theta$ and $\wedge_\theta$ on $\mathscr{A}^*$.



$\vee_\theta$	t	1	p	q	w	z	0	r
t	1	p	p	p	1	p	p	p
1	p	p	p	p	p	p	p	p
p	p	p	p	p	p	p	p	p
q	p	p	p	p	p	p	p	p
w	p	p	p	p	0	r	r	r
z	p	p	p	p	r	r	r	r
0	p	p	p	p	r	r	r	r
r	p	p	p	p	r	r	r	r

$\wedge_\theta$	t	1	p	q	w	z	0	r
t	1	p	p	p	r	r	r	r
1	p	p	p	p	r	r	r	r
p	p	p	p	p	r	r	r	r
q	p	p	p	p	r	r	r	r
w	z	r	r	r	z	r	r	r
z	r	r	r	r	r	r	r	r
0	r	r	r	r	r	r	r	r
r	r	r	r	r	r	r	r	r

Figure 2.

4. From $N_2(L)$ to 3-level inflations

In this section we show that any 2-normalized lattice $\mathscr{B}$ is the 3-level inflation of some lattice. This lattice will be called the skeleton of the algebra $\mathscr{B}$, and we produce it from a quasiorder on B. This will show that the class $N_2(L)$ of 2-normalized lattices is precisely the class of 3-level inflations of lattices.

As in the previous section, we use different symbols for the two binary operations as they occur in different algebras. We begin with any algebra $\mathscr{B} = (B; \sqcup, \sqcap)$ in $N_2(L)$. We want to find a lattice $\mathscr{A}$ such that $\mathscr{B} = \mathscr{A}^*$ for some 3-level inflation of lattice $\mathscr{A}$. In order to produce such a lattice from $\mathscr{B}$, we will use the concept of a quasiorder. We define a relation Q on B by the rule that, for any $x, y \in B$,

$$(*) \qquad (x, y) \in Q \quad \text{if and only if} \quad x \sqcup x \sqcup y = y \sqcup y \sqcup y.$$

Lemma 4.1. *Let $\mathscr{B}$ be any algebra in $N_2(L)$, and let Q be the relation induced on B by $(*)$. Then*

(i) *Q is a quasiorder on B;*

(ii) *$(x, y) \in Q$ if and only if $x \sqcap y \sqcap y = x \sqcap x \sqcap x$ for all $x, y \in B$;*

(iii) *$(x, x \sqcup y) \in Q$, $(y, x \sqcup y) \in Q$, $(x \sqcap y, x) \in Q$ and $(x \sqcap y, y) \in Q$ for all $x, y \in B$.*

P r o o f. We note first that since $\mathscr{B} \in N_2(L)$, it satisfies all 2-normal consequences of the usual commutativity, idempotence, and absorption laws for lattices.

(i) Reflexivity of the relation Q is immediate from the definition. For transitivity, let $(x,y) \in Q$ and $(y,z) \in Q$. From $(*)$ we obtain $x \sqcup x \sqcup y = y \sqcup y \sqcup y$ and $y \sqcup y \sqcup z = z \sqcup z \sqcup z$. Thus, using these equations and 2-normal consequences of idempotence, we obtain $x \sqcup x \sqcup z = x \sqcup z \sqcup z = x \sqcup z \sqcup z \sqcup z = x \sqcup y \sqcup y \sqcup z = x \sqcup x \sqcup y \sqcup z = y \sqcup y \sqcup y \sqcup z = y \sqcup y \sqcup z \sqcup z = z \sqcup z \sqcup z \sqcup z = z \sqcup z \sqcup z$. It follows that $(x,z) \in Q$, as required. Hence, Q is a quasiorder on B.

(ii) Suppose that $(x,y) \in Q$, so $x \sqcup x \sqcup y = y \sqcup y \sqcup y$. Using this equation and 2-normal consequences of idempotence and absorption, we have $x \sqcap y \sqcap y = x \sqcap y \sqcap y \sqcap y = x \sqcap (y \sqcup y \sqcup y) = x \sqcap (x \sqcup x \sqcup y) = x \sqcap (x \sqcup y \sqcup y) = x \sqcap x \sqcap x$. Similarly, we can prove the converse. Thus, $(x,y) \in Q$ if and only if $x \sqcap y \sqcap y = x \sqcap x \sqcap x$.

(iii) By applying 2-normal consequences of idempotence and commutativity, we obtain $x \sqcup x \sqcup (x \sqcup y) = x \sqcup x \sqcup x \sqcup y \sqcup y \sqcup y = (x \sqcup y) \sqcup (x \sqcup y) \sqcup (x \sqcup y)$. Therefore, $(x, x \sqcup y) \in Q$. Analogously we can show $(y, x \sqcup y) \in Q$ and dually also $(x \sqcap y, x) \in Q$ and $(x \sqcap y, y) \in Q$. $\square$

Let $\mathscr{B} = (B; \sqcup, \sqcap)$ be any algebra in $N_2(L)$. The quasiorder Q defined by $(*)$ will be called the *induced quasiorder* of $\mathscr{B}$.

