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Applications of Mathematics, Vol. 54 (2009), No. 3, 251–266

Persistent URL: http://dml.cz/dmlcz/140363

54 (2009) APPLICATIONS OF MATHEMATICS No. 3, 251–266

NUMERICAL STUDY OF NATURAL SUPERCONVERGENCE IN LEAST-SQUARES FINITE ELEMENT METHODS FOR ELLIPTIC PROBLEMS*

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Dedicated to Ivan Hlaváček on the occasion of his 75th birthday

Abstract. Natural superconvergence of the least-squares finite element method is surveyed for the one- and two-dimensional Poisson equation. For two-dimensional problems, both the families of Lagrange elements and Raviart-Thomas elements have been considered on uniform triangular and rectangular meshes. Numerical experiments reveal that many superconvergence properties of the standard Galerkin method are preserved by the least-squares finite element method.

Keywords: least-squares, mixed finite element method, natural superconvergence, Raviart-Thomas element

MSC 2010: 65N30, 65N12

1. Introduction

Superconvergence analysis and *a posteriori* error estimation for finite element methods (FEMs) have been studied for a wide range of problems. For elliptic boundary value problems, there exists abundant mathematical and engineering literature to this subject; see, e.g., monographs and surveys [1], [3], [17], [18], [31], [34], [36], [43], [44], [46], [48], [49] and their references for a bibliography. In particular, superconvergence properties of mixed finite element approximations have been studied in, e.g., the Raviart-Thomas [42] and Brezzi-Douglas-Marini [13] spaces for elliptic problems; cf. [7], [8], [12], [19], [24], [25], [26], [27], [29], [33], [35]. Besides its own theoretical importance, superconvergence analysis and *a posteriori* error estimation

* The second author was supported in part by the US National Science Foundation under Grant DMS-0612908.

have essential applications in numerical approximation of partial differential equations (PDEs) arising from science and engineering, which can provide competitive solution quality estimates with significantly less computational cost.

Recently, the interest in least-squares finite element methods (LSFEMs) has grown continuously. The standard LSFEM transforms the original problem into a system of first-order differential equations, to whose residual an L^2 least-squares principle is applied. This mixed LSFEM possesses many desirable properties, such as the choice of approximating spaces is not subject to the Babuška-Brezzi (BB) condition [2], [11], which ensure LSFEMs successful application to a large variety of problems arising in sciences and engineering. For a review of the method, please refer to [6], [30] and their extensive bibliographies.

Optimal error estimates of LSFEMs for second-order elliptic problems have been established in, e.g., [5], [14], [15], [39], [40], [41]; they are analogous to the error estimates of standard Galerkin finite element methods. There are also several papers in literature devoted to pointwise superconvergence analysis for the least-squares method. For example, superconvergence phenomena for a LSFEM have been observed in numerical experiments of [16] for two-point boundary value problems, which are similar to those for Galerkin methods. In a later article [41], the authors studied error estimates of a least-squares mixed FEM for the one-dimensional self-adjoint equations. Derivative superconvergence at the Gaussian points and function value superconvergence at interelement nodes have been proved. In [38], optimal and superapproximation error estimates in the maximum norm and function value superconvergence at the Lobatto points are established. Nevertheless, research and applications of superconvergence and *a posteriori* error estimation for LSFEMs have not been given adequate importance, though they have become standard practice to Galerkin finite element schemes for different types of differential equations (cf., e.g., [1], [3], [32], [50]). Especially for the two-dimensional elements, there is no pointwise superconvergence result available in literature (cf. also [9] and [10] for any space dimension).

In this paper, our attention is focused on numerical study of *natural superconvergence* for triangular and rectangular least-squares elements for the Poisson equation. Here, by natural superconvergence, we refer to the pointwise superconvergence phenomena without using recovery or postprocessing techniques. We consider both the C^0 Lagrange elements and the Raviart-Thomas elements for the approximation spaces. Theoretical investigation of superconvergence and *a posteriori* error estimation for LSFEMs is an ongoing research project.

This paper is organized as follows. In Section 2, the least-squares finite element formulation is introduced. Some optimal convergence results, and superconvergence results in the one-dimensional setting are reviewed. In Section 3, numerical investigation for some two-dimensional least-squares elements are conducted. Some remarks are made in Section 4.

