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POSTPROCESSING AND HIGHER ORDER CONVERGENCE FOR THE MIXED FINITE ELEMENT APPROXIMATIONS OF THE STOKES EIGENVALUE PROBLEMS

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Dedicated to Ivan Hlaváček on the occasion of his 75th birthday

Abstract. In this paper we propose a method for improving the convergence rate of the mixed finite element approximations for the Stokes eigenvalue problem. It is based on a postprocessing strategy that consists of solving an additional Stokes source problem on an augmented mixed finite element space which can be constructed either by refining the mesh or by using the same mesh but increasing the order of the mixed finite element space.

Keywords: Stokes eigenvalue problem, mixed finite element method, Rayleigh quotient formula, postprocessing

MSC 2010: 65N30, 65N25, 65L15, 65B99

1. Introduction

In this paper we are concerned with the Stokes eigenvalue problem: Find $(\mathbf{u}, p, \lambda)$ such that

$$
\begin{cases} -\Delta \mathbf{u} + \nabla p = \lambda \mathbf{u} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \int_\Omega \mathbf{u}^2 \, \mathrm{d}\Omega = 1, \end{cases} \tag{1.1}
$$

where $\Omega \subset \mathcal{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and Δ, ∇, $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators, respectively.

There are several works for the Stokes eigenvalue problems and their numerical methods such as Babuška and Osborn [2], [3], [22], Křížek [16], Mercier, Osborn, Rappaz, and Raviart [21], etc.

Osborn [22], Mercier, Osborn, Rappaz, and Raviart [21] gave an abstract analysis for the eigenpair approximations by mixed/hybrid finite element methods based on the general theory of compact operators ([10]). Recently, many effective postprocessing methods that improve the convergence rate for the approximations of the eigenvalue problems by the finite element methods have been proposed and analyzed ([1], [11], [17], [18], [19], [20], [23] and [25]). Lin and Lü [19], Lin and Lin [18], Lin, Huang and Li [17] have analyzed the Richardson extrapolation applied to the second order elliptic eigenvalue problems. Chen and Lin [11] also analyzed the Richardson extrapolation for the Stokes eigenvalue problems by the stream function-vorticity-pressure method. Xu and Zhou [25] have given a two-grid discretization technique to improve the convergence rate of the second order elliptic eigenvalue problems and integral eigenvalue problems. Racheva and Andreev [23], Andreev, Lazarov, and Racheva [1] have proposed a postprocessing method that improves the convergence rate for the numerical approximations of $2m$-order selfadjoint eigenvalue problems and biharmonic eigenvalue problems.

In this paper, using the idea of the references above, we propose and analyze a practical postprocessing algorithm which can improve the convergence rate of the eigenpair approximations for the Stokes eigenvalue problem by the mixed finite element method.

The postprocessing procedure can be described as follows: (1) solve the Stokes eigenvalue problem in the original mixed finite element space; (2) solve an additional Stokes source problem in an augmented space using the eigenvalue obtained previously multiplying the corresponding eigenfunction as the load vector. This method can improve the convergence rate of the eigenpair approximations with relatively inexpensive computation because we replace the solution of the Stokes eigenvalue problem by an additional solution of a Stokes source problem on a finer mesh or in a higher order mixed finite element space.

An outline of the paper is as follows. In Section 2, we introduce some preliminaries and notation and state the weak form of the Stokes eigenvalue problem and its corresponding discrete form. Section 3 is devoted to deriving the postprocessing technique and analyzing its efficiency. In Section 4, we propose a practical computational algorithm to implement the postprocessing method. In Section 5, we give a numerical result to confirm the theoretical analysis. Some concluding remarks are given in the last section.

2. Preliminaries and notation

In this paper we use the standard notation ([8], [9] and [20]) for the Sobolev spaces $H^r(\Omega)$ (standard interpolation spaces for real numbers r) and their associated inner products $(\cdot,\cdot)_r$, norms $\|\cdot\|_r$ and seminorms $|\cdot|_r$ for $r \geqslant 0$. The Sobolev space $H^0(\Omega)$ coincides with $L^2(\Omega)$, in which case the norm and inner product are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. In addition, the subspace of $L^2(\Omega)$ denoted by $L_0^2(\Omega)$ consists of the functions on $L^2(\Omega)$ having mean value zero. We also use the vector valued functions $(H^r(\Omega))^2$ just as in [9] and [14].

