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APPROXIMATION AND NUMERICAL REALIZATION OF 3D CONTACT PROBLEMS WITH GIVEN FRICTION AND A COEFFICIENT OF FRICTION DEPENDING ON THE SOLUTION*

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Abstract. The paper presents the analysis, approximation and numerical realization of 3D contact problems for an elastic body unilaterally supported by a rigid half space taking into account friction on the common surface. Friction obeys the simplest Tresca model (a slip bound is given a priori) but with a coefficient of friction $\mathcal{F}$ which depends on a solution. It is shown that a solution exists for a large class of $\mathcal{F}$ and is unique provided that $\mathcal{F}$ is Lipschitz continuous with a sufficiently small modulus of the Lipschitz continuity. The problem is discretized by finite elements, and convergence of discrete solutions is established. Finally, methods for numerical realization are described and several model examples illustrate the efficiency of the proposed approach.

Keywords: unilateral contact and friction, solution-dependent coefficient of friction

MSC 2010: 65N30

1. Introduction

The aim of this paper is to analyze, discretize and solve a mathematical model describing 3D contact problems for an elastic body unilaterally supported by a rigid foundation taking into account the influence of friction on the contacting parts. We shall consider the simplest model of friction, the so-called Tresca model in which the threshold slip is a priori given (see [3]). Although this model of friction is in a certain manner unphysical (unilateral and friction conditions are uncoupled), it plays an important role in the numerical realization of the more realistic Coulomb law of friction ([8], [9]). In the classical Tresca model the threshold slip is expressed as

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the product $\mathcal{F}g$, where g is a non-negative function and $\mathcal{F}$ is a coefficient of friction which does not depend on the solution. In some problems, however, $\mathcal{F}$ can be of the form $\mathcal{F} := \mathcal{F}(\|\boldsymbol{u}_t\|)$, i.e., the coefficient of friction depends on the magnitude of the tangential contact displacement. The paper deals just with this case. The same 2D problem has been already studied in [6]. Its extension to the 3D-case, however, is not straightforward, in particular as far as the numerical treatment is concerned. Indeed, the Lagrange multipliers regularizing the frictional term are now subject to quadratic constraints so that the resulting minimization problem involves quadratic constraints, as well. The theoretical analysis of discrete contact problems with Coulomb friction and a coefficient depending on the solution which is based on a penalization and regularization approach is also presented in [7].

The paper is organized as follows: Section 2 presents the classical and weak formulation of our problem. The existence result which is based on a fixed-point reformulation of the problem is established in Section 3. It is shown that there exists at least one solution for any continuous and bounded coefficient of friction $\mathcal{F}$. In addition, the solution is unique provided that $\mathcal{F}$ is Lipschitz continuous with a sufficiently small modulus of the Lipschitz continuity. A finite element approximation is studied in Section 4 together with the convergence of discrete solutions. The method of successive approximations serves as a main tool for numerical realization of this problem. In Section 5 we describe an efficient way of solving one iterative step which is represented by a contact problem with the Tresca model of friction with a coefficient of friction which does not depend on the solution. Finally, results of several model examples are shown in Section 6.

Throughout the paper we use the following notation: $\|\boldsymbol{x}\|$ and $\boldsymbol{x}^\top \boldsymbol{y}$ stand for the Euclidean norm of a vector $\boldsymbol{x} \in \mathbb{R}^3$and the scalar product of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, respectively. By $H^k(\Omega)$, $H^k(\Gamma)$, k a non-negative integer, $\Gamma \subseteq \partial\Omega$ we denote the classical Sobolev spaces of functions defined in Ω, Γ with the norms $\|\cdot\|_{k,\Omega}$, $\|\cdot\|_{k,\Gamma}$, respectively. Further, $|\cdot|_{k,\Omega}$, $|\cdot|_{k,\Gamma}$ are the corresponding seminorms. If X is a Banach space then the Cartesian product $(X)^3$ and its elements will be denoted by bold letters. Norms and seminorms in $\boldsymbol{X}$ are defined in a standard way.

2. Setting of the problem

Let us consider an elastic body occupying a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ which is split into three relatively open, non-empty, non-overlapping parts $\Gamma_{\boldsymbol{u}}$, $\Gamma_{\boldsymbol{p}}$, and Γ_c such that $\partial\Omega = \overline{\Gamma}_{\boldsymbol{u}} \cup \overline{\Gamma}_{\boldsymbol{p}} \cup \overline{\Gamma}_c$. The zero displacements are prescribed on $\Gamma_{\boldsymbol{u}}$ while surface tractions of density $\boldsymbol{p} = (p_1, p_2, p_3)^\top \in \boldsymbol{L}^2(\Gamma_{\boldsymbol{p}})$ act on $\Gamma_{\boldsymbol{p}}$. The body is unilaterally supported by a rigid foundation S along Γ_c. For the sake of simplicity of our presentation we shall suppose that S is the half-space

$\mathbb{R}^2 \times \mathbb{R}^1_-$ and there is no gap between S and Ω for the undeformed configuration. Besides unilateral constraints imposed on the deformation of Ω on Γ_c, we shall take into account the effects of friction represented by the model with *given friction* in which a given slip bound g is multiplied by a coefficient of friction $\mathcal{F}$ which *depends* on the norm of the tangential component of the displacement vector on Γ_c. Finally, the body is subject to volume forces of density $\boldsymbol{f} = (f_1, f_2, f_3)^\top \in \boldsymbol{L}^2(\Omega)$. Our aim is to find an *equilibrium* state of Ω.

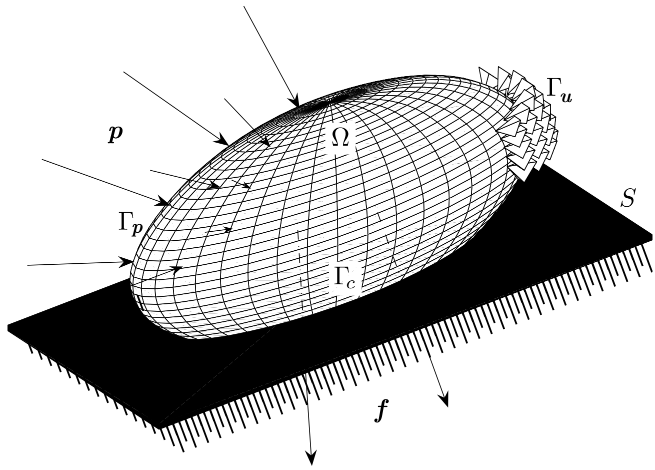


Figure 1. Geometry of the model.

The *classical formulation* of the above problem consists in finding a displacement vector $\boldsymbol{u} = (u_1, u_2, u_3)^\top$ which satisfies the equilibrium equations and the boundary conditions (2.1)–(2.5)[1]:

(*equilibrium equations*)

$$\frac{\partial \tau_{ij}}{\partial x_j}(\boldsymbol{u}) + f_i = 0 \quad \text{in } \Omega, \quad i = 1, 2, 3; \tag{2.1}$$

(*kinematical boundary conditions*)

$$u_i = 0 \quad \text{on } \Gamma_{\boldsymbol{u}}, \quad i = 1, 2, 3; \tag{2.2}$$

(*static boundary conditions*)

$$T_i(\boldsymbol{u}) = p_i \quad \text{on } \Gamma_{\boldsymbol{p}}, \quad i = 1, 2, 3; \tag{2.3}$$

[1] Here and in what follows the Einstein summation convention will be adopted.

(*unilateral conditions*)

$$u_n \leqslant 0, \quad T_n(\boldsymbol{u}) \leqslant 0, \quad u_n T_n(\boldsymbol{u}) = 0 \quad \text{on } \Gamma_c; \tag{2.4}$$

(*friction conditions*)

$$\begin{cases} \boldsymbol{u}_t = \boldsymbol{0} \Longrightarrow \|\boldsymbol{T}_t(\boldsymbol{u})\| \leqslant \mathcal{F}(0) g & \text{on } \Gamma_c; \\ \boldsymbol{u}_t \neq \boldsymbol{0} \Longrightarrow \boldsymbol{T}_t(\boldsymbol{u}) = -\mathcal{F}(\|\boldsymbol{u}_t\|) g \dfrac{\boldsymbol{u}_t}{\|\boldsymbol{u}_t\|} & \text{on } \Gamma_c. \end{cases} \tag{2.5}$$

The symbol $\boldsymbol{\tau}(\boldsymbol{u}) = (\tau_{ij}(\boldsymbol{u}))_{i,j=1}^3$ stands for the symmetric stress tensor which is related to the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^3$ by means of linear Hooke's law:

$$\tau_{ij}(\boldsymbol{u}) = c_{ijkl} \varepsilon_{kl}(\boldsymbol{u}), \quad i, j = 1, 2, 3,$$

where

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i, j = 1, 2, 3,$$

and $c_{ijkl} \in L^\infty(\Omega)$, $i, j, k, l = 1, 2, 3$, are linear elasticity coefficients. They satisfy the following symmetry and ellipticity conditions:

$$c_{ijkl}(\boldsymbol{x}) = c_{jikl}(\boldsymbol{x}) = c_{klij}(\boldsymbol{x}) \quad \text{for a.a. } \boldsymbol{x} \in \Omega; \tag{2.6}$$

$$\exists\, c_{\text{ell}} > 0\colon\ c_{ijkl}(\boldsymbol{x}) \xi_{ij} \xi_{kl} \geqslant c_{\text{ell}} \xi_{ij} \xi_{ij} \quad \text{for a.a. } \boldsymbol{x} \in \Omega \text{ and all } \xi_{ij} = \xi_{ji} \in \mathbb{R}^1. \tag{2.7}$$

Further, $\boldsymbol{n}$ is the unit outward normal to Ω on $\partial\Omega$, $u_n = \boldsymbol{u}^\top \boldsymbol{n}$, $\boldsymbol{u}_t = \boldsymbol{u} - u_n \boldsymbol{n}$ stand for the normal and tangential components of a displacement vector $\boldsymbol{u}$ on Γ_c, respectively, and $\boldsymbol{T}(\boldsymbol{u}) = (T_1(\boldsymbol{u}), T_2(\boldsymbol{u}), T_3(\boldsymbol{u}))^\top$ is a stress vector whose components are $T_i(\boldsymbol{u}) = \tau_{ij}(\boldsymbol{u}) n_j$, $i = 1, 2, 3$. The symbols $T_n(\boldsymbol{u}) = (\boldsymbol{T}(\boldsymbol{u}))^\top \boldsymbol{n}$, $\boldsymbol{T}_t(\boldsymbol{u}) = \boldsymbol{T}(\boldsymbol{u}) - T_n(\boldsymbol{u}) \boldsymbol{n}$ denote the normal, tangential component of a stress vector $\boldsymbol{T}(\boldsymbol{u})$ on Γ_c, respectively. Finally, $\mathcal{F}$ is a *continuous, positive, bounded* function in $\mathbb{R}^1_+$ which defines the coefficient of friction depending on the magnitude $\|\boldsymbol{u}_t\|$ on Γ_c, and $g \in L^2(\Gamma_c)$, $g \geqslant 0$, is a given slip bound.

Let us notice that due to the special geometry of Ω and S we have $v_n = -v_3$ and $\boldsymbol{v}_t = (v_1, v_2, 0)^\top$ on Γ_c.