2. Formulation and review

Consider the Poisson equation with the Dirichlet homogeneous boundary condition

$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0,1)^d, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $d = 1$ or 2 is the spatial parameter and $f \in L^2(\Omega)$ is sufficiently smooth. Assume that the elliptic problem has a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$, where, and throughout this paper, we use the standard notation for the Sobolev spaces and associated norms. The problem (2.1) may be transformed into a first-order equation system

$$\begin{cases} \mathbf{p} - \nabla u = \mathbf{0} & \text{in } \Omega, \\ -\nabla \cdot \mathbf{p} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where ∇ and $\nabla\cdot$ are the gradient and divergence operators, respectively. Here vectors and scalars are shown in bold and plain fonts, respectively. Define the space

$$\mathbf{H}(\Omega) = H(\mathrm{div};\Omega) \times H_0^1(\Omega),$$

where

$$H(\mathrm{div};\Omega) = \{\mathbf{q} \in [L^2(\Omega)]^d \colon\ \nabla \cdot \mathbf{q} \in L^2(\Omega)\}$$

has the corresponding norm

$$\|\mathbf{q}\|_{H(\mathrm{div};\Omega)} = (\|\nabla \cdot \mathbf{q}\|^2_{L^2(\Omega)} + \|\mathbf{q}\|^2_{L^2(\Omega)})^{1/2}.$$

For $\mathbf{u} = [\mathbf{p}, u]^T \in \mathbf{H}(\Omega)$, let

$$\mathcal{A}\mathbf{u} = \begin{bmatrix} \mathbf{p} - \nabla u \\ -\nabla \cdot \mathbf{p} \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} \mathbf{0} \\ f \end{bmatrix}.$$

Equations (2.2) thus read

$$\mathcal{A}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega.$$

2.1. Least-squares finite element discretization

The least-squares functional $\mathcal{J}$ is defined as

$$\mathcal{J}(\mathbf{v}; f) = \frac{1}{2}\|\mathcal{A}\mathbf{v} - \mathbf{f}\|^2_{L^2(\Omega)} = \frac{1}{2}(\mathcal{A}\mathbf{v} - \mathbf{f}, \mathcal{A}\mathbf{v} - \mathbf{f}),$$

where $(\mathbf{u}, \mathbf{v}) = \int_0^1 \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}x$ is the standard inner product. A minimizer $\mathbf{u}$ of the functional $\mathcal{J}$ satisfies

$$\lim_{t \to 0} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{J}(\mathbf{u} + t\mathbf{v}; f) = (\mathcal{A}\mathbf{u} - \mathbf{f}, \mathcal{A}\mathbf{v}) = 0 \quad \forall \, \mathbf{v} \in \mathbf{H}(\Omega).$$

The least-squares variational formulation of (2.2) follows: Find $\mathbf{u} \in \mathbf{H}(\Omega)$ such that

$$B(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \, \mathbf{v} \in \mathbf{H}(\Omega), \tag{2.3}$$

where the bilinear form B and the linear functional L are defined as

$$B(\mathbf{u}, \mathbf{v}) = (\mathcal{A}\mathbf{u}, \mathcal{A}\mathbf{v}) = \int_\Omega \big((\mathbf{p} - \nabla u) \cdot (\mathbf{q} - \nabla v) + (-\nabla \cdot \mathbf{p})(-\nabla \cdot \mathbf{q})\big) \, \mathrm{d}\Omega,$$
$$L(\mathbf{v}) = (\mathbf{f}, \mathcal{A}\mathbf{v}) = \int_\Omega f(-\nabla \cdot \mathbf{q}) \, \mathrm{d}\Omega,$$

with $\mathbf{u} = [\mathbf{p}, u]^T$ and $\mathbf{v} = [\mathbf{q}, v]^T$. The following coercivity result for the bilinear form can be obtained, cf. [15], [40], [41].

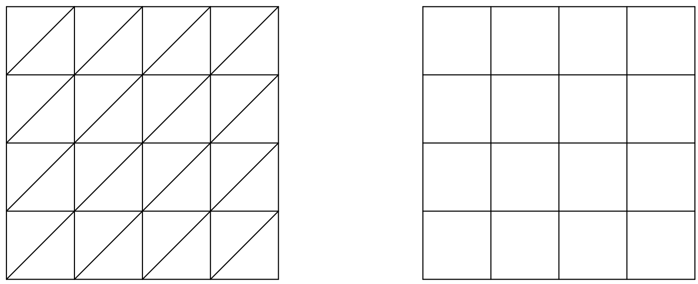
Proposition 2.1. *There exists a constant $\alpha > 0$ such that*

$$B(\mathbf{v}, \mathbf{v}) \geqslant \alpha(\|\mathbf{q}\|^2_{H(\mathrm{div};\Omega)} + \|v\|^2_{H^1(\Omega)})$$

for all $\mathbf{v} = [\mathbf{q}, v]^T \in \mathbf{H}(\Omega)$.

By the Lax-Milgram lemma, problem (2.3) has a unique solution in $\mathbf{H}(\Omega)$.