We now turn to some general information on quasiordered sets. Let $(B;Q)$ be any quasiordered set. We denote by E_Q the relation $Q \cap Q^{-1}$, so that $(a,b) \in E_Q$ if and only if both (a,b) and (b,a) are in Q. This relation E_Q is clearly an equivalence relation on B. We use the notation $[b]_{E_Q}$ for the equivalence class of an element b in this relation, and B/E_Q for the set of equivalence classes of B under E_Q.

Now let $a, b \in B$. In a lattice, a and b have exactly one least upper bound and exactly one greatest lower bound, but in an arbitrary quasiorder this is no longer necessarily true. An element s of B is called a *Q-upper bound of a and b* if $(a,s) \in Q$ and $(b,s) \in Q$. We call s a *minimal Q-upper bound* for a and b if we also have $(s,v) \in Q$ for all Q-upper bounds v of a and b. *Q-lower bounds* and *maximal Q-lower bounds* are then defined dually. We shall denote by $J(a,b)$ the set of all minimal Q-upper bounds of elements a and b, and dually by $M(a,b)$ the set of all maximal Q-lower bounds of a and b. It is easy to see that $M(b,b) = J(b,b) = [b]_{E_Q}$, and that both $J(a,b)$ and $M(a,b)$ are equivalence classes of E_Q if they are non-empty.

We can think of the sets $J(a,b)$ and $M(a,b)$ as the sets of possible joins and meets respectively for the elements a and b. The next lemma shows that in our special case of the quasiorder Q induced on an algebra $\mathscr{B}$ in $N_2(L)$ these sets are always non-empty. This will allow us to construct a lattice to use for our 3-level inflation.

Lemma 4.2. *Let $\mathscr{B} = (B; \sqcup, \sqcap)$ be any algebra in $N_2(L)$, with Q its induced quasiorder. Let $a, b \in B$. Then $a \sqcup b \in J(a, b)$ and $a \sqcap b \in M(a, b)$.*

Proof. From Lemma 4.1 we know that $(a, a \sqcup b) \in Q$ and $(b, a \sqcup b) \in Q$. Thus, $a \sqcup b$ is a Q-upper bound for a and b. Now let v be any Q-upper bound of a and b, so that $a \sqcup a \sqcup v = v \sqcup v \sqcup v$ and $b \sqcup b \sqcup v = v \sqcup v \sqcup v$. Then

$$\begin{aligned}(a \sqcup b) \sqcup (a \sqcup b) \sqcup v &= (a \sqcup b) \sqcup (a \sqcup b) \sqcup v \sqcup v = a \sqcup a \sqcup v \sqcup b \sqcup b \sqcup v \\ &= v \sqcup v \sqcup v \sqcup v \sqcup v \sqcup v = v \sqcup v \sqcup v,\end{aligned}$$

showing that $(a \sqcup b, v) \in Q$. This proves that $a \sqcup b$ is a minimal Q-upper bound for a and b, and hence is in $J(a, b)$. It can be shown dually that $a \sqcap b \in M(a, b)$. In particular, this shows that the sets $M(a, b)$ and $J(a, b)$ are non-empty, so that $J(a, b)$ and $M(a, b)$ are equivalence classes of E_Q. □

In the general case of a quasiorder Q on a set B, a relation $\leqslant_Q$ can be defined on the set B/E_Q of equivalence classes by the rule that

$$[a]_{E_Q} \leqslant_Q [b]_{E_Q} \quad \text{iff} \quad (a, b) \in Q.$$

It is well-known that this relation $\leqslant_Q$ is a partial order (reflexive, antisymmetric and transitive) on B/E_Q. The following result is a special case of the well-known fact that this partial order determines a lattice (see for instance [1]).

Lemma 4.3. *Let $\mathscr{B} = (B; \sqcup, \sqcap)$ be any algebra in $N_2(L)$, with Q its induced quasiorder. Then the partially ordered set $(B/E_Q, \leqslant_Q)$ is a lattice, with*

$$[a]_{E_Q} \sqcup_Q [b]_{E_Q} = J(a, b) \quad \text{and} \quad [a]_{E_Q} \sqcap_Q [b]_{E_Q} = M(a, b)$$

for any $a, b \in B$. Consequently, the algebra $\mathscr{B}/E_Q = (B/E_Q; \sqcup_Q, \sqcap_Q)$ is a lattice.