Let

$$V = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \Gamma_{\boldsymbol{u}}\},$$
$$\boldsymbol{V} = (V)^3,$$

and let $\boldsymbol{K}$ be a closed convex set of kinematically admissible displacements:

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{V}\colon\ v_n \leqslant 0 \text{ a.e. on } \Gamma_c\}.$$

Definition 2.1. By a *weak solution* to a contact problem with given friction and a solution-dependent coefficient of friction $\mathcal{F}$ we mean any displacement vector $\boldsymbol{u}$ satisfying the following *implicit variational inequality of elliptic type*:

$$(\mathcal{P})\qquad \begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{K} \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + \displaystyle\int_{\Gamma_c} \mathcal{F}(\|\boldsymbol{u}_t\|) g(\|\boldsymbol{v}_t\| - \|\boldsymbol{u}_t\|)\,\mathrm{d}S \geqslant \boldsymbol{F}(\boldsymbol{v}-\boldsymbol{u}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{K}, \end{cases}$$

where

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &:= \int_\Omega \tau_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}, & \boldsymbol{u}, \boldsymbol{v} &\in \boldsymbol{V}, \\ \boldsymbol{F}(\boldsymbol{v}) &:= \int_\Omega f_i v_i\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_{\boldsymbol{p}}} p_i v_i\,\mathrm{d}S, & \boldsymbol{v} &= (v_1, v_2, v_3)^\top \in \boldsymbol{V}. \end{aligned}$$

3. Existence result

In this section we derive an equivalent fixed-point formulation of our problem. With its aid we prove the existence of at least one solution and give conditions guaranteeing its uniqueness.

First, we introduce some notation. Let γ be the trace operator on Γ_c:

$$\gamma v = v_{|\Gamma_c}, \quad v \in V,$$

and let $\boldsymbol{\gamma}$, γ_n and $\boldsymbol{\gamma}_t$ be defined by

$$\boldsymbol{\gamma v} = (\gamma v_1, \gamma v_2, \gamma v_3)^\top, \quad \gamma_n \boldsymbol{v} = (\boldsymbol{\gamma v})_n, \quad \boldsymbol{\gamma}_t \boldsymbol{v} = (\boldsymbol{\gamma v})_t, \quad \boldsymbol{v} = (v_1, v_2, v_3)^\top \in \boldsymbol{V}.$$

By $H^{1/2}(\Gamma_c)$ we denote the space of traces on Γ_c of all functions from V, by $H^{1/2}_+(\Gamma_c)$ its subset of all non-negative elements:

$$\begin{aligned} &H^{1/2}(\Gamma_c) = \gamma V, \\ &H^{1/2}_+(\Gamma_c) = \{\psi \in H^{1/2}(\Gamma_c)\colon\ \psi \geqslant 0 \text{ a.e. on } \Gamma_c\}. \end{aligned}$$

The trace space $H^{1/2}(\Gamma_c)$ is a Banach space equipped with the norm

$$\|\psi\|_{1/2,\Gamma_c} = \inf_{\substack{v \in V \\ \gamma v = \psi}} |v|_{1,\Omega}, \quad \psi \in H^{1/2}(\Gamma_c). \tag{3.1}$$

According to our notation, $\boldsymbol{H}^{1/2}(\Gamma_c)$ is the trace space on Γ_c of functions from $\boldsymbol{V}$ with the norm

$$\|\boldsymbol{\psi}\|_{1/2,\Gamma_c} = \inf_{\substack{\boldsymbol{v} \in \boldsymbol{V} \\ \boldsymbol{\gamma v} = \boldsymbol{\psi}}} |\boldsymbol{v}|_{1,\Omega}, \quad \boldsymbol{\psi} \in \boldsymbol{H}^{1/2}(\Gamma_c). \tag{3.2}$$

From the definition of the norms it immediately follows that

$$\| \| \boldsymbol{\psi} \| \|_{1/2,\Gamma_c} \leqslant \| \boldsymbol{\psi} \|_{1/2,\Gamma_c} \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{H}^{1/2}(\Gamma_c). \tag{3.3}$$

In the sequel we shall need the following auxiliary results.

Lemma 3.1.

(i) *If* $\boldsymbol{v} \in \boldsymbol{H}^1(D)$ *then* $\|\boldsymbol{v}\| \in H^1(D)$ *and*

$$\| \|\boldsymbol{v}\| \|_{1,D} \leqslant \|\boldsymbol{v}\|_{1,D}, \quad D = \Omega, \Gamma_c;$$

(ii) *if* $\boldsymbol{v} \in \boldsymbol{V}$ *then* $\gamma_n \boldsymbol{v} \in H^{1/2}(\Gamma_c)$, $\boldsymbol{\gamma}_t \boldsymbol{v} \in \boldsymbol{H}^{1/2}(\Gamma_c)$, *and* $\|\boldsymbol{\gamma}_t \boldsymbol{v}\| \in H^{1/2}(\Gamma_c)$;

(iii) *if* $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ *in* $\boldsymbol{H}^1(\Omega)$, $\boldsymbol{v}^k, \boldsymbol{v} \in \boldsymbol{V}$, $k \to \infty$, *then*

$$\begin{aligned} \boldsymbol{\gamma}_t \boldsymbol{v}^k &\rightharpoonup \boldsymbol{\gamma}_t \boldsymbol{v} && \text{in } \boldsymbol{H}^{1/2}(\Gamma_c), \ k \to \infty, \\ \|\boldsymbol{\gamma}_t \boldsymbol{v}^k\| &\rightharpoonup \|\boldsymbol{\gamma}_t \boldsymbol{v}\| && \text{in } H^{1/2}(\Gamma_c), \ k \to \infty. \end{aligned}$$

For the proofs we refer to [11].

With any $\varphi \in H^{1/2}_+(\Gamma_c)$ we associate the following *auxiliary* problem:

$$(\mathcal{P}(\varphi)) \qquad \begin{cases} \text{Find } \boldsymbol{u} := \boldsymbol{u}(\varphi) \in \boldsymbol{K} \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + \displaystyle\int_{\Gamma_c} \mathcal{F}(\varphi) g(\|\boldsymbol{v}_t\| - \|\boldsymbol{u}_t\|) \,\mathrm{d}S \geqslant \boldsymbol{F}(\boldsymbol{v} - \boldsymbol{u}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{K}. \end{cases}$$

It is known (see [4]) that $(\mathcal{P}(\varphi))$ has a unique solution for every $\varphi \in H^{1/2}_+(\Gamma_c)$. Thus one can define a mapping $\Psi\colon H^{1/2}_+(\Gamma_c) \to H^{1/2}_+(\Gamma_c)$ by

$$\Psi\colon \varphi \mapsto \|\boldsymbol{\gamma}_t(\boldsymbol{u}(\varphi))\|, \quad \varphi \in H^{1/2}_+(\Gamma_c), \tag{3.4}$$

where $\boldsymbol{u}(\varphi) \in \boldsymbol{K}$ solves $(\mathcal{P}(\varphi))$.

Comparing $(\mathcal{P})$ and (3.4), we arrive at the following alternative (and equivalent) definition.

Definition 3.2. By a *weak solution* to a contact problem with given friction and the solution-dependent coefficient of friction $\mathcal{F}$ we mean any function $\boldsymbol{u} \in \boldsymbol{K}$ solving $(\mathcal{P}(\|\boldsymbol{\gamma}_t \boldsymbol{u}\|))$, i.e. $\|\boldsymbol{\gamma}_t \boldsymbol{u}\|$ is a fixed point of Ψ in $H^{1/2}_+(\Gamma_c)$:

$$\Psi(\|\boldsymbol{\gamma}_t \boldsymbol{u}\|) = \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \quad \text{on } \Gamma_c.$$

To prove the existence of at least one fixed point we examine the basic properties of Ψ. Denote

$$B_R = \{\psi \in H^{1/2}_+(\Gamma_c)\colon \|\psi\|_{1/2,\Gamma_c} \leqslant R\}$$

for every $R > 0$.

Lemma 3.3. *The mapping Ψ maps B_R into itself with*

$$R := \frac{\|\boldsymbol{F}\|_{(\boldsymbol{H}^1(\Omega))'}}{c_{\rm ell}c_K},$$

where $c_{\rm ell} > 0$ is the constant in (2.7) and $c_K > 0$ is the constant from Korn's inequality:

$$c_K\|\boldsymbol{v}\|^2_{1,\Omega} \leqslant \int_\Omega \varepsilon_{ij}(\boldsymbol{v})\varepsilon_{ij}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}.$$

Proof. Let $\varphi\in H^{1/2}_+(\Gamma_c)$ be arbitrary but fixed and denote by $\boldsymbol{u} := \boldsymbol{u}(\varphi)$ the solution to $(\mathcal{P}(\varphi))$. Inserting $\boldsymbol{v} := \boldsymbol{0}\in\boldsymbol{K}$ into $(\mathcal{P}(\varphi))$, we obtain

$$-a(\boldsymbol{u},\boldsymbol{u}) - \int_{\Gamma_c}\mathcal{F}(\varphi)g\|\boldsymbol{u}_t\|\,\mathrm{d}S \geqslant -\boldsymbol{F}(\boldsymbol{u}).$$

Therefore,

$$\begin{aligned}(3.5)\qquad c_{\rm ell}c_K\|\boldsymbol{u}\|^2_{1,\Omega} &\leqslant \int_\Omega c_{ijkl}\varepsilon_{kl}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_c}\mathcal{F}(\varphi)g\|\boldsymbol{u}_t\|\,\mathrm{d}S\\ &\leqslant \boldsymbol{F}(\boldsymbol{u}) \leqslant \|\boldsymbol{F}\|_{(\boldsymbol{H}^1(\Omega))'}\|\boldsymbol{u}\|_{1,\Omega}\end{aligned}$$

in virtue of (2.7) and Korn's inequality. From (3.2) and (3.3) one has

$$(3.6)\qquad \boldsymbol{u}\in\boldsymbol{V} \Longrightarrow \|\|\gamma_t\boldsymbol{u}\|\|_{1/2,\Gamma_c} \leqslant \|\gamma_t\boldsymbol{u}\|_{1/2,\Gamma_c} \leqslant \|\boldsymbol{\gamma}\boldsymbol{u}\|_{1/2,\Gamma_c} \leqslant \|\boldsymbol{u}\|_{1,\Omega}.$$

From this and (3.5) we obtain the assertion of the lemma. □

Next we show that the mapping Ψ is weakly continuous in $H^{1/2}_+(\Gamma_c)$.

Lemma 3.4. *Let $\varphi\in H^{1/2}_+(\Gamma_c)$, $\{\varphi^k\}\subset H^{1/2}_+(\Gamma_c)$ be such that*

$$\varphi^k \rightharpoonup \varphi \quad \textit{in } H^{1/2}(\Gamma_c),\ \ k\to\infty.$$

Then

$$\Psi(\varphi^k) \rightharpoonup \Psi(\varphi) \quad \textit{in } H^{1/2}(\Gamma_c),\ \ k\to\infty.$$

Proof. Let $\boldsymbol{u}^k := \boldsymbol{u}(\varphi^k)\in\boldsymbol{K}$ be a solution to $(\mathcal{P}(\varphi^k))$, $k\in\mathbb{N}$:

$$a(\boldsymbol{u}^k,\boldsymbol{v}-\boldsymbol{u}^k) + \int_{\Gamma_c}\mathcal{F}(\varphi^k)g(\|\boldsymbol{v}_t\| - \|\boldsymbol{u}^k_t\|)\,\mathrm{d}S \geqslant \boldsymbol{F}(\boldsymbol{v}-\boldsymbol{u}^k) \quad \forall\,\boldsymbol{v}\in\boldsymbol{K}.$$

From (3.5) we see that $\{\boldsymbol{u}^k\}$ is bounded in $\boldsymbol{H}^1(\Omega)$. Thus there exist a subsequence $\{\boldsymbol{u}^l\} \subseteq \{\boldsymbol{u}^k\}$ and a function $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$\boldsymbol{u}^l \rightharpoonup \boldsymbol{u} \quad \text{in } \boldsymbol{H}^1(\Omega), \ l \to \infty.$$