Let $\mathcal{T}_h = \{K_i\}_{i=1}^N$ be a triangulation of Ω, where K_i is the ith element. When $d = 2$, we consider only the uniform rectangular mesh and the triangular mesh of the regular pattern in this paper (see Fig. 1). Set the mesh parameter $h = \max\limits_{1 \leqslant i \leqslant N} h_i$,



Triangular mesh of regular pattern Rectangular mesh

Figure 1. Finite element meshes.

where h_i is the diameter of K_i. Define V_h and W_h as finite dimensional subspaces of $H_0^1(\Omega)$ and $H(\mathrm{div};\Omega)$ which consist of piecewise polynomials. We denote by $P_k(K_i)$ the space of all polynomials of degree not greater than k restricted to the element K_i, and by $Q_{k,r}(K_i)$ the space of polynomials of degree less than or equal to k in the first variable and to r in the second. We shall use Q_k for $Q_{k,k}$. Then for the Lagrange elements we set

$$V_h = \{v_h \in C^0(\Omega)\colon\ v_h|_{K_i} \in \Psi_k(K_i)\ \ \forall\, K_i \in \mathcal{T}_h,\ \ v|_{\partial\Omega} = 0\},$$
$$W_h = \{\mathbf{q}_h \in H(\mathrm{div};\Omega)\colon\ \mathbf{q}_h|_{K_i} \in [\Psi_r(K_i)]^d\ \ \forall\, K_i \in \mathcal{T}_h\},$$

where $\Psi_k(K_i)$ is taken as $P_k(K_i)$ for triangular elements, and as $Q_k(K_i)$ for rectangular ones. Another choice of the finite element space is, among others, the Raviart-Thomas space. On triangular elements, the kth order Raviart-Thomas space (RT_k) is defined by

$$V_h = \{v_h \in L^2(\Omega)\colon\ v_h|_{K_i} \in P_k(K_i)\ \ \forall\, K_i \in \mathcal{T}_h,\ \ v|_{\partial\Omega} = 0\},$$
$$W_h = \left\{\mathbf{q}_h \in H(\mathrm{div};\Omega)\colon\ \mathbf{q}_h|_{K_i} \in [P_k(K_i)]^2 \oplus \begin{bmatrix} x \\ y \end{bmatrix} P_k(K_i)\ \ \forall\, K_i \in \mathcal{T}_h\right\}.$$

The degrees of freedom of $W_h(K_i)$ are given by the moments

$$\int_e \mathbf{q}_h \cdot \mathbf{n}\, w \,\mathrm{d}s \quad \forall\, w \in P_k(e),\ \ e \in \partial K_i,$$
$$\int_{K_i} \mathbf{q}_h \cdot \mathbf{r} \,\mathrm{d}\Omega \quad \forall\, \mathbf{r} \in (P_{k-1}(K_i))^2,$$

where $\mathbf{n}$ is the outward unit normal vector to ∂K_i. On rectangular elements, the RT_k is defined by

$$V_h = \{v_h \in L^2(\Omega)\colon\ v_h|_{K_i} \in Q_k(K_i)\ \ \forall\, K_i \in \mathcal{T}_h,\ \ v|_{\partial\Omega} = 0\},$$
$$W_h = \{\mathbf{q}_h \in H(\mathrm{div};\Omega)\colon\ \mathbf{q}_h|_{K_i} \in Q_{k+1,k}(K_i) \times Q_{k,k+1}(K_i)\ \ \forall\, K_i \in \mathcal{T}_h\}.$$

In this case, the degrees of freedom of $W_h(K_i)$ are given by

$$\int_e \mathbf{q}_h \cdot \mathbf{n}\, w\, \mathrm{d}s \quad \forall\, w \in P_k(e),\ \ e \in \partial K_i,$$
$$\int_{K_i} \mathbf{q}_h \cdot \mathbf{r}\, \mathrm{d}\Omega \quad \forall\, \mathbf{r} = [r_1, r_2] \in Q_{k-1,k}(K_i) \times Q_{k,k-1}(K_i).$$

The Raviart-Thomas spaces defined above consist of all vector fields whose normal components are continuous across the edges. They satisfy also the BB-condition which, however, is not required for well-posedness of the LSFEM. Notice that, for RT_k, the degree of polynomial basis functions in W_h is $k+1$ for each variable.

The finite element approximation to problem (2.3) is posed as follows: find $\mathbf{u}_h \in W_h \times V_h$ such that

$$B(\mathbf{u}_h, \mathbf{v}_h) = L(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in W_h \times V_h. \tag{2.4}$$

By Proposition 2.1 and the Lax-Milgram lemma, problem (2.4) has a unique solution. Moreover, by (2.3) and (2.4), the following Galerkin orthogonality property holds:

$$B(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0 \quad \forall\, \mathbf{v}_h \in W_h \times V_h. \tag{2.5}$$

2.2. Review of error estimation

We next review some error estimates results for LSFEMs in literature. The following results can be found in, e.g., [15], [40], [41].