We have shown so far that for any $\mathscr{B}$ in $N_2(L)$, the quotient algebra $\mathscr{B}/E_Q = (B/E_Q; \sqcup_Q, \sqcap_Q)$ is a lattice. Now we pick one element from each E_Q-class in B/E_Q, and use these elements to form a new set $A \subseteq B$. This selection can be made by a choice function α on B. We can define operations $\vee$ and $\wedge$ on this set A by $p \vee q = \alpha([p]_{E_Q} \sqcup_Q [q]_{E_Q})$ and $p \wedge q = \alpha([p]_{E_Q} \sqcap_Q [q]_{E_Q})$ for all $p, q \in A$. Clearly, these definitions make $\mathscr{A} = (A; \vee, \wedge)$ into a lattice which is isomorphic to $\mathscr{B}/E_Q$. The new lattice $\mathscr{A}$ will be called the *lattice skeleton* of the original algebra $\mathscr{B}$.

Now we want to inflate the lattice skeleton $\mathscr{A}$ to a new algebra $\mathscr{A}^* = \mathrm{Inf}_3(\mathscr{A}, \theta) = (A^*; \vee_\theta, \wedge_\theta)$ using the construction from Section 3. We do this by adding to each $a \in A$ the set $C_a = [a]_{E_Q}$, so that the base set A^* of $\mathscr{A}^*$ is the same as the base set B of the original algebra $\mathscr{B}$. As required by our construction, we must divide C_a into three sets, for each $a \in A$. To do this we use the following concept introduced in [2].

Let $\mathscr{D}$ be any algebra of type τ and let $d \in D$. The element d is always the output of some term operations $t^{\mathscr{D}}$ on $\mathscr{D}$, in particular, of variable terms. If the maximum depth of any term t for which d is obtainable as an output of $t^{\mathscr{D}}$ is 0 or 1, then we assign d a level of 0 or 1, respectively. Otherwise, we assign d a level of 2. It is clear from the definition of levels of elements in an algebra that applying any operations to elements of given levels increases the level of the output by at least 1 (to a maximum level of 2).

Now, using $\mathscr{B}$ to determine the levels of the elements in A^*, we set $C_a^j = \{b \in C_a \colon b \text{ has level } j \text{ in } \mathscr{B}\}$ for $j = 0, 1, 2$. Hence, we have $C_a = \bigcup\{C_a^j \colon j = 0, 1, 2\}$.

Lemma 4.4. *Let $\mathscr{B} \in N_2(L)$, with Q its induced quasiorder. Let $\mathscr{A}$ be the lattice skeleton of $\mathscr{B}$ and let $\mathscr{A}^* = \mathrm{Inf}_3(\mathscr{A}, \theta)$. Then for all $a \in A^*$ the set C_a^2 has size* 1.

Proof. Let $\mathscr{B} \in N_2(L)$, with Q its induced quasiorder. Let $\mathscr{A}$ be the lattice skeleton of $\mathscr{B}$ and let $\mathscr{A}^* = \mathrm{Inf}_3(\mathscr{A}, \theta)$. Let $a \in A^*$. Suppose that we have elements $p, q \in C_a^2$. Since $p, q \in C_a = [a]_{E_Q}$, we have both $(p, q) \in Q$ and $(q, p) \in Q$. Using $(*)$ and the 2-normal consequences of idempotence and commutativity, we have $q \sqcap q \sqcap q = q \sqcup q \sqcup q = p \sqcup p \sqcup q = q \sqcup q \sqcup p = p \sqcup p \sqcup p = p \sqcap p \sqcap p$. Since $p, q \in C_a^2$, we can write $p = f^{\mathscr{A}^*}(a_1, a_2)$ and $q = g^{\mathscr{A}^*}(a_3, a_4)$ for some $f^{\mathscr{A}^*}, g^{\mathscr{A}^*} \in \{\sqcap, \sqcup\}$ and some $a_1, a_2, a_3, a_4 \in A^*$, where a_1 or a_2 has level 1 and a_3 or a_4 has level 1.

There are four cases to consider. If a_1 and a_3 have level 1, then we can express $a_1 = h^{\mathscr{A}^*}(b_1, b_2)$ and $a_3 = l^{\mathscr{A}^*}(b_3, b_4)$ for some $h^{\mathscr{A}^*}, l^{\mathscr{A}^*} \in \{\sqcap, \sqcup\}$ and some $b_1, b_2, b_3, b_4 \in A^*$. Then, using the above equations, associativity, and 2-normal consequences of commutativity and idempotence,