We prove that $\boldsymbol{u}$ solves $(\mathcal{P}(\varphi))$. First, $\boldsymbol{u} \in \boldsymbol{K}$ and

$$\limsup_{l\to\infty} a(\boldsymbol{u}^l, \boldsymbol{v} - \boldsymbol{u}^l) \leqslant a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \quad \forall \boldsymbol{v} \in \boldsymbol{K},$$
$$\lim_{l\to\infty} \boldsymbol{F}(\boldsymbol{v} - \boldsymbol{u}^l) = \boldsymbol{F}(\boldsymbol{v} - \boldsymbol{u}) \qquad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

Since $\mathcal{F}$ is continuous and $H^{1/2}(\Gamma_c)$ is compactly embedded into $L^2(\Gamma_c)$, one can pass to a subsequence of $\{\varphi^l\}$ (denoted by the same symbol) such that

$$\mathcal{F}(\varphi^l) \to \mathcal{F}(\varphi) \quad \text{a.e. on } \Gamma_c, \ l \to \infty. \tag{3.7}$$

From (iii) of Lemma 3.1 we know that

$$\|\boldsymbol{\gamma}_t \boldsymbol{u}^l\| \rightharpoonup \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \quad \text{in } H^{1/2}(\Gamma_c), \ l \to \infty,$$

which yields

$$\|\boldsymbol{\gamma}_t \boldsymbol{u}^l\| \to \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \quad \text{in } L^2(\Gamma_c), \ l \to \infty.$$

This, the Lebesgue dominated convergence theorem, and (3.7) imply

$$\lim_{l\to\infty} \int_{\Gamma_c} \mathcal{F}(\varphi^l) g(\|\boldsymbol{v}_t\| - \|\boldsymbol{u}_t^l\|) \,\mathrm{d}S = \int_{\Gamma_c} \mathcal{F}(\varphi) g(\|\boldsymbol{v}_t\| - \|\boldsymbol{u}_t\|) \,\mathrm{d}S.$$

Letting $l \to \infty$ in $(\mathcal{P}(\varphi^l))$ and using the previous results, we see that

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + \int_{\Gamma_c} \mathcal{F}(\varphi) g(\|\boldsymbol{v}_t\| - \|\boldsymbol{u}_t\|) \,\mathrm{d}S \geqslant \boldsymbol{F}(\boldsymbol{v} - \boldsymbol{u}) \quad \forall \boldsymbol{v} \in \boldsymbol{K},$$

i.e. $\boldsymbol{u}$ solves $(\mathcal{P}(\varphi))$. Since $(\mathcal{P}(\varphi))$ has a unique solution, the original sequence $\{\boldsymbol{u}^k\}$ tends weakly to $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ and

$$\|\boldsymbol{\gamma}_t \boldsymbol{u}^k\| \rightharpoonup \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \quad \text{in } H^{1/2}(\Gamma_c), \ k \to \infty.$$

□

On the basis of Lemmas 3.3 and 3.4 we obtain the following existence result.

Theorem 3.5. *There exists a weak solution to a contact problem with given friction and a solution-dependent coefficient of friction.*

Proof. It follows from the weak version of the Schauder fixed-point theorem (see [8]). □

Next we show that Ψ is Lipschitz continuous in the $L^2(\Gamma_c)$-norm provided that $\mathcal{F}$ is Lipschitz continuous in $\mathbb{R}^1_+$ and $g \in L^\infty(\Gamma_c)$.

Theorem 3.6. *Let $g \in L^\infty(\Gamma_c)$, $g \geqslant 0$ a.e. on Γ_c, and $c_L > 0$ be such that*

$$|\mathcal{F}(x_1) - \mathcal{F}(\overline{x}_1)| \leqslant c_L |x_1 - \overline{x}_1| \quad \forall\, x_1, \overline{x}_1 \in \mathbb{R}^1_+.$$

Then

$$\|\Psi(\varphi) - \Psi(\overline{\varphi})\|_{0,\Gamma_c} \leqslant c_L \frac{c_T^2 \|g\|_{\infty,\Gamma_c}}{c_{\text{ell}} c_K} \|\varphi - \overline{\varphi}\|_{0,\Gamma_c} \quad \forall\, \varphi, \overline{\varphi} \in H^{1/2}_+(\Gamma_c),$$

where c_T is the norm of the trace mapping $\boldsymbol{\gamma}_t \colon \boldsymbol{V} \to \boldsymbol{L}^2(\Gamma_c)$ and c_{ell}, c_K are the constants from (2.7) and Korn's inequality, respectively.

Proof. Let $\varphi, \overline{\varphi} \in H^{1/2}_+(\Gamma_c)$ be given and let $\boldsymbol{u}$, $\overline{\boldsymbol{u}}$ be the respective solutions of $(\mathcal{P}(\varphi))$, $(\mathcal{P}(\overline{\varphi}))$:

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + \int_{\Gamma_c} \mathcal{F}(\varphi) g(\|\boldsymbol{v}_t\| - \|\boldsymbol{u}_t\|) \,\mathrm{d}S \geqslant \boldsymbol{F}(\boldsymbol{v} - \boldsymbol{u}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{K},$$
$$a(\overline{\boldsymbol{u}}, \boldsymbol{v} - \overline{\boldsymbol{u}}) + \int_{\Gamma_c} \mathcal{F}(\overline{\varphi}) g(\|\boldsymbol{v}_t\| - \|\overline{\boldsymbol{u}}_t\|) \,\mathrm{d}S \geqslant \boldsymbol{F}(\boldsymbol{v} - \overline{\boldsymbol{u}}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{K}.$$

Inserting $\boldsymbol{v} := \overline{\boldsymbol{u}}$ into the first and $\boldsymbol{v} := \boldsymbol{u}$ into the second inequality and summing them, we obtain

$$a(\boldsymbol{u} - \overline{\boldsymbol{u}}, \overline{\boldsymbol{u}} - \boldsymbol{u}) + \int_{\Gamma_c} (\mathcal{F}(\varphi) - \mathcal{F}(\overline{\varphi})) g(\|\overline{\boldsymbol{u}}_t\| - \|\boldsymbol{u}_t\|) \,\mathrm{d}S \geqslant 0. \tag{3.8}$$

It is readily seen that

$$\big\| \|\boldsymbol{\gamma}_t \overline{\boldsymbol{u}}\| - \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \big\|_{0,\Gamma_c} \leqslant \|\boldsymbol{\gamma}_t \overline{\boldsymbol{u}} - \boldsymbol{\gamma}_t \boldsymbol{u}\|_{0,\Gamma_c} \leqslant c_T \|\overline{\boldsymbol{u}} - \boldsymbol{u}\|_{1,\Omega}. \tag{3.9}$$

From (2.7), Korn's inequality, (3.8), and (3.9) we obtain

$$\begin{aligned} c_{\text{ell}} c_K \|\boldsymbol{u} - \overline{\boldsymbol{u}}\|^2_{1,\Omega} &\leqslant a(\boldsymbol{u} - \overline{\boldsymbol{u}}, \boldsymbol{u} - \overline{\boldsymbol{u}}) \\ &\leqslant \|g\|_{\infty,\Gamma_c} \|\mathcal{F}(\varphi) - \mathcal{F}(\overline{\varphi})\|_{0,\Gamma_c} \big\| \|\boldsymbol{\gamma}_t \overline{\boldsymbol{u}}\| - \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \big\|_{0,\Gamma_c} \\ &\leqslant c_L c_T \|g\|_{\infty,\Gamma_c} \|\varphi - \overline{\varphi}\|_{0,\Gamma_c} \|\overline{\boldsymbol{u}} - \boldsymbol{u}\|_{1,\Omega}. \end{aligned} \tag{3.10}$$

Finally, (3.9) and (3.10) yield

$$\big|\|\boldsymbol{\gamma}_t\boldsymbol{u}\| - \|\boldsymbol{\gamma}_t\overline{\boldsymbol{u}}\|\big|_{0,\Gamma_c} \leqslant c_T\|\boldsymbol{u}-\overline{\boldsymbol{u}}\|_{1,\Omega} \leqslant c_L\frac{c_T^2\|g\|_{\infty,\Gamma_c}}{c_{\text{ell}}c_K}\|\varphi-\overline{\varphi}\|_{0,\Gamma_c}.$$

□

Corollary 3.7. *If $c_L \cdot c_T^2\|g\|_{\infty,\Gamma_c}/c_{\text{ell}}c_K < 1$ then the mapping $\Psi\colon H_+^{1/2}(\Gamma_c) \to H_+^{1/2}(\Gamma_c)$ is contractive in the $L^2(\Gamma_c)$-norm. Consequently, Ψ has a unique fixed point and the method of successive approximations*

$$\begin{cases} \varphi^0 \in H_+^{1/2}(\Gamma_c) \ \text{given}; \\ \text{for } k=1,2,\ldots \ \text{set } \varphi^k := \Psi(\varphi^{k-1}) \end{cases} \tag{3.11}$$

is convergent in the $L^2(\Gamma_c)$-norm for any choice of φ^0.

4. Finite element approximation

This section deals with a discretization of problem $(\mathcal{P})$ by a finite element method. We establish the existence as well as the uniqueness of discrete solutions in a way similar to the continuous case. Then we shall study convergence of discrete solutions and as a by-product we obtain an alternative proof of the existence of a solution to $(\mathcal{P})$.

To avoid the use of curved elements we shall suppose that Ω is a *polyhedron.* Let $\{\mathcal{T}_h\}$, $h \to 0+$, be a *regular system of partitions* of $\overline{\Omega}$ into tetrahedra such that every partition $\mathcal{T}_h$ is compatible with the decomposition of $\partial\Omega$ into $\Gamma_{\boldsymbol{u}}$, $\Gamma_{\boldsymbol{p}}$, and Γ_c and such that $\{\mathcal{T}_{h|_{\overline{\Gamma}_c}}\}$, $h \to 0+$, is a *strongly regular system of triangulations* of $\overline{\Gamma}_c$ (see [1]). With any $\mathcal{T}_h$ the following sets will be associated:

$$\begin{aligned} V_h &= \{v_h \in C(\overline{\Omega})\colon\ v_{h|_T} \in P_1(T)\ \forall\, T \in \mathcal{T}_h,\ v_h = 0 \text{ on } \Gamma_{\boldsymbol{u}}\}, \\ \boldsymbol{V}_h &= (V_h)^3, \\ \boldsymbol{K}_h &= \{\boldsymbol{v}_h \in \boldsymbol{V}_h\colon\ v_{hn}(\boldsymbol{a}_i) \leqslant 0\ \forall\, \boldsymbol{a}_i \in \mathcal{N}_h\}, \\ \mathcal{V}_h &= V_{h|_{\Gamma_c}}, \\ \mathcal{V}_h^+ &= \{\varphi_h \in \mathcal{V}_h\colon\ \varphi_h(\boldsymbol{a}_i) \geqslant 0\ \forall\, \boldsymbol{a}_i \in \mathcal{N}_h\}, \end{aligned}$$

where $\mathcal{N}_h$ is the set of all contact nodes, i.e. the nodes of $\mathcal{T}_h$ lying on $\overline{\Gamma}_c\backslash\overline{\Gamma}_{\boldsymbol{u}}$. Obviously, $\boldsymbol{K}_h \subset \boldsymbol{K}$ and $\mathcal{V}_h^+ \subset H_+^{1/2}(\Gamma_c)$ for all $h > 0$.