The corresponding weak form of (1.1) is: Find $(\mathbf{u}, p, \lambda) \in \mathbf{V} \times W \times \mathcal{R}$ such that $s(\mathbf{u},\mathbf{u}) = 1$ and

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \lambda s(\mathbf{u},\mathbf{v}) & \forall\, \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u},q) = 0 & \forall\, q \in W, \end{cases} \tag{2.1}$$

where $\mathbf{V} = (H_0^1(\Omega))^2$, $W = L_0^2(\Omega)$, and

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u}\nabla\mathbf{v}\,\mathrm{d}\Omega,$$
$$b(\mathbf{v},p) = -\int_\Omega \nabla\cdot\mathbf{v}p\,\mathrm{d}\Omega,$$
$$s(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}\mathbf{v}\,\mathrm{d}\Omega.$$

From [3] we know that the eigenvalue problem (2.1) has an eigenvalue sequence $\{\lambda_j\}$:

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots, \qquad \lim_{k\to\infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$(\mathbf{u}_1, p_1), (\mathbf{u}_2, p_2), \ldots, (\mathbf{u}_k, p_k), \ldots,$$

where $s(\mathbf{u}_i, \mathbf{u}_j) = \delta_{ij}$.

For simplicity, we only consider simple eigenvalues in this paper. We know that $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $s(\cdot,\cdot)$ have the following properties ([14]):

$$|a(\mathbf{u},\mathbf{v})| \leqslant \|u\|_1\|v\|_1, \tag{2.2}$$
$$a(\mathbf{u},\mathbf{u}) \geqslant C\|u\|_1^2, \tag{2.3}$$
$$|s(\mathbf{u},\mathbf{v})| \leqslant C\|\mathbf{u}\|_0\|\mathbf{v}\|_0, \tag{2.4}$$
$$s(\mathbf{u},\mathbf{u}) \geqslant C\|\mathbf{u}\|_0^2, \tag{2.5}$$
$$\sup_{0\neq\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_1} \geqslant C_1\|q\|_0, \tag{2.6}$$
$$\|\mathbf{u}\|_1 + \|p\|_0 \leqslant C \sup_{0\neq(\mathbf{v},q)\in\mathbf{V}\times W} \frac{a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + b(\mathbf{u},q)}{\|\mathbf{v}\|_1 + \|q\|_0}, \tag{2.7}$$

where $C_1 > 0$. In this paper, C and C_i denote constants independent of the mesh size h and sometimes dependent on the eigenvalue λ and may be of different values at their different occurrences.

For the eigenvalue there exists the Rayleigh quotient expression

$$\lambda = \frac{a(\mathbf{u},\mathbf{u})}{s(\mathbf{u},\mathbf{u})}. \tag{2.8}$$

Now, let us define the finite element approximations of the problem (2.1). The well-posedness of the discrete weak form of (2.1) can be guaranteed by the fact that the corresponding approximation spaces satisfy the Babuška-Brezzi condition ([9] and [14]). Let $\mathcal{T}_h$ be a partition of Ω into finite elements (triangles or quadrilaterals), which is quasi-uniform and has a mesh size h. Associated with the partition $\mathcal{T}_h$, we define the finite element spaces $\mathbf{V}_h \subset \mathbf{V}$ and $W_h \subset W$ of piecewise polynomials of degree k ([9] and [14]). Let P_r be the set of polynomials of degree not greater than r with $r \geqslant 0$. Assume that the polynomial space P_k, $k \geqslant 1$, is used for the construction of $\mathbf{V}$, and that P_{k-1} is used for the construction of W_h. The two finite element spaces $\mathbf{V}_h$ and W_h are assumed to satisfy the following approximation properties:

$$\begin{cases} \inf\limits_{\mathbf{v}\in\mathbf{V}_h} (\|\mathbf{u}-\mathbf{v}\|_0 + h\|\mathbf{u}-\mathbf{v}\|_1) \leqslant Ch^{m+1}\|\mathbf{u}\|_{m+1}, & 0 \leqslant m \leqslant k, \\ \inf\limits_{q\in W_h} \|p-q\|_0 \leqslant Ch^m\|p\|_m, & 0 \leqslant m \leqslant k, \end{cases} \tag{2.9}$$

for any $\mathbf{u} \in (H^{m+1}(\Omega))^2$ and $p \in H^m(\Omega)$. Since the finite element spaces are subspaces of $(H_0^1(\Omega))^2$, the functions in $\mathbf{V}_h$ are continuous and $k \geqslant 1$.