Proposition 2.2. *Let $s = \min(k, r)$ and assume that $u \in H^{s+1}(\Omega)$ and $\mathbf{p} \in [H^{s+1)}(\Omega)]^d$. Then*

$$\|u - u_h\|_{H^1(\Omega)} + \|\mathbf{p} - \mathbf{p}_h\|_{H(\mathrm{div};\Omega)} \leqslant Ch^s(\|u\|_{H^{s+1}(\Omega)} + \|\mathbf{p}\|_{H^{s+1}(\Omega)}).$$

In this paper, C is used to denote a generic positive constant that is independent of u, $\mathbf{p}$, and h. The following better estimate holds [41].

Proposition 2.3. *Assume that $u \in H^{k+1}(\Omega)$ and $\mathbf{p} \in [H^r(\Omega)]^d$. If $r = k+1$, then*

$$\|u - u_h\|_{L^2(\Omega)} + \|\mathbf{p} - \mathbf{p}_h\|_{H(\mathrm{div};\Omega)} \leqslant Ch^r(\|u\|_{H^r(\Omega)} + \|\mathbf{p}\|_{H^r(\Omega)}).$$

Moreover, for the one-dimensional problems [41] and two-dimensional finite element spaces with the *grid decomposition property* (GDP) [5], the following estimate can be obtained.

Proposition 2.4. *Let* $s = \min(k, r)$ *and assume that* $u \in H^{s+1}(\Omega)$ *and* $\mathbf{p} \in [H^{s+1}(\Omega)]^d$. *Then*

$$\|u - u_h\|_{L^2(\Omega)} + \|\mathbf{p} - \mathbf{p}_h\|_{L^2(\Omega)} \leqslant Ch^{s+1}(\|u\|_{H^{s+1}(\Omega)} + \|\mathbf{p}\|_{H^{s+1}(\Omega)}).$$

The convergence and natural superconvergence for the one-dimensional LSFEMs have been studied in details. In particular, the results in Propositions 2.2 and 2.4 can be improved when $k \neq r$. The following estimates are given in [41].

Proposition 2.5. *Let* $\kappa = \min(k, r+1)$ *and* $\varrho = \min(k+1, r)$. *Assume that* $u \in H^{k+1}(\Omega)$ *and* $p \in H^{r+1}(\Omega)$. *Then*

$$\|u - u_h\|_{H^1(\Omega)} \leqslant Ch^{\kappa}(\|u\|_{H^{\kappa+1}(\Omega)} + \|p\|_{H^{\kappa}(\Omega)}),$$
$$\|p - p_h\|_{H^1(\Omega)} \leqslant Ch^{\varrho}(\|u\|_{H^{\varrho}(\Omega)} + \|p\|_{H^{\varrho+1}(\Omega)}).$$

Proposition 2.6. *Let* $\kappa = \min(k, r+1)$ *and* $\varrho = \min(k+1, r)$. *Assume that* $u \in H^{k+1}(\Omega)$ *and* $p \in H^{r+1}(\Omega)$. *Then*

$$\|u - u_h\|_{L^2(\Omega)} \leqslant Ch^{\kappa+1}(\|u\|_{H^{\kappa+1}(\Omega)} + \|p\|_{H^{\kappa}(\Omega)}) \quad \text{for } r > 1,$$
$$\|p - p_h\|_{L^2(\Omega)} \leqslant Ch^{\varrho+1}(\|u\|_{H^{\varrho}(\Omega)} + \|p\|_{H^{\varrho+1}(\Omega)}) \quad \text{for } k > 1.$$

When $|k - r| = 1$, the estimates in Propositions 2.5 and 2.6 are optimal, since the order of convergence corresponds to the theoretical order of the Galerkin method. When $|k - r| > 1$, nevertheless, the estimates are no longer optimal. See numerical results in [9], [10], and [38] for detailed examples.

The superconvergence phenomena at interelement nodes and the elemental Gaussian points have been observed in [16] and analyzed in [41]; cf. also [38].

Proposition 2.7. *Let* $s = \min(k, r)$ *and let* x_i *be a meshpoint. Assume that* $u \in H^{s+1}(\Omega)$ *and* $p \in H^{s+1}(\Omega)$. *Then*

$$|(u - u_h)(x_i)| + |(p - p_h)(x_i)| \leqslant Ch^{2s}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

Proposition 2.8. *Let* $s = \min(k, r)$ *and suppose that* $u \in H^{s+1}(\Omega)$ *and* $p \in H^{s+1}(\Omega)$. *Let* F_i *be the affine mapping from* $[-1, 1]$ *to* e_i *and let* $g_{j,k}$ *be the*

jth Gaussian point of order k in $[-1,1]$, $1 \leqslant j \leqslant k$. Then for $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant k$, and $1 \leqslant \varrho \leqslant r$,

$$|(u-u_h)'(F_i(g_{j,k}))| + |(p-p_h)'(F_i(g_{\varrho,r}))| \leqslant Ch^{s+1}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

The following optimal and superapproximation estimates in the maximum norm have recently been established in [38].