For every $\varphi_h \in \mathcal{V}_h^+$ we shall consider the following *discrete* problem:

$$(\mathcal{P}(\varphi_h))_h \qquad \begin{cases} \text{Find } \boldsymbol{u}_h := \boldsymbol{u}_h(\varphi_h) \in \boldsymbol{K}_h \text{ such that} \\ a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + \displaystyle\int_{\Gamma_c} \mathcal{F}(\varphi_h) g(\|\boldsymbol{v}_{ht}\| - \|\boldsymbol{u}_{ht}\|)\,\mathrm{d}S \geqslant \boldsymbol{F}(\boldsymbol{v}_h - \boldsymbol{u}_h) \\ \hfill \forall\, \boldsymbol{v}_h \in \boldsymbol{K}_h. \end{cases}$$

Again $(\mathcal{P}(\varphi_h))_h$ has a unique solution for any $\varphi_h \in \mathcal{V}_h^+$ and one can define a mapping Ψ_h by

$$\Psi_h(\varphi_h) = r_h\|\boldsymbol{\gamma}_t(\boldsymbol{u}_h(\varphi_h))\|, \quad \varphi_h \in \mathcal{V}_h^+,$$

where $\boldsymbol{u}_h(\varphi_h) \in \boldsymbol{K}_h$ is the solution to $(\mathcal{P}(\varphi_h))_h$ and $r_h\colon H^{1/2}(\Gamma_c) \to \mathcal{V}_h$ is a linear interpolation operator with the following approximation property: there exists a constant $c_r > 0$ independent of $h_{\Gamma_c} := \max\limits_{F\in\mathcal{T}_{h|\overline{\Gamma}_c}} \operatorname{diam}(F)$ such that

$$(4.1) \qquad \|\psi - r_h\psi\|_{\mu,\Gamma_c} \leqslant c_r h_{\Gamma_c}^{1-\mu}\|\psi\|_{1,\Gamma_c} \quad \forall\, \psi \in H^1(\Gamma_c)$$

for $\mu = 0$ and $1/2$ which preserves monotonicity, i.e.

$$(4.2) \qquad \psi \geqslant 0 \text{ on } \Gamma_c,\ \psi \in H^{1/2}(\Gamma_c) \Longrightarrow r_h\psi \in \mathcal{V}_h^+.$$

For an example of r_h satisfying (4.1) and (4.2) we refer to [2]. The mapping $\Psi_h\colon \mathcal{V}_h^+ \to \mathcal{V}_h^+$ can be viewed as a discretization of Ψ defined by (3.4).

Definition 4.1. By a *discrete solution* to $(\mathcal{P})$ we mean any function $\boldsymbol{u}_h \in \boldsymbol{K}_h$ solving $(\mathcal{P}(r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|))_h$, i.e. $r_h\|\boldsymbol{u}_{ht}\| := r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|$ is a fixed point of Ψ_h in $\mathcal{V}_h^+$.

Lemma 4.2. *The mapping Ψ_h is continuous and maps $\mathcal{V}_h^+ \cap B_{\tilde R}$ into $\mathcal{V}_h^+ \cap B_{\tilde R}$ for some $\tilde R > 0$ which does not depend on h.*

Proof. Let $\varphi_h \in \mathcal{V}_h^+$ be arbitrary but fixed and let $\boldsymbol{u}_h := \boldsymbol{u}_h(\varphi_h)$ be a solution of $(\mathcal{P}(\varphi_h))_h$. The approximation property (4.1), (i) of Lemma 3.1, (3.3), and the inverse inequality between $H^1(\Gamma_c)$ and $H^{1/2}(\Gamma_c)$ give

$$(4.3) \qquad \begin{aligned} \big\|r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|\big\|_{1/2,\Gamma_c} &\leqslant \big\|r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\| - \|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|\big\|_{1/2,\Gamma_c} + \|\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|\|_{1/2,\Gamma_c} \\ &\leqslant c_r h_{\Gamma_c}^{1/2}\|\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|\|_{1,\Gamma_c} + \|\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|\|_{1/2,\Gamma_c} \\ &\leqslant c_r h_{\Gamma_c}^{1/2}\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|_{1,\Gamma_c} + \|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|_{1/2,\Gamma_c} \\ &\leqslant c_r\overline{c}_{\mathrm{inv}}\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|_{1/2,\Gamma_c} + \|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|_{1/2,\Gamma_c}, \end{aligned}$$

where the constants c_r and $\overline{c}_{\rm inv}$ do not depend on h. Arguing exactly as in Lemma 3.3 (see (3.5) and (3.6)), one can show that

$$\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|_{1/2,\Gamma_c} \leqslant \|\boldsymbol{u}_h\|_{1,\Omega} \leqslant \frac{\|\boldsymbol{F}\|_{(\boldsymbol{H}^1(\Omega))'}}{c_{\rm ell}c_K},$$

where $c_{\rm ell}$, c_K are the same as in Lemma 3.3 and independent of h. From this and (4.3) we see that Ψ_h maps $\mathcal{V}_h^+ \cap B_{\tilde{R}}$ into $\mathcal{V}_h^+ \cap B_{\tilde{R}}$ with

$$\tilde{R} := (1 + c_r\overline{c}_{\rm inv})\frac{\|\boldsymbol{F}\|_{(\boldsymbol{H}^1(\Omega))'}}{c_{\rm ell}c_K}.$$

Next we show that Ψ_h is continuous in $\mathcal{V}_h^+$. Let

$$\varphi_h^k \to \varphi_h \quad \text{in } H^{1/2}(\Gamma_c), \;\; \varphi_h^k, \varphi_h \in \mathcal{V}_h^+, \;\; k \to \infty,$$

and denote by $\boldsymbol{u}_h^k := \boldsymbol{u}_h(\varphi_h^k) \in \boldsymbol{K}_h$ solutions to $(\mathcal{P}(\varphi_h^k))_h$. Arguing as in Lemma 3.4, we have

$$\boldsymbol{\gamma}_t\boldsymbol{u}_h^k \to \boldsymbol{\gamma}_t\boldsymbol{u}_h \quad \text{in } \boldsymbol{L}^2(\Gamma_c), \;\; k \to \infty, \tag{4.4}$$

where $\boldsymbol{u}_h := \boldsymbol{u}_h(\varphi_h)$ solves $(\mathcal{P}(\varphi_h))_h$. We already know that $\{r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^k\|\}$ is bounded in the $H^{1/2}(\Gamma_c)$-norm. Thus there exist a subsequence $\{r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l\|\} \subseteq \{r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^k\|\}$ and a function $\varphi \in H^{1/2}(\Gamma_c)$ such that

$$r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l\| \rightharpoonup \varphi \quad \text{in } H^{1/2}(\Gamma_c), \;\; l \to \infty.$$

Since r_h preserves monotonicity (see (4.2)), it is readily seen that

$$\big\|r_h(\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l\| - \|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|)\big\|_{0,\Gamma_c}\| \leqslant \big\|r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l - \boldsymbol{\gamma}_t\boldsymbol{u}_h\|\big\|_{0,\Gamma_c} \quad \forall\, l.$$

Hence,

$$\begin{aligned}
\big\|r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l\| - r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|\big\|_{0,\Gamma_c} &\leqslant \big\|r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l - \boldsymbol{\gamma}_t\boldsymbol{u}_h\|\big\|_{0,\Gamma_c} \\
&\leqslant \big\|r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l - \boldsymbol{\gamma}_t\boldsymbol{u}_h\| - \|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l - \boldsymbol{\gamma}_t\boldsymbol{u}_h\|\big\|_{0,\Gamma_c} + \big\|\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l - \boldsymbol{\gamma}_t\boldsymbol{u}_h\|\big\|_{0,\Gamma_c} \\
&\leqslant c_r h_{\Gamma_c}\big\|\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l - \boldsymbol{\gamma}_t\boldsymbol{u}_h\|\big\|_{1,\Gamma_c} + \big\|\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l - \boldsymbol{\gamma}_t\boldsymbol{u}_h\|\big\|_{0,\Gamma_c} \\
&\leqslant c_r h_{\Gamma_c}\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l - \boldsymbol{\gamma}_t\boldsymbol{u}_h\|_{1,\Gamma_c} + \|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l - \boldsymbol{\gamma}_t\boldsymbol{u}_h\|_{0,\Gamma_c} \\
&\leqslant c_r c_{\rm inv}\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l - \boldsymbol{\gamma}_t\boldsymbol{u}_h\|_{0,\Gamma_c} + \|\boldsymbol{\gamma}_t\boldsymbol{u}_h^l - \boldsymbol{\gamma}_t\boldsymbol{u}_h\|_{0,\Gamma_c} \overset{l\to\infty}{\longrightarrow} 0
\end{aligned} \tag{4.5}$$

in virtue of (4.1), (i) of Lemma 3.1, the inverse inequality between $H^1(\Gamma_c)$ and $L^2(\Gamma_c)$ and (4.4). Thus $\varphi = r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|$ on Γ_c and

$$r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h^k\| \to r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\| \quad \text{in } H^{1/2}(\Gamma_c), \;\; k \to \infty,$$

since $\mathcal{V}_h$ is finite-dimensional for every $h > 0$ fixed. □

From Lemma 4.2 and the Brouwer fixed-point theorem we arrive at the following result.

Theorem 4.3. *There exists a discrete solution to* $(\mathcal{P})$.

Under additional assumptions on $\mathcal{F}$ and g one obtains the following uniqueness result.

Theorem 4.4. *Let* $g \in L^\infty(\Gamma_c)$, $g \geqslant 0$ *a.e. on* Γ_c, *and let* $c_L > 0$ *be such that*

$$|\mathcal{F}(x_1) - \mathcal{F}(\overline{x}_1)| \leqslant c_L |x_1 - \overline{x}_1| \quad \forall\, x_1, \overline{x}_1 \in \mathbb{R}^1_+.$$

Then there exists a positive constant c *which does not depend on* h *and such that*

$$\|\Psi_h(\varphi_h) - \Psi_h(\overline{\varphi}_h)\|_{0,\Gamma_c} \leqslant c c_L \|\varphi_h - \overline{\varphi}_h\|_{0,\Gamma_c} \quad \forall\, \varphi_h, \overline{\varphi}_h \in \mathcal{V}_h^+.$$

Proof. In the same way as in Theorem 3.6 it can be shown (see (3.10)) that

$$\|\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h\|_{1,\Omega} \leqslant c_L \frac{c_T \|g\|_{\infty,\Gamma_c}}{c_{\mathrm{ell}} c_K} \|\varphi_h - \overline{\varphi}_h\|_{0,\Gamma_c}, \tag{4.6}$$

where $\boldsymbol{u}_h$, $\overline{\boldsymbol{u}}_h$ are the solutions to $(\mathcal{P}(\varphi_h))_h$, $(\mathcal{P}(\overline{\varphi}_h))_h$, respectively, for $\varphi_h, \overline{\varphi}_h \in \mathcal{V}_h^+$ given. Moreover, we know (cf. (4.5) and (3.9)) that

$$\begin{aligned} \big\| r_h \|\boldsymbol{\gamma}_t \boldsymbol{u}_h\| - r_h \|\boldsymbol{\gamma}_t \overline{\boldsymbol{u}}_h\| \big\|_{0,\Gamma_c} &\leqslant (1 + c_r c_{\mathrm{inv}}) \|\boldsymbol{\gamma}_t \boldsymbol{u}_h - \boldsymbol{\gamma}_t \overline{\boldsymbol{u}}_h\|_{0,\Gamma_c} \\ &\leqslant c_T (1 + c_r c_{\mathrm{inv}}) \|\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h\|_{1,\Omega}, \end{aligned}$$

where c_T, c_r, c_{inv} are independent of h. From this and (4.6) we see that the assertion of the theorem holds with

$$c := \frac{c_T^2 (1 + c_r c_{\mathrm{inv}}) \|g\|_{\infty,\Gamma_c}}{c_{\mathrm{ell}} c_K}.$$

□

Corollary 4.5. *Let $h > 0$ be fixed. If $cc_L < 1$ then the mapping $\Psi_h\colon \mathcal{V}_h^+ \to \mathcal{V}_h^+$ is contractive. Consequently, Ψ_h has a unique fixed point and the method of successive approximations*

$$\text{(4.7)} \qquad \begin{cases} \varphi_h^0 \in \mathcal{V}_h^+ \text{ given;} \\ \text{for } k = 1, 2, \dots \text{ set } \varphi_h^k := \Psi_h(\varphi_h^{k-1}) \end{cases}$$

is convergent for any choice of φ_h^0.

