Applications of Mathematics

Akio Ito; Nobuyuki Kenmochi; Noriaki Yamazaki

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Applications of Mathematics, Vol. 53 (2008), No. 5, 433–454

Persistent URL: http://dml.cz/dmlcz/140333

53 (2008) APPLICATIONS OF MATHEMATICS No. 5, 433–454

A PHASE-FIELD MODEL OF GRAIN BOUNDARY MOTION

Akio Ito, Hiroshima, Nobuyuki Kenmochi, Chiba,
Noriaki Yamazaki, Muroran

Dedicated to Jürgen Sprekels on the occasion of his 60th birthday

Abstract. We consider a phase-field model of grain structure evolution, which appears in materials sciences. In this paper we study the grain boundary motion model of Kobayashi-Warren-Carter type, which contains a singular diffusivity. The main objective of this paper is to show the existence of solutions in a generalized sense. Moreover, we show the uniqueness of solutions for the model in one-dimensional space.

Keywords: grain boundary motion, singular diffusion, subdifferential

MSC 2010: 35K45, 35K55, 35R35

1. Introduction

In this paper we consider the following phase-field model of grain structure evolution, denoted by (P):

$$
\text{(P)}\qquad \begin{cases} \eta_t - \kappa\Delta\eta + g(\eta) + \alpha'(\eta)|\nabla\theta| = 0 & \text{a.e. in } Q_T := \Omega\times(0,T),\\ \alpha_0(\eta)\theta_t - \nu\Delta\theta - \operatorname{div}\Big(\alpha(\eta)\dfrac{\nabla\theta}{|\nabla\theta|}\Big) = 0 & \text{a.e. in } Q_T,\\ \dfrac{\partial\eta}{\partial n} = 0,\ \ \theta = 0 & \text{a.e. on } \Sigma_T := \Gamma\times(0,T),\\ \eta(x,0) = \eta_0(x),\ \ \theta(x,0) = \theta_0(x) & \text{for a.a. } x\in\Omega, \end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 1)$ with smooth boundary $\Gamma := \partial\Omega$, $T > 0$ is a fixed finite time, $\kappa > 0$ and $\nu > 0$ are given small constants, $g(\cdot)$, $\alpha(\cdot)$ and $\alpha_0(\cdot)$ are given functions on $\mathbb{R}$, $\partial/\partial n$ is the outward normal derivative on Γ, and $\eta_0(x)$, $\theta_0(x)$ are given initial data.

The above model of two dimensional grain structure was proposed in Kobayashi et al [18], where the variable θ is an indicator of the mean orientation of the crystalline and the variable η is an order parameter for the degree of crystalline orientation: $\eta \equiv 1$ implies a completely oriented state and $\eta \equiv 0$ is a state where no meaningful value of orientation exists. The model (P) is derived from the free energy functional of the following form:

$$\text{(1.1)} \qquad \mathcal{F}(\eta,\theta) := \frac{\kappa}{2}\int_\Omega |\nabla\eta|^2\,\mathrm{d}x + \int_\Omega \hat{g}(\eta)\,\mathrm{d}x + \frac{\nu}{2}\int_\Omega |\nabla\theta|^2\,\mathrm{d}x + \int_\Omega \alpha(\eta)|\nabla\theta|\,\mathrm{d}x.$$

Moreover, in [18] some numerical experiments for (P) are given in the case where $\hat{g}(\eta) := \frac{1}{2}(1-\eta)^2$, $\alpha_0(\eta) = \alpha(\eta) = \eta^2$ and Ω is a bounded domain in $\mathbb{R}^2$. However, no theoretical results have been established there. For some related work, we refer to [8], [12], [20], [22].

In connection with this subject, the singular diffusion equations

$$u_t = \operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big), \quad \text{or, more generally,} \quad u_t = \frac{1}{b(x)}\operatorname{div}\Big(a(x)\frac{\nabla u}{|\nabla u|}\Big),$$

kindred to the second equation of (P), have been studied by a lot of mathematicians from various view-points (cf. [1], [2], [3], [6], [11], [17]).

Recently, Ito et al [13] showed the existence-uniqueness of solutions to the one-dimensional grain boundary model of Kobayashi-Warren-Carter type, with $-\kappa\Delta\eta$ replaced by $-(\sigma\eta_t + \kappa\eta)_{xx}$, $0 < \sigma < \infty$, in the first equation.