$$\begin{aligned}
p &= f^{\mathscr{A}^*}(a_1, a_2) = f^{\mathscr{A}^*}(h^{\mathscr{A}^*}(b_1, b_2), a_2) \\
&= f^{\mathscr{A}^*}(f^{\mathscr{A}^*}(h^{\mathscr{A}^*}(b_1, b_2), h^{\mathscr{A}^*}(b_1, b_2)), a_2) \\
&= f^{\mathscr{A}^*}(f^{\mathscr{A}^*}(a_1, a_1), a_2) \\
&= f^{\mathscr{A}^*}(f^{\mathscr{A}^*}(f^{\mathscr{A}^*}(a_1, a_2), f^{\mathscr{A}^*}(a_1, a_2)), f^{\mathscr{A}^*}(a_1, a_2)) \\
&= f^{\mathscr{A}^*}(f^{\mathscr{A}^*}(p, p), p) = g^{\mathscr{A}^*}(g^{\mathscr{A}^*}(q, q), q) \\
&= g^{\mathscr{A}^*}(g^{\mathscr{A}^*}(g^{\mathscr{A}^*}(a_3, a_4), g^{\mathscr{A}^*}(a_3, a_4)), g^{\mathscr{A}^*}(a_3, a_4)) \\
&= g^{\mathscr{A}^*}(g^{\mathscr{A}^*}(a_3, a_3), a_4) \\
&= g^{\mathscr{A}^*}(g^{\mathscr{A}^*}(l^{\mathscr{A}^*}(b_3, b_4), l^{\mathscr{A}^*}(b_3, b_4)), a_4) \\
&= g^{\mathscr{A}^*}(l^{\mathscr{A}^*}(b_3, b_4), a_4) = g^{\mathscr{A}^*}(a_3, a_4) = q.
\end{aligned}$$

Thus, we obtain $p = q$. The other three cases are handled similarly. $\square$

Now we can use Lemma 4.4 to complete our construction of our original algebra $\mathscr{B} \in N_2(L)$ as a 3-level inflation of its lattice skeleton $\mathscr{A}$. For any $a \in A^*$, let $\varphi(C_{\bar{a}}) = b$, where b is the only element of $C^2_{\bar{a}}$. We define two operations $\vee_\theta$ and $\wedge_\theta$ on A^* by setting, for any $a_1, a_2 \in A^*$,

$$a_1 \vee_\theta a_2 = \begin{cases} a_1 \sqcup a_2 & \text{if } a_1 \in C^0_{\bar{a}_1} \text{ and } a_2 \in C^0_{\bar{a}_2}, \\ \varphi(C_{\bar{a}_1 \vee \bar{a}_2}) & \text{otherwise;} \end{cases}$$

$$a_1 \wedge_\theta a_2 = \begin{cases} a_1 \sqcap a_2 & \text{if } a_1 \in C^0_{\bar{a}_1} \text{ and } a_2 \in C^0_{\bar{a}_2}, \\ \varphi(C_{\bar{a}_1 \wedge \bar{a}_2}) & \text{otherwise.} \end{cases}$$

We need to verify that the above definition satisfies the conditions of our 3-level inflation construction given in Section 3. To do this, we must show that if a_1 and a_2 both have level 0, then $a_1 \sqcup a_2$ must be an element of $C^1_{\bar{a}_1 \vee \bar{a}_2} \cup C^2_{\bar{a}_1 \vee \bar{a}_2}$ and $a_1 \sqcap a_2$ must be an element of $C^1_{\bar{a}_1 \wedge \bar{a}_2} \cup C^2_{\bar{a}_1 \wedge \bar{a}_2}$. First, we require the following lemma:

Lemma 4.5. *For any $a_1, a_2 \in A^*$, $a_1 \sqcup a_2 \in C_{\bar{a}_1 \vee \bar{a}_2}$ and $a_1 \sqcap a_2 \in C_{\bar{a}_1 \wedge \bar{a}_2}$.*

Proof. Let a_1, a_2 be any elements of A^*. We will show $a_1 \sqcup a_2 \in C_{\bar{a}_1 \vee \bar{a}_2}$; the proof for $a_1 \sqcap a_2$ is similar. By Lemma 4.2, $a_1 \sqcup a_2 \in J(a_1, a_2)$, and by Lemma 4.3, $J(a_1, a_2) = [a_1]_{E_Q} \sqcup_Q [a_2]_{E_Q}$. Since $a_1 \in A^*$, $a_1 \in C_{\bar{a}_1} = [\bar{a}_1]_{E_Q}$. Hence, $[a_1]_{E_Q} = [\bar{a}_1]_{E_Q}$. Similarly, $[a_2]_{E_Q} = [\bar{a}_2]_{E_Q}$ and so $[a_1]_{E_Q} \sqcup_Q [a_2]_{E_Q} = [\bar{a}_1]_{E_Q} \sqcup_Q [\bar{a}_2]_{E_Q}$. By our construction of $\mathscr{A}$ we have $\bar{a}_1 \vee \bar{a}_2 \in [\bar{a}_1]_{E_Q} \sqcup_Q [\bar{a}_2]_{E_Q}$ and thus, $[\bar{a}_1]_{E_Q} \sqcup_Q [\bar{a}_2]_{E_Q} = [\bar{a}_1 \vee \bar{a}_2]_{E_Q}$. Now, $[\bar{a}_1 \vee \bar{a}_2]_{E_Q} = C_{\bar{a}_1 \vee \bar{a}_2}$. Hence, $J(a_1, a_2) = C_{\bar{a}_1 \vee \bar{a}_2}$, and so $a_1 \sqcup a_2 \in C_{\bar{a}_1 \vee \bar{a}_2}$. □