Let us suppose that $\boldsymbol{K} \cap \boldsymbol{C}^\infty(\overline{\Omega})$ is dense in $\boldsymbol{K}$ in the $\boldsymbol{H}^1(\Omega)$-norm (some cases when this assumption is satisfied are studied in [8]). Let $\{\boldsymbol{u}_h\}$, $h \to 0+$, be a sequence of discrete solutions to $(\mathcal{P})$ and let $\overline{\boldsymbol{v}} \in \boldsymbol{K}$ be arbitrary but fixed. Our density assumption ensures the existence of a sequence $\{\overline{\boldsymbol{v}}_h\}$, $\overline{\boldsymbol{v}}_h \in \boldsymbol{K}_h$, such that

$$\text{(4.8)} \qquad \overline{\boldsymbol{v}}_h \to \overline{\boldsymbol{v}} \quad \text{in } \boldsymbol{H}^1(\Omega),\ h \to 0+.$$

Since $\{\boldsymbol{u}_h\}$ is bounded in $\boldsymbol{H}^1(\Omega)$ and $\boldsymbol{u}_h \in \boldsymbol{K}_h \subset \boldsymbol{K}$ $\forall\, h > 0$, one can pass to a subsequence $\{\boldsymbol{u}_{h'}\} \subseteq \{\boldsymbol{u}_h\}$ and find a function $\boldsymbol{u} \in \boldsymbol{K}$ such that

$$\boldsymbol{u}_{h'} \rightharpoonup \boldsymbol{u} \quad \text{in } \boldsymbol{H}^1(\Omega),\ h' \to 0+.$$

This together with (4.8) yields

$$\begin{gathered} \limsup_{h' \to 0+} a(\boldsymbol{u}_{h'}, \overline{\boldsymbol{v}}_{h'} - \boldsymbol{u}_{h'}) \leqslant a(\boldsymbol{u}, \overline{\boldsymbol{v}} - \boldsymbol{u}), \\ \lim_{h' \to 0+} \boldsymbol{F}(\overline{\boldsymbol{v}}_{h'} - \boldsymbol{u}_{h'}) = \boldsymbol{F}(\overline{\boldsymbol{v}} - \boldsymbol{u}), \\ \|\boldsymbol{\gamma}_t \boldsymbol{u}_{h'}\| \to \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \quad \text{in } L^2(\Gamma_c),\ h' \to 0+. \end{gathered}$$

Using the last relation, (4.1), (i) of Lemma 3.1, the inverse inequality between $H^1(\Gamma_c)$ and $H^{1/2}(\Gamma_c)$ and the boundedness of $\{\boldsymbol{\gamma}_t \boldsymbol{u}_{h'}\}$ in the $\boldsymbol{H}^{1/2}(\Gamma_c)$-norm, we obtain

$$\begin{aligned} \text{(4.9)} \qquad & \big\| r_{h'} \|\boldsymbol{u}_{h't}\| - \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \big\|_{0,\Gamma_c} \\ & \leqslant \big\| r_{h'} \|\boldsymbol{u}_{h't}\| - \|\boldsymbol{\gamma}_t \boldsymbol{u}_{h'}\| \big\|_{0,\Gamma_c} + \big\| \|\boldsymbol{\gamma}_t \boldsymbol{u}_{h'}\| - \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \big\|_{0,\Gamma_c} \\ & \leqslant c_r h'_{\Gamma_c} \big\| \|\boldsymbol{\gamma}_t \boldsymbol{u}_{h'}\| \big\|_{1,\Gamma_c} + \big\| \|\boldsymbol{\gamma}_t \boldsymbol{u}_{h'}\| - \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \big\|_{0,\Gamma_c} \\ & \leqslant c_r h'_{\Gamma_c} \|\boldsymbol{\gamma}_t \boldsymbol{u}_{h'}\|_{1,\Gamma_c} + \big\| \|\boldsymbol{\gamma}_t \boldsymbol{u}_{h'}\| - \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \big\|_{0,\Gamma_c} \\ & \leqslant c_r \overline{c}_{\mathrm{inv}} (h'_{\Gamma_c})^{1/2} \|\boldsymbol{\gamma}_t \boldsymbol{u}_{h'}\|_{1/2,\Gamma_c} + \big\| \|\boldsymbol{\gamma}_t \boldsymbol{u}_{h'}\| - \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \big\|_{0,\Gamma_c} \xrightarrow{h' \to 0+} 0. \end{aligned}$$

Hence, for an appropriate subsequence of $\{\boldsymbol{u}_{h'}\}$ denoted by the same symbol we have

$$r_{h'} \|\boldsymbol{u}_{h't}\| \to \|\boldsymbol{\gamma}_t \boldsymbol{u}\| \quad \text{a.e. on } \Gamma_c,\ h' \to 0+\,.$$

The above results imply

$$\lim_{h'\to 0+}\int_{\Gamma_c}\mathcal{F}(r_{h'}\|\boldsymbol{u}_{h't}\|)g(\|\overline{\boldsymbol{v}}_{h't}\|-\|\boldsymbol{u}_{h't}\|)\,\mathrm{d}S=\int_{\Gamma_c}\mathcal{F}(\|\boldsymbol{u}_t\|)g(\|\overline{\boldsymbol{v}}_t\|-\|\boldsymbol{u}_t\|)\,\mathrm{d}S.$$

Consequently, $\boldsymbol{u}\in\boldsymbol{K}$ satisfies

$$a(\boldsymbol{u},\overline{\boldsymbol{v}}-\boldsymbol{u})+\int_{\Gamma_c}\mathcal{F}(\|\boldsymbol{u}_t\|)g(\|\overline{\boldsymbol{v}}_t\|-\|\boldsymbol{u}_t\|)\,\mathrm{d}S\geqslant\boldsymbol{F}(\overline{\boldsymbol{v}}-\boldsymbol{u}).$$

Since $\overline{\boldsymbol{v}}\in\boldsymbol{K}$ was arbitrary, the function $\boldsymbol{u}$ solves $(\mathcal{P})$ and $\|\boldsymbol{\gamma}_t\boldsymbol{u}\|$ is a fixed point of Ψ. Finally, from the boundedness of $\{r_{h'}\|\boldsymbol{u}_{h't}\|\}$ in the $H^{1/2}(\Gamma_c)$-norm and (4.9) it follows that $\{r_{h'}\|\boldsymbol{u}_{h't}\|\}$ tends weakly to $\|\boldsymbol{\gamma}_t\boldsymbol{u}\|$ in the $H^{1/2}(\Gamma_c)$-norm.

The result is summarized in the following theorem.

Theorem 4.6. *Let $\boldsymbol{K}\cap\boldsymbol{C}^\infty(\overline{\Omega})$ be dense in $\boldsymbol{K}$ in the $\boldsymbol{H}^1(\Omega)$-norm, let $\{\boldsymbol{u}_h\}$, $h\to 0+$, be a sequence of discrete solutions to $(\mathcal{P})$. Then for any sequence $\{r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|\}$, $h\to 0+$, of fixed points of Ψ_h there exists a subsequence of $\{\boldsymbol{u}_h\}$ (denoted by the same symbol) such that*

$$\begin{cases}\boldsymbol{u}_h\rightharpoonup\boldsymbol{u}\ \text{in}\ \boldsymbol{H}^1(\Omega),\ \ h\to 0+,\\ r_h\|\boldsymbol{\gamma}_t\boldsymbol{u}_h\|\rightharpoonup\|\boldsymbol{\gamma}_t\boldsymbol{u}\|\ \text{in}\ H^{1/2}(\Gamma_c),\ \ h\to 0+,\end{cases}\tag{4.10}$$

where $\boldsymbol{u}$ solves $(\mathcal{P})$ and $\|\boldsymbol{\gamma}_t\boldsymbol{u}\|$ is the respective fixed point of Ψ. In addition, if $(\mathcal{P})$ has a unique solution, (4.10) *holds for the whole sequences.*

5. Mixed variational formulation

A natural way how to find a fixed point of the mapping Ψ is to use the *method of successive approximations* (3.11). Since the main step in this method is a contact problem with given friction and a coefficient which does not depend on a solution, we focus on it now.

It is known (see [4]) that the solution $\boldsymbol{u}$ of $(\mathcal{P}(\varphi))$, $\varphi\in H_+^{1/2}(\Gamma_c)$, can be equivalently characterized as a solution of the following minimization problem:

$$\begin{cases}\text{Find } \boldsymbol{u}:=\boldsymbol{u}(\varphi)\in\boldsymbol{K}\text{ such that}\\ J_\varphi(\boldsymbol{u})\leqslant J_\varphi(\boldsymbol{v})\ \ \forall\,\boldsymbol{v}\in\boldsymbol{K},\end{cases}$$

where

$$J_\varphi(\boldsymbol{v})=\frac{1}{2}a(\boldsymbol{v},\boldsymbol{v})-\boldsymbol{F}(\boldsymbol{v})+j_\varphi(\boldsymbol{v}),\quad \boldsymbol{v}\in\boldsymbol{V},$$

with

$$j_\varphi(\boldsymbol{v}) = \int_{\Gamma_c} \mathcal{F}(\varphi) g \|\boldsymbol{v}_t\| \,\mathrm{d}S, \quad \boldsymbol{v} \in \boldsymbol{V}.$$

This is a constrained minimization problem for the non-differentiable total potential energy functional J_φ. To release the unilateral constraint $u_n \leqslant 0$ on Γ_c and to regularize the non-differentiable term j_φ we shall use a mixed variational formulation.

Let $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ be two unit orthogonal vectors in the tangential plane to Γ_c. Then the triplet $\{\boldsymbol{n}, \boldsymbol{t}_1, \boldsymbol{t}_2\}$ forms an orthonormal basis in $\mathbb{R}^3$ and any vector function $\boldsymbol{v}\colon \Gamma_c \to \mathbb{R}^3$ can be represented in the coordinate system $\{\boldsymbol{n}, \boldsymbol{t}_1, \boldsymbol{t}_2\}$ as

$$\boldsymbol{v}(\boldsymbol{x}) = (v_n(\boldsymbol{x}), \boldsymbol{v}_t(\boldsymbol{x}))^\top \in \mathbb{R} \times \mathbb{R}^2, \quad \boldsymbol{x} \in \Gamma_c,$$

where $v_n(\boldsymbol{x}) = (\boldsymbol{v}(\boldsymbol{x}))^\top \boldsymbol{n}$, $\boldsymbol{v}_t(\boldsymbol{x}) = (v_{t_1}(\boldsymbol{x}), v_{t_2}(\boldsymbol{x}))^\top$, $v_{t_j}(\boldsymbol{x}) = (\boldsymbol{v}(\boldsymbol{x}))^\top \boldsymbol{t}_j$, $j = 1, 2$. This representation will be used for the traces of displacement vectors on Γ_c. In accordance with the previous notation, the symbol $\|\boldsymbol{v}_t\|$ stands for the Euclidean norm of $\boldsymbol{v}_t$:

$$\|\boldsymbol{v}_t\| = ((v_{t_1})^2 + (v_{t_2})^2)^{1/2} \quad \text{on } \Gamma_c.$$

Next, let $\varphi \in H_+^{1/2}(\Gamma_c)$ be given and let us set

$$\Lambda_n = \{\mu \in (H^{1/2}(\Gamma_c))'\colon \langle \mu, \psi \rangle_{1/2,\Gamma_c} \geqslant 0 \ \forall \psi \in H_+^{1/2}(\Gamma_c)\},$$
$$\boldsymbol{\Lambda}_t(\varphi) = \{\boldsymbol{\mu}_t \in (L^2(\Gamma_c))^2\colon \|\boldsymbol{\mu}_t\| \leqslant \mathcal{F}(\varphi) g \text{ a.e. on } \Gamma_c\},$$

where $(H^{1/2}(\Gamma_c))'$ stands for the topological dual space of $H^{1/2}(\Gamma_c)$, $\langle \cdot, \cdot \rangle_{1/2,\Gamma_c}$ denotes the respective duality pairing and $g \in L^2(\Gamma_c)$, $g \geqslant 0$ a.e. on Γ_c, is a given slip bound.