In this paper, we shall show the existence of a weak solution to (P) in any dimension of space and the uniqueness in dimension one.

The plan of this paper is as follows. In Section 2, we mention the main theorems of this paper. In Section 3, we prepare some auxiliary results, and in Section 4, we solve the approximating systems to (P). In the final section, we show the existence of a solution of (P) by discussing the convergence of approximate solutions and the uniqueness of solution in one dimensional space.

2. Main results

Throughout this paper, we use the following notation:

(1) We denote by $\|\cdot\|_X$ the norm of a Banach space X. In particular, the norm of $L^\infty := L^\infty(\Omega)$ will be denoted by $\|\cdot\|_\infty$.

(2) We denote by $H := L^2(\Omega)$ with the usual real Hilbert space structure. The inner product and norm in H are denoted by $(\cdot,\cdot)$ and by $\|\cdot\|_H$, respectively. Also, $H^1 := H^1(\Omega)$, $H^1_0 := H^1_0(\Omega)$ and $H^2 := H^2(\Omega)$ are the usual Sobolev spaces.

(3) Let ψ be a proper (i.e., not identically equal to infinity), l.s.c. (lower semi-continuous) and convex function on H whose effective domain is denoted by

$D(\psi) := \{z \in H;\ \psi(z) < +\infty\}$. We denote by $\partial\psi$ the subdifferential of ψ in H, i.e., $\partial\psi$ is an operator from H into 2^H, and is defined by $z^* \in \partial\psi(z)$ if and only if

$$z \in D(\psi) \quad \text{and} \quad (z^*, y - z) \leqslant \psi(y) - \psi(z) \quad \text{for all } y \in H.$$

The domain $D(\partial\psi)$ of $\partial\psi$ is the set $\{z \in H;\ \partial\psi(z) \neq \emptyset\}$. For the fundamental properties of subdifferentials, we refer to the textbooks [4], [5], [7], [15].

Let us now give some assumptions on the data. Throughout this paper, the following conditions are always assumed:

(A1) α_0 is a Lipschitz continuous function on $\mathbb{R}$ such that $\alpha_0 \geqslant \delta_0$ on $\mathbb{R}$ for a positive constant δ_0. We denote by $L(\alpha_0)$ the Lipschitz constant.

(A2) α is a non-negative function in $C^1(\mathbb{R})$, whose derivative α' is non-decreasing and bounded on $\mathbb{R}$ such that $\alpha'(0) = 0$. We denote by $L(\alpha)$ the Lipschitz constant.

(A3) g is a Lipschitz continuous function on $\mathbb{R}$. Its Lipschitz constant is denoted by $L(g)$. We assume that $g \leqslant 0$ on $(-\infty, 0]$ and $g \geqslant 0$ on $[1, \infty)$. Also, we denote by $\hat{g}$ a primitive of g, and assume that $\hat{g}$ is non-negative on $\mathbb{R}$.

(A4) $\eta_0 \in H^1$ with $0 \leqslant \eta_0 \leqslant 1$ a.e. on Ω, and $\theta_0 \in H_0^1$.

Next, we give the notion of a solution to (P).

Definition 2.1. A pair $[\eta, \theta]$ of functions $\eta\colon [0,T] \to H^1$ and $\theta\colon [0,T] \to H_0^1$ is a solution to (P) on $[0,T]$, if the following conditions (1)–(5) are satisfied:

(1) $\eta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$.

(2) $\theta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;H_0^1)$.

(3) The following parabolic equation holds:

$$\eta'(t) - \kappa\Delta_N\eta(t) + g(\eta(t)) + \alpha'(\eta(t))|\nabla\theta(t)| = 0 \ \text{ in } H \quad \text{for a.a. } t \in (0,T), \tag{2.1}$$

where $\eta' := \mathrm{d}\eta/\mathrm{d}t$ and $\Delta_N\colon D(\Delta_N) := \{z \in H^2;\ \partial z/\partial n = 0 \text{ a.e. on } \Gamma\} \to H$ is the Laplacian with homogeneous Neumann boundary condition.