We know that the convergence rate of the eigenpair approximations by the finite element methods depends on the regularities of the exact eigenfunctions. The exact eigenfunctions of the Stokes problem belong only to the space $(H^1(\Omega))^2 \times H^0(\Omega)$ on general domains. Nevertheless, for the domains with smooth boundary, the exact eigenfunctions have additional regularities. In this case we need to use isoparametric mixed finite element methods to fit the domain more exactly ([18] and [12]). The goal of this paper is to propose and analyze a postprocessing method which can improve the convergence rate for both the eigenvalue and the eigenfunction approximations. The assumption that Ω is a convex polygonal domain can make the expression of the main idea of this paper more transparent. However, we need to note that this assumption limits the regularity of the exact eigenfunctions and makes the analysis of the convergence rates much more complicated. It is well known ([15] and [13]) that for a given $\mathbf{f} \in (H^\gamma(\Omega))^2$ the solution $(\mathbf{u}, p)$ of the corresponding Stokes problem

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) & \forall\, \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u},q) = 0 & \forall\, q \in W \end{cases} \tag{2.10}$$

has the regularity ([4], [5], [7], [13] and [15])

$$\|\mathbf{u}\|_{2+\gamma} + \|p\|_{1+\gamma} \leqslant C\|\mathbf{f}\|_{\gamma} \quad \forall\, \mathbf{f} \in (H^{\gamma}(\Omega))^2, \tag{2.11}$$

where $0 < \gamma \leqslant 1$ is a parameter that depends on the largest interior angle of $\partial\Omega$ ([4] and [15]).

Now, let us define the approximation of an eigenpair $(\mathbf{u}, p, \lambda)$ by the mixed finite element method as $(\mathbf{u}_h, p_h, \lambda_h) \in \mathbf{V}_h \times W_h \times \mathcal{R}$ such that $s(\mathbf{u}_h, \mathbf{u}_h) = 1$ and

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) = \lambda_h s(\mathbf{u}_h, \mathbf{v}) & \forall\, \mathbf{v} \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q) = 0 & \forall\, q \in W_h. \end{cases} \tag{2.12}$$

From (2.12) we know that the Rayleigh quotient for λ_h is

$$\lambda_h = \frac{a(\mathbf{u}_h, \mathbf{u}_h)}{s(\mathbf{u}_h, \mathbf{u}_h)}. \tag{2.13}$$

We know from [3] that the Stokes eigenvalue problem (2.12) has eigenvalues

$$0 < (\lambda_1)_h \leqslant (\lambda_2)_h \leqslant \ldots \leqslant (\lambda_k)_h \leqslant \ldots \leqslant (\lambda_N)_h,$$

and the corresponding eigenfunctions

$$((\mathbf{u}_1)_h, (p_1)_h), ((\mathbf{u}_2)_h, (p_2)_h), \ldots, ((\mathbf{u}_k)_h, (p_k)_h), \ldots, ((\mathbf{u}_N)_h, (p_N)_h),$$

where $s((\mathbf{u}_i)_h, (\mathbf{u}_j)_h) = \delta_{ij}$, $1 \leqslant i, j \leqslant N$. If the pair of finite element spaces $\mathbf{V}_h$ and W_h satisfies the Babuška-Brezzi condition

$$\inf_{0 \neq q \in W_h} \sup_{0 \neq \mathbf{v} \in \mathbf{V}_h} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_1 \|q\|_0} \geqslant C > 0, \tag{2.14}$$

the eigenvalue approximation λ_h and the corresponding eigenfunction approximation $(\mathbf{u}_h, p_h)$ have the following bound ([22], [21], [9], and [14]):

$$|\lambda - \lambda_h| \leqslant C\Big(\inf_{\mathbf{v} \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_1 + \inf_{q \in W_h} \|p - q\|_0\Big)^2, \tag{2.15}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + h\|\mathbf{u} - \mathbf{u}_h\|_1 \leqslant Ch\Big(\inf_{\mathbf{v} \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_1 + \inf_{q \in W_h} \|p - q\|_0\Big), \tag{2.16}$$

$$\|p - p_h\|_0 \leqslant C\Big(\inf_{\mathbf{v} \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_1 + \inf_{q \in W_h} \|p - q\|_0\Big). \tag{2.17}$$

In particular, if $(\mathbf{u}, p) \in (H^{k+1}(\Omega))^2 \times H^{k+1}(\Omega)$, then (2.9) yields the following error estimates:

$$|\lambda - \lambda_h| \leqslant Ch^{2k}(\|\mathbf{u}\|_{k+1} + \|p\|_k)^2, \tag{2.18}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + h\|\mathbf{u} - \mathbf{u}_h\|_1 + h\|p - p_h\|_0 \leqslant Ch^{k+1}(\|\mathbf{u}\|_{k+1} + \|p\|_k). \tag{2.19}$$

If the domain is convex and polygonal, we have

$$|\lambda - \lambda_h| \leqslant Ch^{2s}(\|\mathbf{u}\|_{s+1} + \|p\|_s)^2, \tag{2.20}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + h\|\mathbf{u} - \mathbf{u}_h\|_1 + h\|p - p_h\|_0 \leqslant Ch^{s+1}(\|\mathbf{u}\|_{s+1} + \|p\|_s), \tag{2.21}$$

where and hereafter in this paper $s := \min\{1 + \gamma, k\}$.