Proposition 2.9. *Let $s = \min(k,r)$ and assume that $u \in W_\infty^{s+1}(\Omega)$ and $p \in W_\infty^{s+1}(\Omega)$. Then*

$$\|u-u_h\|_{L^\infty(\Omega)} + \|p-p_h\|_{L^\infty(\Omega)} \leqslant Ch^{s+1}(\|u\|_{W_\infty^{s+1}(\Omega)} + \|p\|_{W_\infty^{s+1}(\Omega)}), \tag{2.6}$$

$$\|u-u_h\|_{W_\infty^1(\Omega)} + \|p-p_h\|_{W_\infty^1(\Omega)} \leqslant Ch^{s}(\|u\|_{W_\infty^{s+1}(\Omega)} + \|p\|_{W_\infty^{s+1}(\Omega)}). \tag{2.7}$$

Proposition 2.10. *Let $s = \min(k,r)$ and assume that $u \in W_\infty^{s+1}(\Omega)$ and $p \in W_\infty^{s+1}(\Omega)$. Let $\mathbf{N}_h\mathbf{u} = [M_hp, N_hu]^T$ be the projection of $\mathbf{u}$ into $W_h \times V_h$ so that $((M_hp-p)', q') = 0$ for all $q \in W_h$ and $((N_hu-u)', v') = 0$ for all $v \in V_h$. Then*

$$\begin{aligned}\|(N_hu-u_h)'\|_{L^\infty(\Omega)} &+ \|(M_hp-p_h)'\|_{L^\infty(\Omega)} \\ &\leqslant Ch^{s+1}(\|u\|_{W_\infty^{s+1}(\Omega)} + \|p\|_{W_\infty^{s+1}(\Omega)}).\end{aligned} \tag{2.8}$$

When $s > 1$, then

$$\begin{aligned}\|N_hu-u_h\|_{L^\infty(\Omega)} &+ \|M_hp-p_h\|_{L^\infty(\Omega)} \\ &\leqslant Ch^{s+2}(\|u\|_{W_\infty^{s+1}(\Omega)} + \|p\|_{W_\infty^{s+1}(\Omega)}).\end{aligned} \tag{2.9}$$

Propositions 2.9 and 2.10 lead to the following superconvergence error estimate at the Lobatto points [38].

Proposition 2.11. *Let $s = \min(k,r)$ and assume that $u \in H^{s+1}(\Omega)$ and $p \in H^{s+1}(\Omega)$. Let F_i be the affine mapping from $[-1,1]$ to e_i and let $l_{j,k}$ be the jth Lobatto point of order k in $[-1,1]$, $1 \leqslant j \leqslant k-1$. Then for $s > 1$, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant k-1$, and $1 \leqslant \varrho \leqslant r-1$ we have*

$$|(u-u_h)(F_i(l_{j,k}))| + |(p-p_h)(F_i(l_{\varrho,r}))| \leqslant Ch^{s+2}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

Remark 2.1. The convergence results in Propositions 2.2–2.11 hold for general elliptic two-point boundary value problems. The superconvergence estimates (Propositions 2.7, 2.8, and 2.11) are analogous to those for the standard Galerkin method; cf., e.g., [21], [22], [45]. However, the estimation cannot be improved when $k \neq r$. See [38] for details.

Natural superconvergence for the two-dimensional LSFEMs has not been reported in literature. Nevertheless, pointwise superconvergence has been well studied for the Galerkin method, see, e.g., [4], [17], [31], [36], [37], [44], [47]. In particular, for rectangular elements, derivative superconvergence is achieved along the corresponding Gaussian lines, and function value superconvergence is obtained at tensor product of the Lobatto points. In a triangular mesh of the regular pattern, derivative superconvergence points are along tangential directions at the midpoints of edges for elements of odd degrees, and at the second order Gaussian points of edges for quadratic elements; while function value superconvergence points are the vertices and midpoints of edges for elements of even degrees. Some pointwise superconvergence results have also been proved for the Raviart-Thomas and Brezzi-Douglas-Marini elements, see, e.g., [23], [24], [25], [28], [29], [33]. In the next section, we will investigate natural superconvergence for the two-dimensional least-squares Lagrange and Raviart-Thomas elements in uniform rectangular and regular triangular meshes.