(4) For any $z \in H_0^1$ and a.a. $t \in (0,T)$, the following variational inequality holds:

$$\begin{aligned}(\alpha_0(\eta(t))\theta'(t), \theta(t) - z) &+ \nu(\nabla\theta(t), \nabla\theta(t) - \nabla z) \\ &+ \int_\Omega \alpha(\eta(x,t))|\nabla\theta(x,t)|\,\mathrm{d}x \leqslant \int_\Omega \alpha(\eta(x,t))|\nabla z(x)|\,\mathrm{d}x,\end{aligned} \tag{2.2}$$

where $\theta' := \mathrm{d}\theta/\mathrm{d}t$.

(5) $\eta(0) = \eta_0$ and $\theta(0) = \theta_0$ in H.

Our main results of this paper are stated as follows:

Theorem 2.1. *Assume* (A1)–(A4) *hold. Then, there is at least one solution* $[\eta, \theta]$ *of* (P) *in the sense of Definition* 2.1, *and* η *satisfies*

$$0 \leqslant \eta \leqslant 1 \quad \text{a.e. on } Q_T. \tag{2.3}$$

Theorem 2.2. *Assume* (A1)–(A4) *hold, and the space dimension of* Ω *is one, say,* $\Omega = (-L, L)$ *for a positive number* L. *Then the solution* $[\eta, \theta]$ *obtained by Theorem* 2.1 *is unique.*

The main idea for the proof of Theorem 2.1 is to use the subdifferential technique in order to handle the variational inequality (2.2). In fact, we introduce a proper, l.s.c. and convex function $\varphi(\eta(t); \cdot)$ on H, depending on $\eta \in W^{1,2}(0, T; H)$, which is defined by

$$\varphi(\eta(t); z) := \begin{cases} \dfrac{\nu}{2} \displaystyle\int_\Omega |\nabla z|^2 \,\mathrm{d}x + \int_\Omega \alpha(\eta(t))|\nabla z| \,\mathrm{d}x, & \text{if } z \in H_0^1, \\ \infty, & \text{otherwise.} \end{cases} \tag{2.4}$$

By $\partial\varphi(\eta(t); z)$ we denote the subdifferential of $\varphi(\eta(t); z)$ with respect to $z \in H$. It is easily checked that with this function the variational inequality (2.2) is written in the form

$$\alpha_0(\eta(t))\theta'(t) + \partial\varphi(\eta(t); \theta(t)) \ni 0 \quad \text{in } H \quad \text{for a.a. } t \in (0, T). \tag{2.5}$$

The first theorem will be proved by discussing the convergence of the following approximate problems $(\mathrm{P})_\varepsilon$ with real parameter $\varepsilon \in (0, 1]$, as $\varepsilon \downarrow 0$:

$$(\mathrm{P})_\varepsilon \quad \begin{cases} \eta'(t) - \kappa\Delta_N\eta(t) + g(\eta(t)) + \alpha'(\eta(t))|\nabla\theta(t)| = 0 \text{ in } H & \text{for a.a. } t \in (0, T), \\ \alpha_0((\varrho_\varepsilon * \eta)(t))\theta'(t) + \partial\varphi(\eta(t); \theta(t)) \ni 0 \text{ in } H & \text{for a.a. } t \in (0, T), \\ \eta(x, 0) = \eta_0(x), \quad \theta(x, 0) = \theta_0(x) & \text{for a.a. } x \in \Omega, \end{cases}$$

where ϱ_ε is the usual one-dimensional mollifier with support $[-\varepsilon, \varepsilon]$ in time, and $(\varrho_\varepsilon * \eta)$ is the convolution of ϱ_ε and $\tilde{\eta}$, namely

$$(\varrho_\varepsilon * \eta)(x, t) := \int_{-\infty}^{\infty} \varrho_\varepsilon(t - s)\tilde{\eta}(x, s) \,\mathrm{d}s \quad \text{for } x \in \Omega, \ t \in [0, T], \tag{2.6}$$

where $\tilde{\eta}$ is the extension of η to $\Omega \times \mathbb{R}$ given by

$$\tilde{\eta}(x, t) := \begin{cases} \eta(x, 0) & \text{for } x \in \Omega, \ t < 0, \\ \eta(x, t) & \text{for } x \in \Omega, \ 0 \leqslant t \leqslant T, \\ \eta(x, T) & \text{for } x \in \Omega, \ t > T. \end{cases}$$