It is easy to see that

$$\min_{\boldsymbol{v} \in \boldsymbol{K}} J_\varphi(\boldsymbol{v}) = \min_{\boldsymbol{v} \in \boldsymbol{V}} \sup_{\substack{\mu_n \in \Lambda_n \\ \boldsymbol{\mu}_t \in \boldsymbol{\Lambda}_t(\varphi)}} \mathcal{L}(\boldsymbol{v}, \mu_n, \boldsymbol{\mu}_t),$$

where $\mathcal{L}\colon \boldsymbol{V} \times \Lambda_n \times \boldsymbol{\Lambda}_t(\varphi) \to \mathbb{R}^1$ is the Lagrangian defined by

$$\mathcal{L}(\boldsymbol{v}, \mu_n, \boldsymbol{\mu}_t) = \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{v}) + \langle \mu_n, v_n \rangle_{1/2,\Gamma_c} + \int_{\Gamma_c} \boldsymbol{\mu}_t^\top \boldsymbol{v}_t \,\mathrm{d}S,$$
$$(\boldsymbol{v}, \mu_n, \boldsymbol{\mu}_t)^\top \in \boldsymbol{V} \times \Lambda_n \times \boldsymbol{\Lambda}_t(\varphi).$$

By a *mixed variational formulation* of $(\mathcal{P}(\varphi))$ we mean a problem of finding a *saddle-point* of $\mathcal{L}$ on $\boldsymbol{V} \times \Lambda_n \times \boldsymbol{\Lambda}_t(\varphi)$:

$$\begin{cases} \text{Find } (\boldsymbol{w}, \lambda_n, \boldsymbol{\lambda}_t)^\top \in \boldsymbol{V} \times \Lambda_n \times \boldsymbol{\Lambda}_t(\varphi) \text{ such that} \\ \mathcal{L}(\boldsymbol{w}, \mu_n, \boldsymbol{\mu}_t) \leqslant \mathcal{L}(\boldsymbol{w}, \lambda_n, \boldsymbol{\lambda}_t) \leqslant \mathcal{L}(\boldsymbol{v}, \lambda_n, \boldsymbol{\lambda}_t) \ \forall (\boldsymbol{v}, \mu_n, \boldsymbol{\mu}_t)^\top \in \boldsymbol{V} \times \Lambda_n \times \boldsymbol{\Lambda}_t(\varphi), \end{cases}$$

or equivalently:

$$(\mathcal{M}(\varphi)) \qquad \begin{cases} \text{Find } (\boldsymbol{w}, \lambda_n, \boldsymbol{\lambda}_t)^\top \in \boldsymbol{V} \times \Lambda_n \times \boldsymbol{\Lambda}_t(\varphi) \text{ such that} \\ a(\boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{F}(\boldsymbol{v}) - \langle \lambda_n, v_n \rangle_{1/2,\Gamma_c} - \displaystyle\int_{\Gamma_c} \boldsymbol{\lambda}_t^\top \boldsymbol{v}_t \, \mathrm{d}S \;\; \forall \, \boldsymbol{v} \in \boldsymbol{V}, \\ \langle \mu_n - \lambda_n, w_n \rangle_{1/2,\Gamma_c} + \displaystyle\int_{\Gamma_c} (\boldsymbol{\mu}_t - \boldsymbol{\lambda}_t)^\top \boldsymbol{w}_t \, \mathrm{d}S \leqslant 0 \\ \hfill \forall \, (\mu_n, \boldsymbol{\mu}_t)^\top \in \Lambda_n \times \boldsymbol{\Lambda}_t(\varphi). \end{cases}$$

The following result is a standard one.

Theorem 5.1. *There exists a unique solution* $(\boldsymbol{w}, \lambda_n, \boldsymbol{\lambda}_t)^\top$ *of* $(\mathcal{M}(\varphi))$. *In addition,*

$$\boldsymbol{w} = \boldsymbol{u}, \quad \lambda_n = -T_n(\boldsymbol{u}), \quad \boldsymbol{\lambda}_t = -\boldsymbol{T}_t(\boldsymbol{u}),$$

where $\boldsymbol{u} \in \boldsymbol{K}$ *solves* $(\mathcal{P}(\varphi))$.

Next, we describe an approximation of $(\mathcal{M}(\varphi))$. Recall that the sets $\boldsymbol{V}_h$ and $\mathcal{V}_h^+$ have been already defined in Section 4. Further, let $\{\mathcal{T}_H\}$, $H \to 0+$, be a family of regular partitions of $\overline{\Gamma}_c$ into rectangles R whose diameters do not exceed H. With any $\mathcal{T}_H$ we associate the space of piecewise-constant functions

$$L_H = \{\mu_H \in L^2(\Gamma_c) \colon \; \mu_{H|_R} \in P_0(R) \; \forall \, R \in \mathcal{T}_H\}.$$

Let $\varphi_h \in \mathcal{V}_h^+$ be fixed. The sets

$$\Lambda_{nH} = \{\mu_{nH} \in L_H \colon \; \mu_{nH} \geqslant 0 \text{ a.e. on } \Gamma_c\},$$
$$\boldsymbol{\Lambda}_{tH}(\varphi_h) = \left\{ \boldsymbol{\mu}_{tH} \in (L_H)^2 \colon \; \|\boldsymbol{\mu}_{tH|_R}\| \leqslant \frac{\int_R \mathcal{F}(\varphi_h) g \, \mathrm{d}S}{\mathrm{meas}_2(R)} \;\; \forall \, R \in \mathcal{T}_H \right\},$$

where $\mathrm{meas}_2(R)$ is the area of R, will be used as the discretizations of Λ_n and $\boldsymbol{\Lambda}_t(\varphi_h)$, respectively.

The discretization of the mixed formulation $(\mathcal{M}(\varphi_h))$ reads as follows:

$$(\mathcal{M}(\varphi_h))_{hH} \begin{cases} \text{Find } (\boldsymbol{w}_h, \lambda_{nH}, \boldsymbol{\lambda}_{tH})^\top \in \boldsymbol{V}_h \times \Lambda_{nH} \times \boldsymbol{\Lambda}_{tH}(\varphi_h) \text{ such that} \\ a(\boldsymbol{w}_h, \boldsymbol{v}_h) = \boldsymbol{F}(\boldsymbol{v}_h) - \displaystyle\int_{\Gamma_c} \lambda_{nH} v_{hn} \, \mathrm{d}S - \int_{\Gamma_c} \boldsymbol{\lambda}_{tH}^\top \boldsymbol{v}_{ht} \, \mathrm{d}S \quad \forall \, \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \displaystyle\int_{\Gamma_c} (\mu_{nH} - \lambda_{nH}) w_{hn} \, \mathrm{d}S + \int_{\Gamma_c} (\boldsymbol{\mu}_{tH} - \boldsymbol{\lambda}_{tH})^\top \boldsymbol{w}_{ht} \, \mathrm{d}S \leqslant 0 \\ \hfill \forall \, (\mu_{nH}, \boldsymbol{\mu}_{tH})^\top \in \Lambda_{nH} \times \boldsymbol{\Lambda}_{tH}(\varphi_h). \end{cases}$$

It is known (see [8], [9]) that $(\mathcal{M}(\varphi_h))_{hH}$ has a unique solution provided that the following *stability condition* is satisfied:

$$
(5.1)\qquad \begin{cases} \text{If } (\mu_{nH}, \boldsymbol{\mu}_{tH})^\top \in \Lambda_{nH} \times \boldsymbol{\Lambda}_{tH}(\varphi_h) \text{ is such that} \\ \displaystyle\int_{\Gamma_c} \mu_{nH} v_{hn} \,\mathrm{d}S + \int_{\Gamma_c} \boldsymbol{\mu}_{tH}^\top \boldsymbol{v}_{ht} \,\mathrm{d}S = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \\ \text{then } (\mu_{nH}, \boldsymbol{\mu}_{tH})^\top = (0, \boldsymbol{0})^\top. \end{cases}
$$

Denote by

$$
\boldsymbol{K}_{hH} = \Big\{ \boldsymbol{v}_h \in \boldsymbol{V}_h \colon \int_R v_{hn} \,\mathrm{d}S \leqslant 0 \ \forall R \in \mathcal{T}_H \Big\},
$$

$$
j_{\varphi_h H}(\boldsymbol{v}_h) = \sup_{\boldsymbol{\mu}_{tH} \in \boldsymbol{\Lambda}_{tH}(\varphi_h)} \int_{\Gamma_c} \boldsymbol{\mu}_{tH}^\top \boldsymbol{v}_{ht} \,\mathrm{d}S, \ \ \boldsymbol{v}_h \in \boldsymbol{V}_h,
$$

the approximations of $\boldsymbol{K}$ and j_{φ_h}, respectively. It is easy to show that the first component $\boldsymbol{w}_h$ of the solution to $(\mathcal{M}(\varphi_h))_{hH}$ solves the variational inequality of the second kind:

$$
\boldsymbol{w}_h \in \boldsymbol{K}_{hH} \colon a(\boldsymbol{w}_h, \boldsymbol{v}_h - \boldsymbol{w}_h) + j_{\varphi_h H}(\boldsymbol{v}_h) - j_{\varphi_h H}(\boldsymbol{w}_h) \geqslant \boldsymbol{F}(\boldsymbol{v}_h - \boldsymbol{w}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{K}_{hH}.
$$

It is worth noticing that $\boldsymbol{K}_{hH}$ is an *external* approximation of $\boldsymbol{K}$, since the non-penetration condition $w_{hn} \leqslant 0$ on Γ_c is satisfied in a weak (integral) sense only.

Next we present the algebraic form of $(\mathcal{M}(\varphi_h))_{hH}$. Let $h, H > 0$ be fixed and suppose that the stability condition (5.1) is satisfied. By $\vec{\mathbf{v}} \in \mathbb{R}^p$, $p = \dim \boldsymbol{V}_h$, we denote the coordinates of $\boldsymbol{v}_h$ with respect to a chosen basis in $\boldsymbol{V}_h$. Analogously, $\vec{\boldsymbol{\mu}}_n$, $\vec{\boldsymbol{\mu}}_{t_1}$, $\vec{\boldsymbol{\mu}}_{t_2} \in \mathbb{R}^r$, $r = \dim L_H$, are the coordinates of μ_{nH}, $\mu_{t_1 H}$, $\mu_{t_2 H}$, respectively, with respect to the basis of L_H consisting of the characteristic functions of $\operatorname{int} R_i$, $R_i \in \mathcal{T}_H$, $i = 1, \ldots, r$. Let

$$
(5.2)\qquad \boldsymbol{\Lambda}^n = \mathbb{R}^r_+,
$$

$$
(5.3)\qquad \boldsymbol{\Lambda}^t(\varphi_h) = \Big\{ (\vec{\boldsymbol{\mu}}_{t_1}, \vec{\boldsymbol{\mu}}_{t_2})^\top = (\mu_{t_1 1}, \ldots, \mu_{t_1 r}, \mu_{t_2 1}, \ldots, \mu_{t_2 r})^\top \in \mathbb{R}^{2r} \colon \|(\mu_{t_1 i}, \mu_{t_2 i})\| \leqslant \frac{\int_{R_i} \mathcal{F}(\varphi_h) g \,\mathrm{d}S}{\operatorname{meas}_2(R_i)} \ \ \forall i = 1, \ldots, r \Big\}
$$

be the algebraic representatives of Λ_{nH}, $\boldsymbol{\Lambda}_{tH}(\varphi_h)$, respectively. Further, let $\mathbb{K}$ be the stiffness matrix, $\vec{\mathbf{f}}$ the load vector, $\mathbb{M}$ the kinematic transformation matrix linking the primal and the dual variables and $\mathbb{B}^1, \mathbb{B}^2, \mathbb{B}^3$ the matrices representing the linear mappings $\boldsymbol{v}_h \mapsto v_{hn}$, $\boldsymbol{v}_h \mapsto v_{ht_1}$, $\boldsymbol{v}_h \mapsto v_{ht_2}$, $\boldsymbol{v}_h \in \boldsymbol{V}_h$, respectively.