3. Numerical experiments

Numerical examples for the one-dimensional LSFEMs can be found in [16], [38]. In this section we consider the two-dimensional test problem (2.1) with the exact solution

$$u(x,y) = (\mathrm{e}^y - \sin 2\pi x)(x - x^2)(y - y^2).$$

The primary objective of the numerical study is to determine whether the least-squares formulation exhibits natural superconvergence properties similar to those of the standard Galerkin method. The discrete problem is set up as described in the preceding sections using the Lagrange and Raviart-Thomas elements, and it is solved on a set of equidistant meshes of decreasing size. A computer algebra system (e.g. Maple) is employed to compute the exact analytical formation of the stiffness matrices and load vectors. Set $e_h = u - u_h$ and $\boldsymbol{\varepsilon}_h = \mathbf{p} - \mathbf{p}_h$ as numerical errors, the derivatives of which are also considered. The results for the rate of convergence are illustrated as the slope of log-log plots of errors against mesh size h in the usual way.

3.1. Lagrange elements

Linear (P_1) and quadratic (P_2) elements on triangular meshes, and bilinear (Q_1) and biquadratic (Q_2) elements on rectangular meshes are tested and presented in Figs. 2 and 3, respectively. Optimal convergence rates of Propositions 2.2 and 2.4 have been observed in the numerical results since both the meshes have the GDP, they, however have not been plotted; cf. the numerical experiments in [5]. Superconvergence for derivatives and function values has been tested at certain points.

In particular, for linear elements, $|\partial_x e_h|$ and $|\partial_y e_h|$ have convergence rate $O(h^2)$ at the midpoints of horizontal and vertical edges, respectively, which is one order higher than a global rate $O(h)$ of $\|e_h\|_{H^1(\Omega)}$. $|\partial_x \varepsilon_{h,1}|$ and $|\partial_y \varepsilon_{h,2}|$ converge with rate $O(h^{1.6})$ at the midpoints of horizontal and vertical edges, respectively, which can be contrasted with the optimal rate $O(h)$ of $\|\varepsilon_h\|_{H(\mathrm{div},\Omega)}$. Here ∂_x and ∂_y are used for $\partial/\partial x$ and $\partial/\partial y$, respectively, and $\varepsilon_{h,l}$ is the lth component of ε_h, $l = 1, 2$. Moreover, $|\partial_x \varepsilon_{h,2}|$ and $|\partial_y \varepsilon_{h,1}|$ converge with the rate of $O(h^{1.4})$. For quadratic elements, the convergence rate of $|\partial_x e_h|$ and $|\partial_y e_h|$ at the local Gaussian points $(g_{2,s}, y_j)$ of horizontal edges (with $g_{2,s} = x_i - \frac{1}{2}\big(1 \pm \frac{1}{\sqrt{3}}\big)h$ in $[x_i - h, x_i]$) and $(x_i, g_{2,s})$ of vertical edges (with $g_{2,s} = y_j - \frac{1}{2}\big(1 \pm \frac{1}{\sqrt{3}}\big)h$ in $[y_j - h, y_j]$) are about $O(h^3)$, as compared with a global rate of $O(h^2)$. The rate for $|e_h|$ is $O(h^4)$ at vertices and midpoints of edges (i.e. the local Lobatto points $l_{i,j}$ of edges in this case), which is one order higher than the optimal rate. Superconvergence of ε_h and its derivatives is nonetheless not observed at the corresponding Lobatto and Gaussian points. See details in Fig. 2.

For bilinear elements, $|\partial_x e_h|$, $|\partial_y e_h|$, and $|\nabla \cdot \varepsilon_h|$ have second order superconvergence rate along the corresponding Gaussian lines (denoted by $g_{i,j}$ in Fig. 3). That is to say, in an element $[x_i - h, x_i] \times [y_j - h, y_j]$, $|\partial_x e_h|$ and $|\partial_x \varepsilon_{h,1}|$ are calculated at some points $g_{i,j}$ along $\big\{x_i - \frac{1}{2}h\big\} \times [y_j - h, y_j]$, and $|\partial_y e_h|$ and $|\partial_y \varepsilon_{h,2}|$ are computed at some other points $g_{i,j}$ along $[x_i - h, x_i] \times \big\{y_j - \frac{1}{2}h\big\}$. For biquadratic elements, $|\partial_x e_h|$ and $|\partial_y e_h|$ converge in rate $O(h^3)$ along Gaussian lines $\big\{x_i - \frac{1}{2}\big(1 \pm \frac{1}{\sqrt{3}}\big)h\big\} \times [y_j - h, y_j]$ and $[x_i - h, x_i] \times \big\{y_j - \frac{1}{2}\big(1 \pm \frac{1}{\sqrt{3}}\big)h\big\}$, respectively. $|\partial_x \varepsilon_{h,1}|$ and $|\partial_y \varepsilon_{h,2}|$ have convergence rate $O(h^{2.2})$ along the corresponding lines, which are still higher than the global rate $O(h^2)$. Superconvergence rate of $O(h^{3.7})$ has been observed for $|e_h|$ at the tensor product of local Lobatto points $l_{i,j}$, which are vertices of the mesh and midpoint of edges. Superconvergence of ε_h is not observed at the Lobatto points. See details in Fig. 3.