3. Postprocessing technique

In this section we present a postprocessing method to improve the accuracy of the eigenvalue and eigenfunction approximations. This postprocessing method consists of solving the original Stokes eigenvalue problem in the k-order mixed finite element space and one additional Stokes source problem in an augmented mixed finite element space. Here, we give two ways to construct the enriched space: on a finer mesh which results from the original mesh by refining and on a $k+1$-order mixed finite element space on the same mesh.

To derive our method, we need first to introduce the error expansions of the eigenvalues by the Rayleigh quotient formula. It is well known that there are the Rayleigh quotient error expansions for the eigenvalues of the second elliptic problems.

Theorem 3.1. *Assume $(\mathbf{u}, p, \lambda)$ is the true solution of the Stokes eigenvalue problem (2.1), $0 \neq \mathbf{w} \in (H_0^1(\Omega))^2$ and $\psi \in L_0^2(\Omega)$ satisfy*

$$b(\mathbf{w}, \psi) = 0. \tag{3.1}$$

Let us define

$$\hat{\lambda} = \frac{a(\mathbf{w}, \mathbf{w})}{s(\mathbf{w}, \mathbf{w})}. \tag{3.2}$$

Then we have

$$\hat{\lambda} - \lambda = \frac{a(\mathbf{w} - \mathbf{u}, \mathbf{w} - \mathbf{u}) + 2b(\mathbf{w} - \mathbf{u}, p - \psi) - \lambda s(\mathbf{w} - \mathbf{u}, \mathbf{w} - \mathbf{u})}{s(\mathbf{w}, \mathbf{w})}. \tag{3.3}$$

P r o o f. From (2.1), (2.12), (2.13), (3.1), (3.2) by direct computation we obtain

$$\begin{aligned}\hat\lambda-\lambda&=\frac{a(\mathbf{w},\mathbf{w})-\lambda s(\mathbf{w},\mathbf{w})}{s(\mathbf{w},\mathbf{w})}\\&=\frac{a(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u})+2a(\mathbf{w},\mathbf{u})-a(\mathbf{u},\mathbf{u})-\lambda s(\mathbf{w},\mathbf{w})}{s(\mathbf{w},\mathbf{w})}\\&=\frac{a(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u})+2\lambda s(\mathbf{w},\mathbf{u})+2b(\mathbf{w},p)-\lambda s(\mathbf{u},\mathbf{u})-\lambda s(\mathbf{w},\mathbf{w})}{s(\mathbf{w},\mathbf{w})}\\&=\frac{a(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u})-\lambda s(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u})+2b(\mathbf{w},p)}{s(\mathbf{w},\mathbf{w})}\\&=\frac{a(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u})-\lambda s(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u})+2b(\mathbf{w}-\mathbf{u},p)}{s(\mathbf{w},\mathbf{w})}\\&=\frac{a(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u})-\lambda s(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u})+2b(\mathbf{w}-\mathbf{u},p-\psi)}{s(\mathbf{w},\mathbf{w})}.\end{aligned}$$

This is the desired result and the proof is completed. □

If the eigenpair approximation $(\mathbf{u}_h,p_h,\lambda_h)$ of the Stokes eigenvalue problem (2.1) has been obtained, let us define the Stokes source problem: Find $(\tilde{\mathbf{u}},\tilde p)\in\mathbf{V}\times W$ such that

$$\begin{cases}a(\tilde{\mathbf{u}},\mathbf{v})+b(\mathbf{v},\tilde p)=\lambda_h s(\mathbf{u}_h,\mathbf{v}) & \forall\,\mathbf{v}\in\mathbf{V},\\ b(\tilde{\mathbf{u}},q)=0 & \forall\,q\in W.\end{cases}\tag{3.4}$$

We also define the Rayleigh quotient formula

$$\tilde\lambda=\frac{a(\tilde{\mathbf{u}},\tilde{\mathbf{u}})}{s(\tilde{\mathbf{u}},\tilde{\mathbf{u}})}.\tag{3.5}$$

For $(\tilde{\mathbf{u}},\tilde p,\tilde\lambda)$ we have the following error estimate.