3. Auxiliary problems

In this section, we consider separately the following Cauchy problems:

$$(\mathrm{P1};\theta)\quad \begin{cases} \eta'(t) - \kappa\Delta_N \eta(t) + g(\eta(t)) + \alpha'(\eta(t))|\nabla\theta(t)| = 0 \ \text{ in } H \\ \qquad \text{for a.a. } t \in (0,T), \\ \eta(0) = \eta_0 \ \text{ in } H, \end{cases}$$

where θ is given in $W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1_0)$, and

$$(\mathrm{P2};\tilde{\alpha}_0,\eta)\quad \begin{cases} \tilde{\alpha}_0(t)\theta'(t) + \partial\varphi(\eta(t);\theta(t)) \ni 0 \ \text{ in } H \ \text{ for a.a. } t \in (0,T), \\ \theta(0) = \theta_0 \ \text{ in } H, \end{cases}$$

where $\tilde{\alpha}_0$ is a given function in $L^\infty(Q_T)$ with $\alpha_0 \geqslant \delta_0$ a.e. on Q_T, and η is a given function in $W^{1,2}(0,T;H)$.

Throughout this section, we always make the assumptions (A2)–(A4).

(1) Problem $(\mathrm{P1};\theta)$

Firstly, we consider the problem $(\mathrm{P1};\theta)$.

Proposition 3.1. *Assume that* (A2)–(A4) *are satisfied. Then, we have:*

(a) *For any θ in $W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1_0)$, the problem $(\mathrm{P1};\theta)$ has one and only one solution η in the class $W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$, and the solution η satisfies*

$$0 \leqslant \eta \leqslant 1 \quad \text{a.e. on } Q_T. \tag{3.1}$$

Moreover, the following energy inequality holds:

$$\|\eta'(t)\|_H^2 + \kappa\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\eta(t)\|_H^2 + 2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \hat{g}(\eta(t))\,\mathrm{d}x \leqslant R_1\|\nabla\theta(t)\|_H^2 \quad \text{for a.a. } t \in (0,T), \tag{3.2}$$

where $R_1 := L(\alpha)^2$.

(b) *Let $\{\theta_n\}$ be any sequence in $W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1_0)$ such that $\theta_n \to \theta$ in $L^2(0,T;H^1_0)$, weakly in $W^{1,2}(0,T;H)$ and weakly* in $L^\infty(0,T;H^1_0)$ (as $n \to \infty$). Then, denoting by η_n and η the solutions of $(\mathrm{P1};\theta_n)$ and $(\mathrm{P1};\theta)$ on $[0,T]$, respectively, we have*

$$\eta_n \to \eta \text{ weakly in } W^{1,2}(0,T;H), \text{ weakly in } L^2(0,T;H^2) \text{ and weakly}^* \text{ in } L^\infty(0,T;H^1), \tag{3.3}$$

hence, $\eta_n \to \eta$ in $C([0,T];H) \cap L^2(0,T;H^1)$ as $n \to \infty$.

P r o o f. By the general theory of parabolic PDEs (cf. [10]), there exists a unique solution η in the class $W^{1,2}(0,T;H)\cap L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$. Now, we multiply the equation

$$\eta'(t) - \kappa\Delta_N\eta(t) + g(\eta(t)) + \alpha'(\eta(t))|\nabla\theta(t)| = 0 \quad \text{in } H \text{ for a.a. } t \in (0,T) \tag{3.4}$$

by $\eta'(t)$ to get

$$\|\eta'(t)\|_H^2 + \frac{1}{2}\kappa\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\eta(t)\|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \hat{g}(\eta(t))\,\mathrm{d}x \leqslant \int_\Omega |\alpha'(\eta(t))||\nabla\theta(t)||\eta'(t)|\,\mathrm{d}x \quad \text{for a.a. } t \in (0,T).$$

Hence, with $R_1 := L(\alpha)^2$ we get (3.2).