3.2. Raviart-Thomas elements

The triangular and rectangular RT_0 and RT_1 have also been used for the LSFEM to compute the model problem. Notice that the numerical results by Raviart-Thomas elements might be discontinuous at vertices and/or along edges, where arithmetic averages have been used in calculation of errors. For the RT_0 elements, convergence rate has been investigated at vertices and midpoints of edges for function values and derivatives. No superconvergence has been observed. Similar computation has been conducted for triangular RT_1 elements, which leads to no superconvergence at the aforementioned points either. For rectangular RT_1 elements, it is observed that, at the element center $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$ (i.e. the tensor product of the local Gaussian point), the convergence rates of $|\partial_x e_h|$, $|\partial_y e_h|$, and $|\nabla \cdot \varepsilon_h|$ are all about $O(h^2)$, which is one order higher than the estimate in Proposition 2.2, cf. also [26]. We notice that for

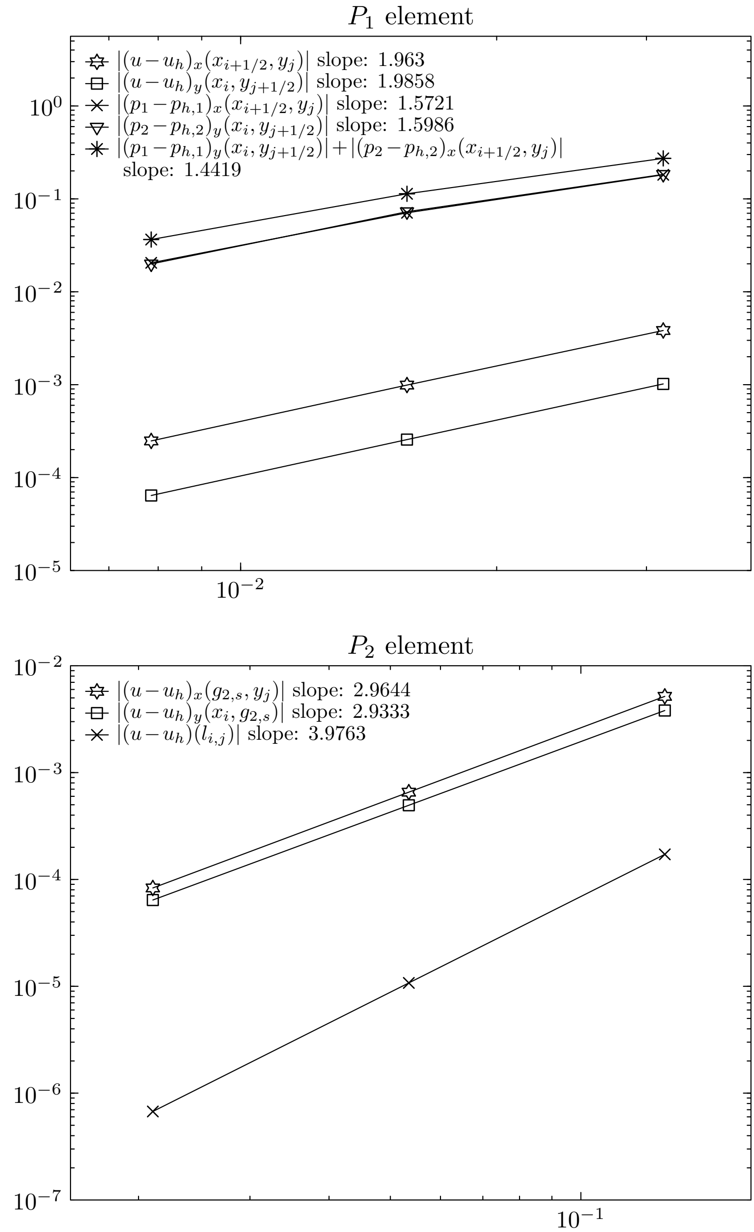


Figure 2. Superconvergence of Lagrange triangular elements.

the standard non-least-squares RT_1 element, the optimal convergence rate for $|\nabla\cdot\boldsymbol{\varepsilon}_h|$ is also $O(h^2)$, cf. [20]. See details in Fig. 4.