Theorem 3.2. *Assume* $(\mathbf{u},p,\lambda)$ *is the true solution of the Stokes eigenvalue problem* (2.1), $(\mathbf{u}_h,p_h,\lambda_h)$ *is the corresponding mixed finite element solution of the discrete Stokes eigenvalue problem* (2.12), $(\tilde{\mathbf{u}},\tilde p)$ *is the true solution of problem* (3.4) *and* $\tilde\lambda$ *is defined by* (3.5). *Then we have the estimate*

$$\|\mathbf{u}-\tilde{\mathbf{u}}\|_1+\|p-\tilde p\|_0\leqslant C(\|\mathbf{u}-\mathbf{u}_h\|_0+|\lambda-\lambda_h|),\tag{3.6}$$

$$|\tilde\lambda-\lambda|\leqslant C(\|\mathbf{u}-\mathbf{u}_h\|_0^2+|\lambda-\lambda_h|^2).\tag{3.7}$$

P r o o f. From the Stokes eigenvalue (2.1) and the Stokes problem (3.4) we have

$$\begin{aligned}a(\tilde{\mathbf{u}}-\mathbf{u},\mathbf{v})+b(\mathbf{v},\tilde p-p)+b(\tilde{\mathbf{u}}-\mathbf{u},q)&\\=s(\lambda_h\mathbf{u}_h-\lambda\mathbf{u},\mathbf{v})&\\=\lambda_h s(\mathbf{u}_h-\mathbf{u},\mathbf{v})+(\lambda_h-\lambda)s(\mathbf{u},\mathbf{v})&\\\leqslant C(\|\mathbf{u}_h-\mathbf{u}\|_0+|\lambda_h-\lambda|)\|\mathbf{v}\|_1.&\end{aligned}\tag{3.8}$$

Then, from (2.7), we have

$$\|\tilde{\mathbf{u}}-\mathbf{u}\|_1+\|\tilde{p}-p\|_0 \leqslant \sup_{0\neq(\mathbf{v},q)\in\mathbf{V}\times W}\frac{a(\tilde{\mathbf{u}}-\mathbf{u},\mathbf{v})+b(\mathbf{v},\tilde{p}-p)+b(\tilde{\mathbf{u}}-\mathbf{u},q)}{\|\mathbf{v}\|_1+\|q\|_0} \leqslant C(\|\mathbf{u}_h-\mathbf{u}\|_0+|\lambda_h-\lambda|). \tag{3.9}$$

From Theorem 3.1 and (3.9) we obtain

$$\begin{aligned}\tilde{\lambda}-\lambda &\leqslant C(\|\tilde{\mathbf{u}}-\mathbf{u}\|_1^2+\|\tilde{\mathbf{u}}-\mathbf{u}\|_1\|\tilde{p}-p\|_0)\\ &\leqslant C(\|\mathbf{u}_h-\mathbf{u}\|_0+|\lambda_h-\lambda|)^2\\ &\leqslant C(\|\mathbf{u}_h-\mathbf{u}\|_0^2+|\lambda-\lambda_h|^2).\end{aligned}$$

So, the proof is complete. □

Based on the result of the convergence rate of the eigenpair approximation, for a smooth domain we can obtain

$$\|\tilde{\mathbf{u}}-\mathbf{u}\|_1+\|\tilde{p}-p\|_0 \leqslant Ch^{k+1}, \tag{3.10}$$

$$|\tilde{\lambda}-\lambda| \leqslant Ch^{2k+2}. \tag{3.11}$$

For a convex polygonal domain, (2.20) and (2.21) yield

$$\|\tilde{\mathbf{u}}-\mathbf{u}\|_1+\|\tilde{p}-p\|_0 \leqslant Ch^{s+1}, \tag{3.12}$$

$$|\tilde{\lambda}-\lambda| \leqslant Ch^{2s+2}. \tag{3.13}$$

This means that $(\tilde{\mathbf{u}},\tilde{p},\tilde{\lambda})$ is a much better approximation of the true solution $(\mathbf{u},p,\lambda)$ of the Stokes eigenvalue problem (2.1) than $(\mathbf{u}_h,p_h,\lambda_h)$.

4. Postprocessing algorithm

Theorem 3.2 has only theoretical value and cannot be used in practice since the exact solution of the Stokes source problem (3.4) is not always known. In order to make it useful, we need to get a sufficiently accurate approximation of the Stokes source problem. Here we discuss two possible ways how to obtain the approximation of the Stokes source problem (3.4). The first way is the "two-grid method" of Xu and Zhou introduced and studied in [25] for second order differential equations and integral equations. The second way proposed and studied by Andreev and Racheva in [23] uses the same mesh but higher order mixed finite element space.