So we have that $a_1 \sqcup a_2 \in C_{\bar{a}_1 \vee \bar{a}_2}$ and $a_1 \sqcap a_2 \in C_{\bar{a}_1 \wedge \bar{a}_2}$ for any $a_1, a_2 \in A^*$. Now if both a_1 and a_2 have level 0, then $a_1 \sqcup a_2$ has at least level 1. Hence, $a_1 \sqcup a_2 \in C^1_{\bar{a}_1 \vee \bar{a}_2} \cup C^2_{\bar{a}_1 \vee \bar{a}_2}$ and $a_1 \sqcap a_2 \in C^1_{\bar{a}_1 \wedge \bar{a}_2} \cup C^2_{\bar{a}_1 \wedge \bar{a}_2}$, as required by our construction in Section 3.

Theorem 4.6. *Any algebra $\mathscr{B} = (B; \sqcup, \sqcap)$ in $N_2(L)$ is a 3-level inflation of its skeleton lattice.*

Proof. Let $\mathscr{B} = (B; \sqcup, \sqcap)$ be any algebra in $N_2(L)$, with Q its induced quasiorder. Let $\mathscr{A}$ be the lattice skeleton of $\mathscr{B}$ and let $\mathscr{A}^* = \mathrm{Inf}_3(\mathscr{A}, \theta) = (A^*; \vee_\theta, \wedge_\theta)$, with operations defined as above. It follows from our construction that A^* and B are equal as sets, and we want to show that we have $a_1 \vee_\theta a_2 = a_1 \sqcup a_2$ and $a_1 \wedge_\theta a_2 = a_1 \sqcap a_2$ for all $a_1, a_2 \in A^*$.

If a_1 and a_2 both have level 0, then by definition, $a_1 \vee_\theta a_2 = a_1 \sqcup a_2$ and $a_1 \wedge_\theta a_2 = a_1 \sqcap a_2$. If at least one of a_1 and a_2 does not have level 0, then $a_1 \vee_\theta a_2 = \varphi(C_{\bar{a}_1 \vee \bar{a}_2})$

is an element of $C^2_{\bar{a}_1 \vee \bar{a}_2}$. By Lemma 4.5 we know that $a_1 \sqcup a_2$ is an element of $C_{\bar{a}_1 \vee \bar{a}_2}$. Since at least one of a_1 and a_2 has level at least 1, $a_1 \sqcup a_2$ has level 2. Therefore, $a_1 \sqcup a_2 \in C^2_{\bar{a}_1 \vee \bar{a}_2}$. So we have that both $a_1 \vee_\theta a_2$ and $a_1 \sqcup a_2$ are elements of $C^2_{\bar{a}_1 \vee \bar{a}_2}$, but by Lemma 4.4, $C^2_{\bar{a}_1 \vee \bar{a}_2}$ has only one element and thus $a_1 \sqcup a_2 = a_1 \vee_\theta a_2$. Similarly, we obtain $a_1 \sqcap a_2 = a_1 \wedge_\theta a_2$. Therefore, $\mathscr{B} = \mathscr{A}^*$. $\square$

Corollary 4.7. *The class $N_2(L)$ of 2-normalized lattices is precisely the class L^* of all 3-level inflations of lattices.*

Example 4.8. Let $\mathscr{B} = (\{t, 1, p, q, w, 0, z, r\}, \sqcup, \sqcap)$ be the eight-element algebra in $N_2(L)$ constructed in Example 3.4, with Q its induced quasiorder. Now, we form the lattice $\mathscr{B}/E_Q = (B/E_Q; \sqcup_Q, \sqcap_Q)$ which will have two elements: $[1]_{E_Q}$ and $[0]_{E_Q}$. To form $\mathscr{A} = (A; \vee, \wedge)$, we select the element t from $[1]_{E_Q}$ $(= [t]_{E_Q})$ and the element 0 from $[0]_{E_Q}$. Finally, we form $\mathscr{A}^* = \mathrm{Inf}_3(\mathscr{A}, \theta) = (A^*; \vee_\theta, \wedge_\theta)$ by setting $C_t = [t]_{E_Q}$ and $C_0 = [0]_{E_Q}$. We use the tables for the operations $\sqcup$ and $\sqcap$ of $\mathscr{B}$ to assign a level to each element in B, obtaining $C^0_t = \{t, q\}$, $C^1_t = \{1\}$, $C^2_t = \{p\}$, $C^0_0 = \{w\}$, $C^1_0 = \{0, z\}$ and $C^2_0 = \{r\}$. We set $\varphi(C_t) = p$ and $\varphi(C_0) = r$. We define $\vee_\theta$ and $\wedge_\theta$ as above such that $t \vee_\theta t = 1$, $t \vee_\theta w = 1$, $w \vee_\theta t = p$, $w \vee_\theta w = 0$, $t \wedge_\theta t = 1$, $t \wedge_\theta w = r$, $w \wedge_\theta t = z$ and $w \wedge_\theta w = z$. This ensures that $\vee_\theta$ has the same table as $\sqcup$ and $\wedge_\theta$ has the same table as $\sqcap$. Fig. 3 gives an overview of this process by showing the progression of our construction from the quotient algebra $\mathscr{B}/E_Q$ to the lattice skeleton $\mathscr{A}$ to the 3-level inflation $\mathscr{A}^*$ of $\mathscr{A}$.