The algebraic form of $(\mathcal{M}(\varphi_h))_{hH}$ reads as follows:

$$(5.4)\qquad \begin{cases} \text{Find } (\vec{\mathbf{w}}, \vec{\boldsymbol{\lambda}})^\top \in \mathbb{R}^p \times \boldsymbol{\Lambda}(\varphi_h) \text{ such that} \\ \mathbb{K}\vec{\mathbf{w}} = \vec{\mathbf{f}} - \mathbb{B}^\top \vec{\boldsymbol{\lambda}}, \\ (\vec{\boldsymbol{\mu}} - \vec{\boldsymbol{\lambda}})^\top \mathbb{B}\vec{\mathbf{w}} \leqslant 0 \quad \forall\, \vec{\boldsymbol{\mu}} \in \boldsymbol{\Lambda}(\varphi_h), \end{cases}$$

where $\vec{\boldsymbol{\mu}} := (\vec{\boldsymbol{\mu}}_n, \vec{\boldsymbol{\mu}}_{t_1}, \vec{\boldsymbol{\mu}}_{t_2})^\top$, $\vec{\boldsymbol{\lambda}} := (\vec{\boldsymbol{\lambda}}_n, \vec{\boldsymbol{\lambda}}_{t_1}, \vec{\boldsymbol{\lambda}}_{t_2})^\top$, $\boldsymbol{\Lambda}(\varphi_h) := \boldsymbol{\Lambda}^n \times \boldsymbol{\Lambda}^t(\varphi_h)$ and

$$\mathbb{B} := \begin{pmatrix} \mathbb{M}\mathbb{B}^1 \\ \mathbb{M}\mathbb{B}^2 \\ \mathbb{M}\mathbb{B}^3 \end{pmatrix},$$

for short.

For numerical realization of (5.4) we shall use the dual approach. From $(5.4)_2$ one can express $\vec{\mathbf{w}}$:

$$\vec{\mathbf{w}} = \mathbb{K}^{-1}(\vec{\mathbf{f}} - \mathbb{B}^\top \vec{\boldsymbol{\lambda}}).$$

Inserting $\vec{\mathbf{w}}$ into $(5.4)_3$, we obtain a new problem in terms of the Lagrange multipliers which is equivalent to the following *quadratic programming problem*:

$$(5.5)\qquad \begin{cases} \text{Find } \vec{\boldsymbol{\lambda}} \in \boldsymbol{\Lambda}(\varphi_h) \text{ such that} \\ \boldsymbol{\mathcal{S}}(\vec{\boldsymbol{\lambda}}) \leqslant \boldsymbol{\mathcal{S}}(\vec{\boldsymbol{\mu}}) \quad \forall\, \vec{\boldsymbol{\mu}} \in \boldsymbol{\Lambda}(\varphi_h), \end{cases}$$

where

$$\boldsymbol{\mathcal{S}}(\vec{\boldsymbol{\mu}}) = \tfrac{1}{2}\vec{\boldsymbol{\mu}}^\top \mathbb{Q}\vec{\boldsymbol{\mu}} - \vec{\mathbf{h}}^\top \vec{\boldsymbol{\mu}}, \quad \vec{\boldsymbol{\mu}} \in \boldsymbol{\Lambda}(\varphi_h),$$

with

$$(5.6)\qquad \mathbb{Q} := \mathbb{B}\mathbb{K}^{-1}\mathbb{B}^\top, \quad \vec{\mathbf{h}} := \mathbb{B}\mathbb{K}^{-1}\vec{\mathbf{f}}.$$

Let us point out that $\boldsymbol{\Lambda}^n$ is defined by the *simple* (box) constraints (5.2) while $\boldsymbol{\Lambda}^t(\varphi_h)$ is determined by the *quadratic constraints* (5.3). Since the quadratic constraints are *separated*, one can use an algorithm that combines the conjugate gradient method with the gradient projections for solving (5.5). For detailed theoretical analysis of this approach we refer to [10].

The iterative process (4.7) based on the dual formulation (5.5) reads as

$$(5.7)\qquad \begin{cases} \text{Let } \varphi_h^0 \in \mathcal{V}_h^+ \text{ be given;} \\ \text{for } \varphi_h^{k-1} \in \mathcal{V}_h^+,\ k = 1, 2, \dots \text{ known, solve:} \\ \vec{\boldsymbol{\lambda}} = \arg\min\{\boldsymbol{\mathcal{S}}(\vec{\boldsymbol{\mu}})\colon\ \vec{\boldsymbol{\mu}} \in \boldsymbol{\Lambda}(\varphi_h^{k-1})\}; \\ \text{set } \vec{\mathbf{w}} = \mathbb{K}^{-1}(\vec{\mathbf{f}} - \mathbb{B}^\top \vec{\boldsymbol{\lambda}}); \\ \quad \varphi_h^k = r_h \|\boldsymbol{\gamma}_t \boldsymbol{w}_h\|; \\ \text{repeat until stopping criterion.} \end{cases}$$

Here $\boldsymbol{w}_h$ denotes the element of $\boldsymbol{V}_h$ whose nodal values are given by $\vec{\mathbf{w}}$.

6. Model examples

We now present numerical results of several model examples. A deformable body Ω will be represented by the brick $(0,3)\times(0,1)\times(0,1)$ (in m) which is fixed along $\Gamma_{\boldsymbol{u}}=\{0\}\times(0,1)\times(0,1)$ and supported by the rigid foundation $S=\mathbb{R}^2\times\mathbb{R}^1_-$, i.e. $\Gamma_c=(0,3)\times(0,1)\times\{0\}$. The rest of the boundary $\partial\Omega$ represents $\Gamma_{\boldsymbol{p}}$, where the body is subject to surface tractions of density $\boldsymbol{p}=(p_1,p_2,p_3)^\top$ (see Fig. 2):

$$\begin{aligned}
&p_1=p_x^1,\ p_2=0,\ p_3=p_z^1 && \text{on } \Gamma_{\boldsymbol{p}}^1=\{\boldsymbol{x}=(x_1,x_2,x_3)^\top\in\Gamma_{\boldsymbol{p}}\colon\ x_1=3\},\\
&p_1=0,\ p_2=0,\ p_3=p_z^2 && \text{on } \Gamma_{\boldsymbol{p}}^2=\{\boldsymbol{x}=(x_1,x_2,x_3)^\top\in\Gamma_{\boldsymbol{p}}\colon\ x_3=1\},\\
&\boldsymbol{p}=\boldsymbol{0} && \text{on } \Gamma_{\boldsymbol{p}}\setminus(\overline{\Gamma_{\boldsymbol{p}}^1\cup\Gamma_{\boldsymbol{p}}^2}),
\end{aligned}$$

where $p_x^1=1.\mathrm{e}7\,[\mathrm{Pa}]$, $p_z^1=2.\mathrm{e}7\,[\mathrm{Pa}]$ and $p_z^2=-3.\mathrm{e}7\,[\mathrm{Pa}]$. The volume forces will be neglected, i.e. $\boldsymbol{f}=\boldsymbol{0}$ in Ω.

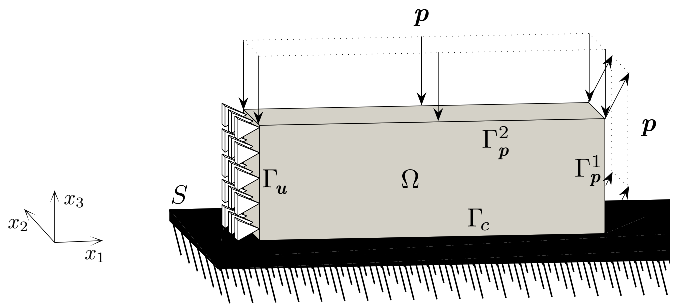


Figure 2. Geometry.

The brick is made of an elastic, isotropic, and homogeneous material characterized by Poisson's ratio $\sigma=0.277$ and Young's modulus $E=21.19\mathrm{e}10\,[\mathrm{Pa}]$ (steel). The coefficient of friction $\mathcal{F}$ is defined by

$$\mathcal{F}(t)=\begin{cases}0.3 & \text{if } t\leqslant 10^{-5};\\[4pt] 0.3-\dfrac{0.1\,\mathrm{param}}{2}(t-10^{-5}) & \text{if } t\in\Big(10^{-5},10^{-5}+\dfrac{2}{\mathrm{param}}\Big);\\[4pt] 0.2 & \text{if } t\geqslant 10^{-5}+\dfrac{2}{\mathrm{param}}.\end{cases}\tag{6.1}$$

Three different values of param were considered, namely param $=2.\mathrm{e}4$, $6.\mathrm{e}4$, and $3.\mathrm{e}5$ (see Fig. 3). The slip bound was chosen to be $g=2.\mathrm{e}7\,[\mathrm{Pa}]$.

To construct partitions $\mathcal{T}_h$ we cut $\overline{\Omega}$ into $3n\times n\times n$ small cubes for $n=4$, 6, 8, 10, 12, 14 and 16. Next, each cube is divided into five tetrahedra as shown in

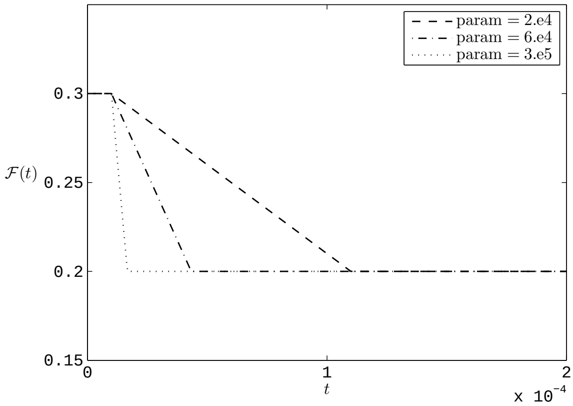


Figure 3. Coefficients of friction.

Fig. 4. Having $\mathcal{T}_h$ at our disposal, we construct the partition $\mathcal{T}_H$ of $\overline{\Gamma}_c$ as shown in Fig. 5: the partition $\mathcal{T}_{h|_{\overline{\Gamma}_c}}$ and its nodes are depicted by the fine lines and the black dots, respectively, while the partition $\mathcal{T}_H$ is "the chessboard" on $\overline{\Gamma}_c$ whose elements are constructed by piecing together eight triangles of $\mathcal{T}_{h|_{\overline{\Gamma}_c}}$ which share a common contact node. One can easily verify the satisfaction of the stability condition (5.1) for such partitions.

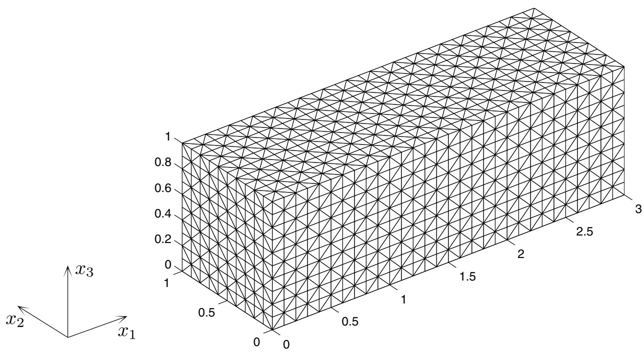


Figure 4. Partition $\mathcal{T}_h$ of $\overline{\Omega}$.