The first way uses a finer mesh (with mesh size $h^{(k+1)/k}$ or $h^{(s+1)/s}$) to get an approximation of $\tilde{\lambda}$ with an error $O(h^{2k+2})$ or $O(h^{2s+2})$. The advantage of this approach is that it uses the same finite element spaces and does not require higher regularity of the exact eigenfunctions. The second way is based on the same finite element mesh $\mathcal{T}_h$ but using a one order higher mixed finite element space. Also, to get an improvement for the approximation of $\tilde{\lambda}_h$ to the error $O(h^4)$ or $O(h^{2+2\gamma})$ from $O(h^2)$, we need to investigate the regularity of the Stokes eigenvalue problem.

We can treat both ways in the same abstract manner. Namely, let us introduce the enriched mixed finite element space $\tilde{\mathbf{V}}_h \times \tilde{W}_h$ such that $\mathbf{V}_h \times W_h \subset \tilde{\mathbf{V}}_h \times \tilde{W}_h \subset (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ and consider the following discrete Stokes problem: Find $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \tilde{\mathbf{V}}_h \times \tilde{W}_h$ such that

$$\begin{cases} a(\tilde{\mathbf{u}}_h, \mathbf{v}) + b(\mathbf{v}, \tilde{p}_h) = \lambda_h s(\mathbf{u}_h, \mathbf{v}) & \forall\, \mathbf{v} \in \tilde{\mathbf{V}}_h, \\ b(\tilde{\mathbf{u}}_h, q) = 0 & \forall\, q \in \tilde{W}_h. \end{cases} \tag{4.1}$$

Here we suppose that the approximation $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \tilde{\mathbf{V}}_h \times \tilde{W}_h$ has the following error estimate: for a smooth domain

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_1 + \|\tilde{p} - \tilde{p}_h\|_0 \leqslant Ch^{k+1}(\|\tilde{\mathbf{u}}\|_{k+2} + \|\tilde{p}\|_{k+1}), \tag{4.2}$$

and for a convex polygonal domain

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_1 + \|\tilde{p} - \tilde{p}_h\|_0 \leqslant Ch^{s+1}(\|\tilde{\mathbf{u}}\|_{s+2} + \|\tilde{p}\|_{s+1}). \tag{4.3}$$

So, we need to define the Rayleigh quotient

$$\tilde{\lambda}_h = \frac{a(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h)}{s(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h)}. \tag{4.4}$$

From the above analysis, we obtain the following error estimate for $(\tilde{\mathbf{u}}_h, \tilde{p}_h, \tilde{\lambda}_h) \in \tilde{\mathbf{V}}_h \times \tilde{W}_h \times \mathcal{R}$.

Theorem 4.1. *Assume $\tilde{\lambda}_h$ is defined by* (4.4), $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$ *is the solution of* (4.1) *and* $(\mathbf{u}, p, \lambda)$ *is the true solution of the Stokes eigenvalue problem* (2.1). *Then we have*

$$|\tilde{\lambda}_h - \lambda| \leqslant C(\|\mathbf{u} - \mathbf{u}_h\|_0 + |\lambda - \lambda_h| + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_1 + \|\tilde{p} - \tilde{p}_h\|_0)^2, \tag{4.5}$$

$$\|\tilde{\mathbf{u}}_h - \mathbf{u}\|_1 + \|\tilde{p}_h - p\|_0 \leqslant C(\|\mathbf{u} - \mathbf{u}_h\|_0 + |\lambda - \lambda_h| + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_1 + \|\tilde{p} - \tilde{p}_h\|_0). \tag{4.6}$$

Proof. First, from (3.6) and the triangle inequality we obtain (4.6). By virtue of $b(\tilde{\mathbf{u}}_h, \tilde{p}_h) = 0$ and (3.3), the following error estimate holds

$$\begin{aligned} |\tilde{\lambda}_h - \lambda| &\leqslant C(\|\tilde{\mathbf{u}}_h - \mathbf{u}\|_1^2 + \|\tilde{p}_h - p\|_0^2) \\ &\leqslant C(\|\mathbf{u} - \mathbf{u}_h\|_0 + |\lambda - \lambda_h| + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_1 + \|\tilde{p} - \tilde{p}_h\|_0)^2. \end{aligned}$$

□

Now, we can present a postprocessing algorithm which can improve the accuracy of the eigenvalue and eigenfunction approximations for the Stokes eigenvalue problem (2.1).