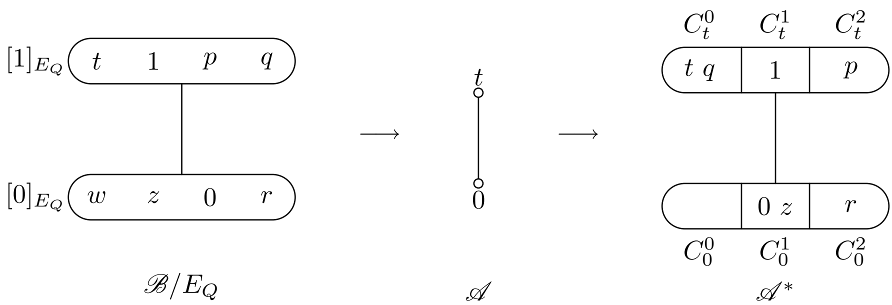


Figure 3.

5. An equational basis for $N_2(L)$

The variety $N_2(L)$ of 2-normalized lattices is defined as the equational class of algebras determined by the set of all 2-normal identities of the variety L of lattices. This means that this set, of all 2-normal identities of the variety of lattices, forms an equational basis for the variety $N_2(L)$. This basis is countably infinite in size. In this section we present a finite basis for $N_2(L)$. We shall return to the convention of using the symbols $\vee$ and $\wedge$ for our two binary operations.

Theorem 5.1. *The set $\Sigma_{N_2(L)}$ consisting of the identities listed below forms a finite basis for the identities of the variety $N_2(L)$:*

(1) Associativity	$x \vee (y \vee z) \approx (x \vee y) \vee z$,	$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$
(2) 2-Normal Commutativity	$x \vee y \vee z \approx x \vee z \vee y$,	$x \wedge y \wedge z \approx x \wedge z \wedge y$
	$x \vee y \vee z \approx y \vee x \vee z$,	$x \wedge y \wedge z \approx y \wedge x \wedge z$
(3) 2-Normal Idempotence	$x \vee y \vee z \vee z \approx x \vee y \vee z$,	$x \wedge y \wedge z \wedge z \approx x \wedge y \wedge z$
	$z \vee (x \wedge y \wedge y) \approx z \vee (x \wedge y)$,	$z \wedge (x \vee y \vee y) \approx z \wedge (x \vee y)$
	$(x \wedge y) \vee z \vee z \approx (x \wedge y) \vee z$,	$(x \vee y) \wedge z \wedge z \approx (x \vee y) \wedge z$
(4) 2-Normal Absorption	$x \vee (x \wedge y) \approx x \vee x \vee x$,	$x \wedge (x \vee y) \approx x \wedge x \wedge x$
(5) Equalization	$x \vee x \vee x \approx x \wedge x \wedge x$	

It is clear that all the identities in $\Sigma_{N_2(L)}$ do hold in $N_2(L)$, since they are 2-normal consequences of identities in the standard basis for the variety L. To prove Theorem 5.1, we will show that given any lattice identity $s \approx t$ such that $v(s), v(t) \geqslant 2$, we can produce a deduction of $s \approx t$ from $\Sigma_{N_2(L)}$ using the standard five rules of deduction. First, we need the following definition:

Definition 5.2. For any term $u \in W_\tau(X)$, let $u' = u \vee u \vee u$.

Note that $u \approx u'$ is a lattice identity. Let Σ be the standard lattice basis and Σ' the set of identities $u' \approx w'$ such that $u \approx w \in \Sigma$. Our strategy to deduce the given identity $s \approx t$ from $\Sigma_{N_2(L)}$ involves the deduction of the three identities $s \approx s'$, $s' \approx t'$, and $t' \approx t$ from $\Sigma_{N_2(L)}$. The proof will be broken up into several lemmas.

Lemma 5.3. *For any term u of depth $\geqslant 2$, we can deduce $u \approx u'$ from $\Sigma_{N_2(L)}$.*

Proof. Let u be any term such that $v(u) \geqslant 2$. Thus, u has the form $f(g(p,q),w)$ or $f(p,g(q,w))$ for some terms p, q and w. We need to show that $f(g(p,q),w) \approx f(g(p,q),w) \vee f(g(p,q),w) \vee f(g(p,q),w)$ and $f(p,g(q,w)) \approx f(p,g(q,w)) \vee f(p,g(q,w)) \vee f(p,g(q,w))$ can be deduced from $\Sigma_{N_2(L)}$. These deductions are long but straightforward and similar to the example given in Lemma 5.4. □

From Lemma 5.3 we know that we can deduce $s \approx s'$ and $t' \approx t$ from $\Sigma_{N_2(L)}$. It suffices for us to prove that $s' \approx t'$ can be deduced from $\Sigma_{N_2(L)}$.