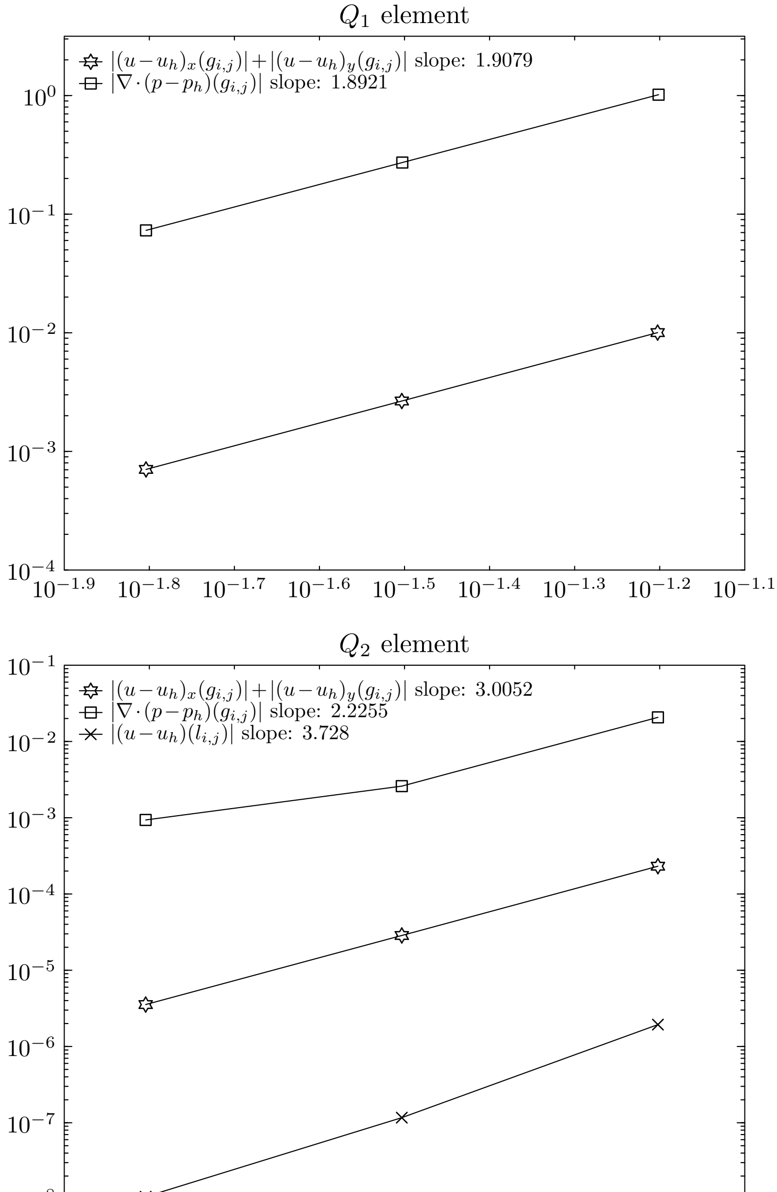


Figure 3. Superconvergence of Lagrange rectangular elements.

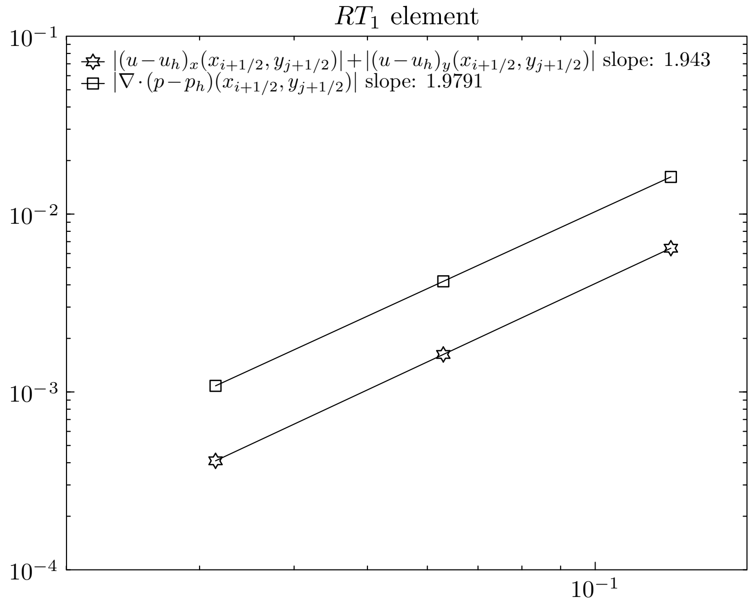


Figure 4. Convergence of RT_1 rectangular elements.

4. Conclusions

In the paper we have considered pointwise superconvergence of the LSFEM for the simple model problem of the Poisson equation. Convergence and superconvergence error estimates for the one-dimensional problems in literature have been reviewed. Numerical investigation has been conducted for the two-dimensional Lagrange and Raviart-Thomas elements. Some, but not all, superconvergence properties of the Galerkin method have been preserved by the LSFEM. A theoretical investigation of natural superconvergence for the two-dimensional LSFEM is an ongoing project.

Acknowledgements. The authors would like to express their appreciation to Michal Křížek and the Institute of Mathematics of the Academy of Sciences of the Czech Republic for hosting the 4th Conference on Superconvergence Phenomena in the Finite Element Method, June 25–28, 2008, Prague, Czech Republic. Their hospitality is gratefully acknowledged.

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