Algorithm 1.
(1) *Solve the Stokes eigenvalue problem* (2.12) *for* $(\mathbf{u}_h, p_h, \lambda_h) \in \mathbf{V}_h \times W_h \times \mathcal{R}$.
(2) *Solve the Stokes problem* (4.1) *and find* $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \tilde{\mathbf{V}}_h \times \tilde{W}_h$.
(3) *Compute*

$$\tilde{\lambda}_h = \frac{a(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h)}{s(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h)}.$$

The pair $(\tilde{\mathbf{u}}_h, \tilde{p}_h, \tilde{\lambda}_h)$ *represents a new (and better than* $(\mathbf{u}_h, p_h, \lambda_h)$*) approximation to* $(\mathbf{u}, p, \lambda)$.

Let us discuss two methods of constructing the augmented mixed finite element space $\tilde{\mathbf{V}}_h \times \tilde{W}_h$ for solving the Stokes source problem (4.1).

Way 1 ("Two-grid method" from [25]): In this case, $\tilde{\mathbf{V}}_h \times \tilde{W}_h$ is the same type of mixed finite element space as $\mathbf{V}_h \times W_h$ on the finer mesh $\tilde{\mathcal{T}}_h$ with mesh size h^β $(\beta > 1)$. Here $\tilde{\mathcal{T}}_h$ is a finer mesh of Ω which can be generated by the refinement just as in the multigrid method ([25]).

First, let us consider the case when the exact eigenfunction is smooth and has the error estimates (2.18) and (2.19). Because the maximum regularity of the solution $(\tilde{\mathbf{u}}, \tilde{p})$ of the Stokes source problem (3.4) is $(H^3(\Omega))^2 \times H^2(\Omega)$, we need to choose $k \leqslant 2$. In this case, we obtain the following improved accuracy for the eigenpair approximation when $\beta = (k+1)/k$ ([25]):

$$|\tilde{\lambda}_h - \lambda| \leqslant Ch^{2k}, \tag{4.7}$$

$$\|\tilde{\mathbf{u}}_h - \mathbf{u}\|_1 + \|\tilde{p}_h - p\|_0 \leqslant Ch^{k+1}. \tag{4.8}$$

When Ω is a convex polygonal domain, the error estimate (2.20), (2.21) and Theorem 4.1 yield

$$|\lambda - \tilde{\lambda}_h| \leqslant C(h^{2s+2} + h^{2\beta s}). \tag{4.9}$$

The optimal parameter is chosen to balance the two terms in the above inequality, i.e., $2s + 2 = 2\beta s$. So, we can obtain the following error estimate for $\beta = (s+1)/s$:

$$|\lambda - \tilde{\lambda}_h| \leqslant Ch^{2s+2}, \tag{4.10}$$

$$\|\tilde{\mathbf{u}}_h - \mathbf{u}\|_1 + \|\tilde{p}_h - p\|_0 \leqslant Ch^{s+1}. \tag{4.11}$$

From this error estimate we see that the postprocessing method can provide the same accuracy when solving the Stokes eigenvalue problem on the finer mesh $\tilde{\mathcal{T}}_h$.

This improvement costs solving the Stokes source problem on a finer mesh with mesh size $O(h^\beta)$. This is better than solving the Stokes eigenvalue problem on the finer mesh directly because solving the Stokes source problem needs much less computation than solving the Stokes eigenvalue problem.

Way 2 ("Two space" method from [23]): In this case, $\tilde{\mathbf{V}}_h \times \tilde{W}_h$ is defined on the same mesh $\mathcal{T}_h$ but with order higher by one than $\mathbf{V}_h \times W_h$. Since the maximum regularity of the solution $(\tilde{\mathbf{u}}, \tilde{p})$ for the Stokes source problem (3.4) is $(H^3(\Omega))^2 \times H^2(\Omega)$, we can only use the first order mixed finite element space to solve the original Stokes eigenvalue problem (2.12), and solve the Stokes source problem (4.1) in the second order mixed finite element space. So, we have only the error estimate for $(\mathbf{u}_h, p_h, \lambda_h)$

$$|\lambda - \lambda_h| \leqslant Ch^2, \tag{4.12}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant Ch. \tag{4.13}$$

First, if the domain Ω is smooth, we have the error estimate

$$|\lambda - \tilde{\lambda}_h| \leqslant Ch^4, \tag{4.14}$$

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_1 + \|p - \tilde{p}_h\|_0 \leqslant Ch^2. \tag{4.15}$$

This is an obvious improvement as compared with (4.12) and (4.13).

When Ω is a convex polygonal domain, the regularity of the Stokes source problem and the error estimate imply

$$|\lambda - \tilde{\lambda}_h| \leqslant Ch^{2+2\gamma}, \tag{4.16}$$

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_1 + \|p - \tilde{p}_h\|_0 \leqslant Ch^{1+\gamma}. \tag{4.17}$$

This estimate is also an obvious improvement as compared with (4.12) and (4.13).