Next, we show (3.1). Let η be the solution of (P1; θ) on $[0,T]$. We multiply (3.4) by $[\eta(t)-1]^+$, where $[\eta(t)-1]^+$ denotes the positive part of the function $\eta(t)-1$. Then we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|[\eta(t)-1]^+\|_H^2 + \kappa\|\nabla[\eta(t)-1]^+\|_H^2 \\ &+(g(\eta(t)),[\eta(t)-1]^+) + (\alpha'(\eta(t))|\nabla\theta(t)|,[\eta(t)-1]^+) = 0 \\ &\text{for a.a. } t \in (0,T). \end{aligned} \tag{3.5}$$

Here, we note from (A3) that

$$g(\eta) \geqslant 0 \quad \text{on the set } \{(x,t)\in Q_T;\ [\eta(t)-1]^+ > 0\},$$

which implies that

$$(g(\eta(t)),[\eta(t)-1]^+) \geqslant 0 \quad \text{for all } t \in (0,T). \tag{3.6}$$

Also, it follows from (A2) that

$$(\alpha'(\eta(t))|\nabla\theta(t)|,[\eta(t)-1]^+) \geqslant 0 \quad \text{for all } t \in (0,T). \tag{3.7}$$

Therefore, we see from (3.5)–(3.7) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|[\eta(t)-1]^+\|_H^2 \leqslant 0 \quad \text{for a.a. } t \in (0,T),$$

hence,

$$\frac{1}{2}\|[\eta(t)-1]^+\|_H^2 \leqslant \frac{1}{2}\|[\eta_0-1]^+\|_H^2 = 0 \quad \text{for all } t \in (0,T).$$

Thus, we have

$$\eta(x,t) \leqslant 1 \quad \text{for all } t \in [0,T] \quad \text{and a.a. } x \in \Omega. \tag{3.8}$$

Next, we multiply (3.4) by $[\eta(t)]^-$, where $[\eta(t)]^-$ denotes the negative part of the function $\eta(t)$. Then, we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|[\eta(t)]^-\|_H^2 + \kappa\|\nabla[\eta(t)]^-\|_H^2 \\ &-(g(\eta(t)),[\eta(t)]^-) - (\alpha'(\eta(t))|\nabla\theta(t)|,[\eta(t)]^-) = 0 \\ &\text{for a.a. } t \in (0,T). \end{aligned} \tag{3.9}$$

Here, we note from (A2) and (A3) that

$$(\alpha'(\eta(t))|\nabla\theta(t)|,[\eta(t)]^-) \leqslant 0, \quad (g(\eta(t)),[\eta(t)]^-) \leqslant 0 \quad \text{for all } t \in (0,T).$$

Therefore, we see from (3.9) that

$$\frac{1}{2}\|[\eta(t)]^-\|_H^2 \leqslant \frac{1}{2}\|[\eta_0]^-\|_H^2 = 0 \quad \text{for all } t \in (0,T).$$

Thus, we have

$$\eta(x,t) \geqslant 0 \quad \text{for all } t \in [0,T] \text{ and a.a. } x \in \Omega. \tag{3.10}$$

Therefore, we infer (3.1) from (3.8) and (3.10). Thus, we have (a).

Next, we prove (b). Since $\{\|\nabla\theta_n\|_H\}$ is bounded in $L^\infty(0,T)$, it follows from (3.1) and the energy inequality (3.2) that $\{\eta_n\}$ is bounded in $W^{1,2}(0,T;H)\cap L^\infty(0,T;H^1)$, and, hence, is bounded in $L^2(0,T;H^2)$ by (3.4). Hence, applying Aubin's compactness theorem (cf. [19]), there is a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ and a function $\tilde\eta \in W^{1,2}(0,T;H)\cap L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ such that

$$\begin{aligned} \eta_{n_k} \to \tilde\eta \quad &\text{in } C([0,T];H)\cap L^2(0,T;H^1), \\ &\text{weakly in } W^{1,2}(0,T;H) \text{ and weakly* in } L^\infty(0,T;H^1) \end{aligned}$$

as $k \to \infty$. These convergences imply immediately that the limit $\tilde\eta$ is a solution of (P1; θ) on $[0,T]$. By the uniqueness of solution of (P1; θ), it follows that $\tilde\eta = \eta$ and (3.3) holds without extracting any subsequence from $\{\eta_n\}$. □

(2) Problem (P2; $\tilde\alpha_0, \eta$)

Secondly, we consider the problem (P2; $\tilde\alpha_0, \eta$).