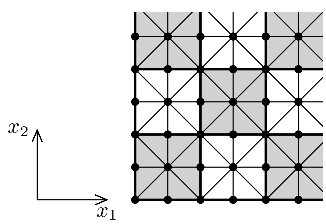


Figure 5. Partition $\mathcal{T}_H$ of $\overline{\Gamma}_c$.

The initial approximation of $\|\boldsymbol{w}_t\|$ for the method of successive approximations was chosen to be $\varphi_h^0 = 0$ on Γ_c. The stopping criterion of the outer (fixed-point) loop is

$$\operatorname{err}(k) := \frac{\|\vec{\boldsymbol{\varphi}}^k - \vec{\boldsymbol{\varphi}}^{k-1}\|}{\|\vec{\boldsymbol{\varphi}}^k\|} \leqslant 10^{-4},$$

where $\vec{\boldsymbol{\varphi}}^k$ is the vector whose components are the values of φ_h^k at the contact nodes and $\|\cdot\|$ stands for the Euclidean norm. The minimization problem (5.5) was realized by the algorithm described in [10] with a minor modification—the proportioning and the expansion steps are performed simultaneously. The stopping criterion of this inner loop is

$$\|\tilde{\mathbf{g}}(\vec{\boldsymbol{\lambda}})\| \leqslant 10^{-6}\|\vec{\mathbf{h}}\|,$$

where $\tilde{\mathbf{g}}(\vec{\boldsymbol{\lambda}})$ is the so-called projected gradient of $\boldsymbol{\mathcal{S}}$ at $\vec{\boldsymbol{\lambda}}$, $\|\cdot\|$ is the Euclidean norm and $\vec{\mathbf{h}}$ is defined in (5.6).

Tab. 1 presents results for different values of n and for the coefficient of friction $\mathcal{F}$ defined by (6.1) with param = 6.e4, while Tab. 2 compares the results for different coefficients $\mathcal{F}$ on the finest mesh ($n = 16$). Here $n_{\rm p}$, $n_{\rm d}$ stand for the total number of the primal and dual variables, respectively, and 'it' is the number of the fixed-point iterations. Further, $n_{\rm mult}$ is the total number of the multiplications by $\mathbb{K}^{-1}$, which is the most expensive part of the algorithm (in fact, we do not compute the matrix $\mathbb{K}^{-1}$, but we first perform the Cholesky factorization and then use the backward-substitution instead). The total computational time is given in seconds and w_{hn}^{+} is the positive part of w_{hn}:

$$w_{hn}^{+} = \max\{0, w_{hn}\} \quad \text{on } \Gamma_c.$$

Hence, the last two columns of the tables can be viewed as a measure of violation of the non-penetration condition. The convergence history of the method of successive approximations for the finest mesh (i.e. the dependence of err on the number of iterations) is depicted in Fig. 6.

n	$n_{\rm p}$	$n_{\rm d}$	it	$n_{\rm mult}$	time	$\|w_{hn}^{+}\|_{0,\Gamma_c}$	$\|w_{hn}^{+}\|_{\infty,\Gamma_c}$
4	900	36	6	2383	13	7.0e−6	1.6e−5
6	2646	81	6	2063	60	1.6e−6	4.6e−6
8	5832	144	6	3381	341	1.4e−6	7.4e−6
10	10890	225	6	3622	1006	1.5e−6	4.0e−6
12	18252	324	6	3985	2565	8.4e−7	2.7e−6
14	28350	441	7	3962	5221	7.7e−7	4.5e−6
16	41616	576	6	4432	11033	8.2e−7	2.2e−6

Table 1. Different meshes.

param	it	$n_{\rm mult}$	time	$\|w_{hn}^{+}\|_{0,\Gamma_c}$	$\|w_{hn}^{+}\|_{\infty,\Gamma_c}$
2.e4	6	3870	9705	8.4e−7	2.3e−6
6.e4	6	4432	11033	8.2e−7	2.2e−6
3.e5	7	4604	11490	8.0e−7	2.2e−6

Table 2. Different coefficients $\mathcal{F}$.

The next figures illustrate the behaviour of $\boldsymbol{w}_{h|\Gamma_c}$ and of the Lagrange multipliers λ_{nH}, $\boldsymbol{\lambda}_{tH}$ for $n = 16$ and $\mathcal{F}$ defined by (6.1) with param = 6.e4. The deformed

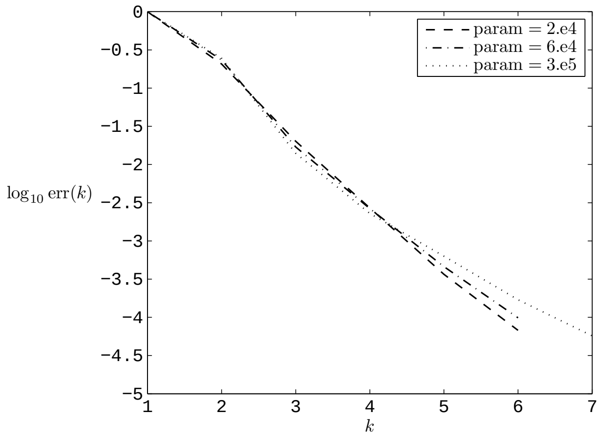


Figure 6. Convergence history.

body is shown in Fig. 7 (the deformation is 500× enlarged). The graphs of $-w_{hn}$ and λ_{nH} on Γ_c are depicted in Figs. 8 a) and 8 b), respectively. Similar graphs but

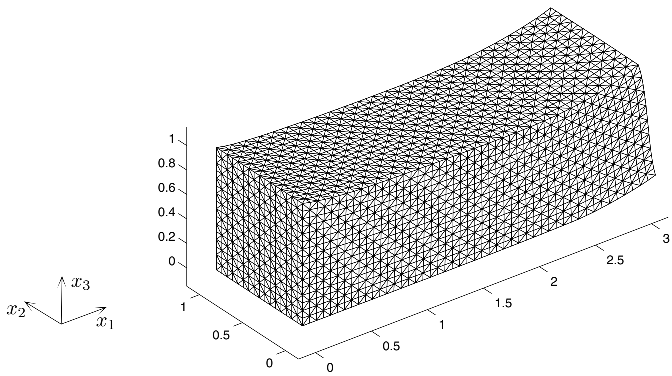
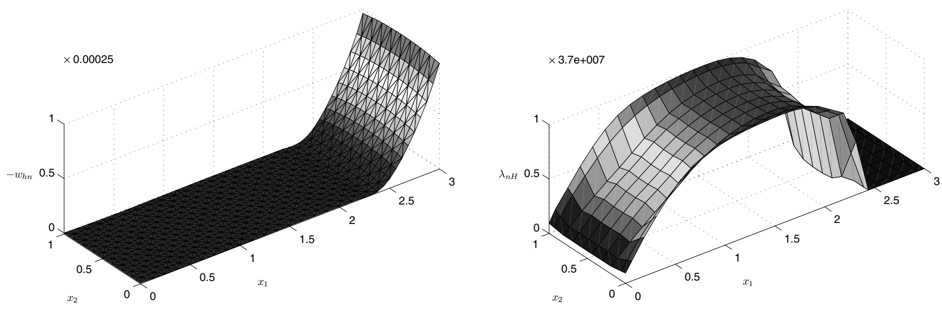


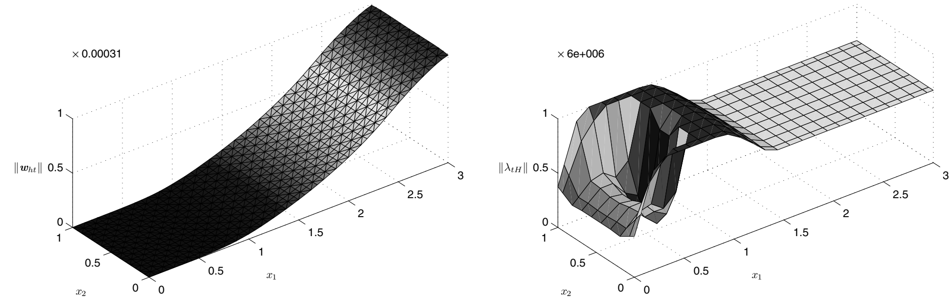
Figure 7. Deformed body.



a) Normal displacements on Γ_c.   b) Lagrange multiplier λ_{nH}.

Figure 8.

for $\|\boldsymbol{w}_{ht}\|$ and $\|\boldsymbol{\lambda}_{tH}\|$ on Γ_c are shown in Figs. 9 a) and 9 b), respectively. Finally, Fig. 10 illustrates the distribution of the coefficient $\mathcal{F}$ along Γ_c and Fig. 11 explains in more detail the behaviour of $\boldsymbol{\lambda}_{tH}$. The radius of each circle whose centre is in the centre of gravity of $R \in \mathcal{T}_H$ is equal to $\int_R \mathcal{F}(\|\boldsymbol{w}_{ht}\|) g \, \mathrm{d}S / \operatorname{meas}_2(R)$ whereas the segment emanating from its centre represents the vector $\boldsymbol{\lambda}_{tH}$ in R.



a) Norm of the tangential displacements on Γ_c. b) Norm of the Lagrange multiplier $\boldsymbol{\lambda}_{tH}$.

Figure 9.

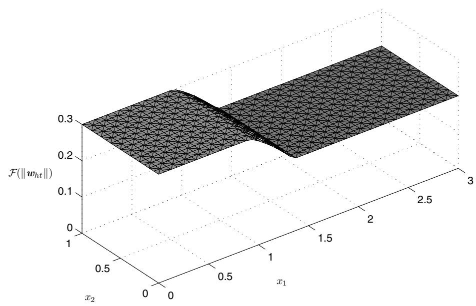


Figure 10. Coefficient $\mathcal{F}$.

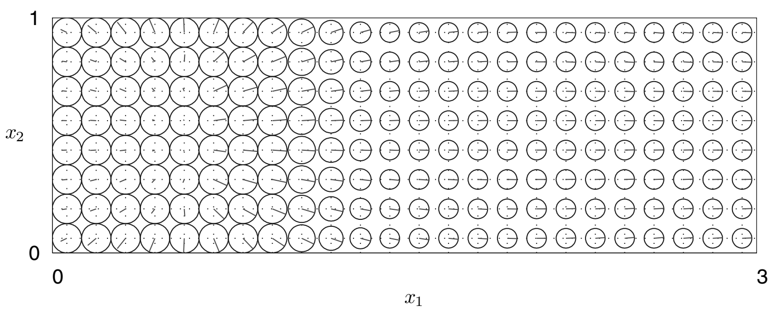


Figure 11. Lagrange multiplier $\boldsymbol{\lambda}_{tH}$.

7. Conclusions

Theoretical analysis, approximation and numerical realization of 3D contact problems with given friction and a coefficient of friction depending on the solution were presented. The mathematical analysis, as well as numerical realization are based on a fixed-point formulation of this problem. We proved the existence of at least one fixed point provided that the coefficient of friction is represented by a continuous, positive and bounded function. Conditions guaranteeing the uniqueness of the fixed point were given. Further, the convergence of the discretized problems was established. The method of successive approximations was proposed as a tool for finding the fixed points. Numerical realization uses the dual formulation of each iterative step. This formulation after a discretization leads to a quadratic programming problem for the Lagrange multipliers on Γ_c subject to simple and separable quadratic constraints.

Several numerical experiments were done. No preconditioning was used in our computations. However, the values of n_{mult} indicate that the matrix $\mathbb{Q}$ in the quadratic programming problem (5.5) is relatively well-conditioned. Moreover, only a small number of the fixed-point iterations practically independent of both the mesh size and the modulus of the Lipschitz continuity of $\mathcal{F}$ is needed to get a solution with a given accuracy. Thus, the cost of solving depends only on the cost of the individual iterative step represented by a contact problem with given friction in which the coefficient of friction does not depend on the solution. Therefore, the method of successive approximations (5.7) combined with the dual formulation of each iterative step turned out to be an efficient method for solving such problems.

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