Since $s' \approx t'$ is a lattice identity, there exists a deduction of $s \approx t$ using the five rules of deduction and the standard lattice basis Σ. We will call this deduction the *given deduction*. We take the given deduction and replace each step $u_j \approx w_j$ by $u'_j \approx w'_j$. We will call the result the *derived list*. We need to show that the derived list is a deduction of $s' \approx t'$ from $\Sigma_{N_2(L)}$ and its consequences. In particular, we want to be able to use identities in Σ' as consequences of $\Sigma_{N_2(L)}$.

Lemma 5.4. *The set Σ' can be deduced from $\Sigma_{N_2(L)}$.*

Proof. Let $u \approx w$ be any element of Σ, so $u' \approx w'$ is an element of Σ'. If $u \approx w$ is associativity, then clearly $u' \approx w'$ can be deduced from $\Sigma_{N_2(L)}$. If $u \approx w$ is commutativity, idempotence or absorption, then the deduction of $u' \approx w'$ from $\Sigma_{N_2(L)}$ is long but straightforward. These deductions make frequent use of the 2-normal commutativity identities (2) and the 2-normal idempotence identities (3). The equalization identity (5) is frequently used when $\wedge$ is the main operation symbol of u or w.

We will provide as an example the deduction of the primed version of the idempotent identity for $\wedge$.

Deduction of $(x \wedge x) \vee (x \wedge x) \vee (x \wedge x) \approx x \vee x \vee x$:

Line	Identity	Justification
1	$x \vee x \vee x \approx x \wedge x \wedge x$	From $\Sigma_{N_2(L)}$.
2	$(x \wedge x) \vee (x \wedge x) \vee (x \wedge x)$ $\approx x \wedge x \wedge x \wedge x \wedge x \wedge x$	Substitution on line 1, x by $x \wedge x$.
3	$x \wedge y \wedge z \wedge z \approx x \wedge y \wedge z$	From $\Sigma_{N_2(L)}$.
4	$x \wedge x \wedge z \wedge z \approx x \wedge x \wedge z$	Substitution on line 3, replace y by x.
5	$x \wedge x \wedge x \wedge x \wedge x \wedge x$ $\approx x \wedge x \wedge x \wedge x$	Substitution on line 4, replace z by $x \wedge x$.
6	$(x \wedge x) \vee (x \wedge x) \vee (x \wedge x)$ $\approx x \wedge x \wedge x \wedge x$	Transitivity on lines 2 and 5.
7	$x \wedge x \wedge x \wedge x \approx x \wedge x \wedge x$	Substitution on line 4, replace z by x.
8	$(x \wedge x) \vee (x \wedge x) \vee (x \wedge x)$ $\approx x \wedge x \wedge x$	Transitivity on lines 6 and 7.
9	$x \wedge x \wedge x \approx x \vee x \vee x$	Symmetry on line 1.
10	$(x \wedge x) \vee (x \wedge x) \vee (x \wedge x)$ $\approx x \vee x \vee x$	Transitivity on lines 8 and 9.

□

As a result of Lemma 5.4, it will suffice to show that our derived list is a deduction of $s' \approx t'$ from $\Sigma_{N_2(L)} \cup \Sigma'$. To show this we need to verify that the justification for each step j in the derived list is the same as the justification for step j in the given deduction. We shall use the following two lemmas to handle two of the cases.

Lemma 5.5. *For any terms u, w, p, q, the identity $f(u,w)' \approx f(p,q)'$ can be deduced from $\Sigma_{N_2(L)} \cup \{f(u',w') \approx f(p',q')\}$.*

P r o o f. We will first consider the case that $f = \vee$. We use $u \vee u \vee u \vee w \vee w \vee w \approx p \vee p \vee p \vee q \vee q \vee q$ as the first line in the deduction of $u \vee w \vee u \vee w \vee u \vee w \approx p \vee q \vee p \vee q \vee p \vee q$. To obtain several identities, we use the $\Sigma_{N_2(L)}$ identity $x \vee y \vee z \approx x \vee z \vee y$ repeatedly along with several applications of the substitution rule. Then we use symmetry and multiple applications of transitivity to obtain $f(u,w)' \approx f(p,q)'$.

For $f = \wedge$, we use the equalization identity (5) at the beginning and end of the deduction. The middle section of the deduction is similar to the case when $f = \vee$, except that we use the identity $x \wedge y \wedge z \approx x \wedge z \wedge y$. $\square$

We will denote by $\mathrm{Subs}(u,x,w)$ the term obtained by replacing every occurrence of the variable x in the term w by the term u.