Proposition 3.2. *Assume that* (A2)–(A4) *are satisfied. Then, we have:*

(c) *Let* $\tilde{\alpha}_0$ *be any function in* $L^\infty(Q_T)$ *such that* $\tilde{\alpha}_0 \geqslant \delta_0$ *a.e. on* Q_T *for a positive constant* δ_0, *and let* η *be any function in* $W^{1,2}(0,T;H)$. *Then* (P2; $\tilde{\alpha}_0, \eta$) *has at least one solution* θ *in the class* $W^{1,2}(0,T;H) \cap L^\infty(0,T;H_0^1)$ *such that* $t \to \varphi(\eta(t);\theta(t))$ *is absolutely continuous on* $[0,T]$ *and the energy inequality*

$$\begin{aligned}(3.11)\qquad \delta_0\|\theta'(t)\|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\eta(t);\theta(t)) &\leqslant L(\alpha)\|\eta'(t)\|_H\|\nabla\theta(t)\|_H \\ &(\leqslant R_2\|\eta'(t)\|_H(\varphi(\eta(t);\theta(t))+1)) \quad \text{for a.a. } t\in(0,T)\end{aligned}$$

holds, where $R_2 := \sqrt{2/\nu}L(\alpha)$. *Moreover, if* $\partial\tilde{\alpha}_0/\partial t \in L^\infty(Q_T)$, *then the solution* θ *of* (P2; $\tilde{\alpha}_0, \eta$) *is unique.*

(d) *Let* $\{\tilde{\alpha}_{0,n}\}$ *be any bounded sequence in* $L^\infty(Q_T)$ *such that* $\tilde{\alpha}_{0,n} \geqslant \delta_0$ *a.e. on* Q_T *for all* $n = 1,2,\ldots$, *where* δ_0 *is a positive constant. Also, let* $\{\eta_n\}$ *be any bounded sequence in* $W^{1,2}(0,T;H)\cap L^\infty(0,T;H^1)$. *Suppose that*

$$(3.12)\qquad \tilde{\alpha}_{0,n} \to \tilde{\alpha}_0 \quad \text{in } L^2(Q_T)$$

and

$$(3.13)\qquad \eta_n \to \eta \ \text{ weakly in } W^{1,2}(0,T;H) \quad \text{and weakly}^* \text{ in } L^\infty(0,T;H^1)$$

as $n\to\infty$. *Denote by* θ_n *a solution of* (P2; $\tilde{\alpha}_{0,n}, \eta_n$) *for each* $n = 1,2,\ldots$. *Then there is a subsequence* $\{\theta_{n_k}\}$ *of* $\{\theta_n\}$ *and a function* $\theta \in W^{1,2}(0,T;H)\cap L^\infty(0,T;H_0^1)$ *such that*

$$(3.14)\qquad \theta_{n_k} \to \theta \ \text{ weakly in } W^{1,2}(0,T;H) \quad \text{and weakly}^* \text{ in } L^\infty(0,T;H_0^1)$$

and

$$(3.15)\qquad \theta_{n_k} \to \theta \ \text{ in } L^2(0,T;H_0^1)$$

as $k\to\infty$. *Moreover, the limit* θ *is a solution of* (P2; $\tilde{\alpha}_0, \eta$) *on* $[0,T]$. *In addition, if* $\partial\tilde{\alpha}_0/\partial t \in L^\infty(Q_T)$, *then* (3.14) *and* (3.15) *hold for the whole sequence* $\{\theta_n\}$.

An essential part of our proof of Proposition 3.2 (c), is contained in the following lemma.

Lemma 3.1. *Assume that* (A2)–(A4) *are satisfied. Let* $\tilde{\alpha}_0$ *be any function in* $C^2(\overline{Q_T})$ *such that* $\tilde{\alpha}_0 \geqslant \delta_0$ *on* Q_T*, where* δ_0 *is a positive constant. Let* η *be any function in* $W^{1,2}(0,T;H)$*. Then the problem* $(\mathrm{P2};\tilde{\alpha}_0,\eta)$ *has one and only one solution* θ *in the class* $W^{1,2}(0,T;H)\cap L^\infty(0,T;H_0^1)$ *such that* $t\to\varphi(\eta(t);\theta(t))$ *is absolutely continuous on* $[0,T]$ *and the energy inequality* (3.11) *holds.*

P r o o f. In order to transform the equation

$$\tilde{\alpha}_0(t)\theta'(t)+\partial\varphi(\eta(t);\theta(t))\ni 0 \quad \text{in } H \quad \text{for a.a. } t\in(0,T) \tag{3.16}$$

to the normal form, we introduce a proper, l.s.c. and convex function $\psi^t(\cdot)$ on H defined by

$$\psi^t(z):=\varphi\Big(\eta(t);\frac{z}{\sqrt{\tilde{\alpha}_0(t)}}\Big),\quad \forall\, z\in H,\ \forall\, t\in[0,T]. \tag{3.17}$$