The improved error estimate above just costs solving the Stokes source problem on the same mesh in the second order mixed finite element space.

5. Numerical results

In this section we give a numerical example to illustrate the efficiency of the postprocessing algorithm derived in this paper. Since we do not know the exact solution of the Stokes eigenvalue problems, the numerical result only shows the behavior of eigenvalue approximations by the postprocessing algorithm.

We consider the Stokes eigenvalue problem (1.1) on the domain $\Omega = (0,1) \times (0,1)$. According to [24] and [11], we can choose a sufficiently accurate first eigenvalue approximation $\lambda = 52.3446911$ as the true first eigenvalue.

Here we give the numerical results of the postprocessing algorithm with the enriched spaces constructed by one order higher mixed finite element. We first solve the Stokes eigenvalue problem (2.12) by the lowest order Bernardi-Raugel mixed finite element ([6], [9] and [14]) and solve the Stokes source problem (3.4) by the $Q_2 - P_1$ mixed finite element on the rectangular meshes ([9] and [14]). Now, we introduce the lowest order Bernadi-Raugel mixed finite element

$$\mathbf{V}_h = \{\mathbf{v} \in (H_0^1(\Omega))^2 \colon \mathbf{v}|_e \in Q_{12} \times Q_{21},\ \forall e \in \mathcal{T}_h\},$$
$$Q_h = \{p \in L_0^2(\Omega) \colon p|_e \in Q_{00},\ \forall e \in \mathcal{T}_h\},$$

and the $Q_2 - P_1$ mixed finite element

$$\mathbf{V}_h = \{\mathbf{v} \in (H_0^1(\Omega))^2 \colon \mathbf{v}|_e \in Q_{22} \times Q_{22},\ \forall e \in \mathcal{T}_h\},$$
$$Q_h = \{p \in L_0^2(\Omega) \colon p|_e \in P_1,\ \forall e \in \mathcal{T}_h\},$$

where $Q_{ij} = \operatorname{span}\{x^k y^l \colon 0 \leqslant k \leqslant i,\ 0 \leqslant l \leqslant j\}$ and $P_i = \operatorname{span}\{x^k y^l;\ k \geqslant 0,\ l \geqslant 0,\ k+l \leqslant i\}$. From the above theoretical analysis, we know that the accuracies of λ_h and $\tilde{\lambda}_h$ are $O(h^2)$ and $O(h^4)$, respectively.

In order to illustrate the convergence rate, we introduce the notation

$$\begin{aligned}
\mathrm{err}_h &= \lambda_h - \lambda,\\
\widetilde{\mathrm{err}}_h &= \tilde{\lambda}_h - \lambda,\\
R_h &= \frac{\log(|\mathrm{err}_{h_1}|/|\mathrm{err}_{h_2}|)}{\log(|h_1/h_2|)},\\
\widetilde{\mathrm{err}}_h &= \tilde{\lambda}_h - \lambda,\\
\tilde{R}_h &= \frac{\log(|\tilde{\mathrm{err}}_{h_1}|/|\tilde{\mathrm{err}}_{h_2}|)}{\log(|h_1/h_2|)}.
\end{aligned}$$

The numerical results are shown in Tab. 1. From Tab. 1, we can find that the postprocessing algorithm can improve the accuracy of the eigenvalue approximations and thus confirm the theoretical analysis.

$M \times N$	4×4	8×8	16×16	32×32	64×64
λ_h	51.531253583	52.093629173	52.278091067	52.327797607	52.340452665
$\tilde{\lambda}_h$	52.484991391	52.355892886	52.345473959	52.34474510	52.344694676
err_h	−8.13438E−1	−2.51062E−1	−6.66000E−2	−1.68935E−2	−4.23844E−3
$\widetilde{\mathrm{err}}_h$	1.403003E−1	1.120179E−2	7.828588E−4	5.399888E−5	3.575866E−6
R_h	–	1.695988266	1.914448462	1.979055271	1.994864197
$\tilde{R}_h$	–	3.646717312	3.838832828	3.857750706	3.916564890

Table 1. The results for the eigenvalue approximations by postprocessing algorithm.

6. Concluding remarks

The method and the result can be extended to the general mixed eigenvalue problems which can be described by (2.1) with $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ satisfying (2.2)–(2.7). We can use the better eigenvalue and eigenfunction approximation $(\tilde{\mathbf{u}}_h, \tilde{p}, \tilde{\lambda}_h)$ to construct an a posteriori error estimator of the eigenpair approximation $(\mathbf{u}_h, p_h, \lambda_h)$ for the Stokes eigenvalue problem ([11]).

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