Lemma 5.6. *For any terms u and w and any variable x, the term $\mathrm{Subs}(u,x,w)'$ is identical with the term $\mathrm{Subs}(u,x,w')$.*

P r o o f. Let u and w be any terms and let x be any variable. First, if w is a variable x, then clearly $\mathrm{Subs}(u,x,w)'$ is identical with $\mathrm{Subs}(u,x,w')$.

Otherwise, $w = f(w_1, w_2)$ is a compound term. Suppose without loss of generality that $f = \vee$. Then

$$\begin{aligned}
\mathrm{Subs}(u,x,w)' &= \mathrm{Subs}(u,x,w_1 \vee w_2)' \\
&= (\mathrm{Subs}(u,x,w_1) \vee \mathrm{Subs}(u,x,w_2))' \\
&= \mathrm{Subs}(u,x,w_1) \vee \mathrm{Subs}(u,x,w_2) \vee \mathrm{Subs}(u,x,w_1) \vee \mathrm{Subs}(u,x,w_2) \\
&\qquad \vee \mathrm{Subs}(u,x,w_1) \vee \mathrm{Subs}(u,x,w_2)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Subs}(u,x,w') &= \mathrm{Subs}(u,x,(w_1 \vee w_2)') \\
&= \mathrm{Subs}(u,x,w_1 \vee w_2 \vee w_1 \vee w_2 \vee w_1 \vee w_2) \\
&= \mathrm{Subs}(u,x,w_1) \vee \mathrm{Subs}(u,x,w_2) \vee \mathrm{Subs}(u,x,w_1) \vee \mathrm{Subs}(u,x,w_2) \\
&\qquad \vee \mathrm{Subs}(u,x,w_1) \vee \mathrm{Subs}(u,x,w_2).
\end{aligned}$$

Hence, $\mathrm{Subs}(u,x,w)'$ is identical with $\mathrm{Subs}(u,x,w')$. $\square$

Now we will prove that the derived list is a deduction of $s' \approx t'$ from $\Sigma_{N_2(L)} \cup \Sigma'$.

Lemma 5.7. *Let $s \approx t$ be any lattice identity such that $v(s), v(t) \geqslant 2$. Then the derived list is a deduction of $s' \approx t'$ from $\Sigma_{N_2(L)} \cup \Sigma'$.*

Proof. We need to verify that the justification for each step j in the derived list is the same as the justification for step j in the given deduction. Consider the identity $u_j \approx w_j$ at any step j in the given deduction. If step j was an instance of an identity from Σ, then step j in the derived list is an instance of the corresponding identity from Σ'. If step j was an instance of the reflexive, symmetric, or transitive rules of deduction, then clearly step j in the derived list is an instance of the same rule.

If step j in the given deduction was an instance of the compatibility rule on two previous steps c and d, then step j was $f(u_c, u_d) \approx f(w_c, w_d)$ deduced from $u_c \approx w_c$ and $u_d \approx w_d$. According to our construction of the derived list, step j in the derived list is $f(u_c, u_d)' \approx f(w_c, w_d)'$. This is not what we obtain from the application of the compatibility rule to steps c and d. Instead, we obtain $f(u_c', u_d') \approx f(w_c', w_d')$. However, by Lemma 5.5 we can produce a deduction of $f(u_c, u_d)' \approx f(w_c, w_d)'$ from $\Sigma_{N_2(L)}$ and the identity $f(u_c', u_d') \approx f(w_c', w_d')$.

If step j in the given deduction was an instance of the substitution rule on a previous step e, then step j in the given deduction was $\mathrm{Subs}(z, x, u_e) \approx \mathrm{Subs}(z, x, w_e)$ and so step j in the derived list is $\mathrm{Subs}(z, x, u_e)' \approx \mathrm{Subs}(z, x, w_e)'$. When we apply the substitution rule to step e in the derived list, we obtain $\mathrm{Subs}(z, x, u_e') \approx \mathrm{Subs}(z, x, w_e')$. By Lemma 5.6, the term $\mathrm{Subs}(z, x, u_e)'$ is identical with the term $\mathrm{Subs}(z, x, u_e')$ and the term $\mathrm{Subs}(z, x, w_e)'$ is identical with the term $\mathrm{Subs}(z, x, w_e')$; hence step j in the derived list is an instance of the substitution rule applied to step e in the derived list.

Thus, the derived list is a deduction of $s' \approx t'$ from $\Sigma_{N_2(L)} \cup \Sigma'$. □

Since by Lemma 5.4 we can deduce Σ' from $\Sigma_{N_2(L)}$, Lemma 5.7 shows that $s' \approx t'$ can be deduced from $\Sigma_{N_2(L)}$. From Lemma 5.3, we have that $s \approx s'$ and $t' \approx t$ can also be deduced from $\Sigma_{N_2(L)}$. This completes the proof of Theorem 5.1.

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