It is easy to check that

$$\begin{gathered}|\varphi(\eta(t);z)-\varphi(\eta(s);z)|\leqslant L(\alpha)\int_s^t\|\eta'(\tau)\|_H\,\mathrm{d}\tau\cdot\|\nabla z\|_H,\\ \forall\, z\in H_0^1,\ \forall\, s,\ \forall\, t\in[0,T]\ \text{with}\ s\leqslant t.\end{gathered} \tag{3.18}$$

With the help of (3.18), we see that

$$\begin{gathered}|\psi^t(z)-\psi^s(z)|\leqslant R_3\int_s^t(\|\eta'(\tau)\|_H+1)\,\mathrm{d}\tau\cdot\|z\|_{H_0^1},\\ \forall\, z\in H_0^1,\ \forall\, s,\ \forall\, t\in[0,T]\ \text{with}\ s\leqslant t,\end{gathered} \tag{3.19}$$

where R_3 is a positive constant depending only on $\tilde{\alpha}_0$ and α. Moreover, by an elementary calculation, we have

$$D(\psi^t)=H_0^1,\quad \partial\psi^t(z)=\frac{1}{\sqrt{\tilde{\alpha}_0(t)}}\partial\varphi\Big(\eta(t);\frac{z}{\sqrt{\tilde{\alpha}_0(t)}}\Big),\quad \forall\, z\in D(\partial\psi^t). \tag{3.20}$$

Now, in terms of the function $u(x,t):=\sqrt{\tilde{\alpha}_0(x,t)}\theta(x,t)$, we see from (3.20) that (3.16) is transformed into the normal form

$$u'(t)+\partial\psi^t(u(t))\ni\frac{\tilde{\alpha}_0'(t)}{2\tilde{\alpha}_0(t)}u(t)\quad \text{in } H\ \text{ for a.a. } t\in(0,T). \tag{3.21}$$

By virtue of the general theory ([14], [21]) for nonlinear evolution equations governed by time-dependent subdifferentials, under the condition (3.19) the Cauchy problem for (3.21) with initial value $u_0:=\sqrt{\tilde{\alpha}_0(0)}\theta_0$ has one and only one solution u in the class $W^{1,2}(0,T;H)\cap L^\infty(0,T;H^1_0)$ such that $\psi^t(u(t))$ is absolutely continuous in $t\in[0,T]$. This shows that the function $\theta := u/\sqrt{\tilde\alpha_0}$ gives a unique solution of $(\mathrm{P2};\tilde\alpha_0,\eta)$ such that $\varphi(\eta(t);\theta(t))$ is absolutely continuous in $t\in[0,T]$.

Finally, we show (3.11). To do so, we multiply (3.16) by θ' to obtain

$$\delta_0\|\theta'(t)\|_H^2+(\theta^*(t),\theta'(t))\leqslant 0 \quad \text{for a.a. } t\in(0,T), \tag{3.22}$$

where $\theta^*(t) := -\tilde\alpha_0(t)\theta'(t)\in\partial\varphi(\eta(t);\theta(t))$ in H for a.a. $t\in(0,T)$. Here, we use the following inequality obtained later:

$$\Big|\frac{\mathrm d}{\mathrm dt}\varphi(\eta(t);\theta(t))-(\theta^*(t),\theta'(t))\Big|\leqslant L(\alpha)\|\eta'(t)\|_H\|\nabla\theta(t)\|_H \quad \text{for a.a. } t\in(0,T). \tag{3.23}$$

We infer (3.11) immediately from (3.22) and (3.23).

The inequality (3.23) can be proved from (3.18) as follows. For any $s,t\in(0,T)$ with $s<t$, we observe that

$$\begin{aligned}&\varphi(\eta(t);\theta(t))-\varphi(\eta(s);\theta(s))\\&\quad=\varphi(\eta(t);\theta(t))-\varphi(\eta(t);\theta(s))+\varphi(\eta(t);\theta(s))-\varphi(\eta(s);\theta(s))\\&\quad\leqslant(\theta^*(t),\theta(t)-\theta(s))+L(\alpha)\int_s^t\|\eta'(\tau)\|_H\,\mathrm d\tau\cdot\|\nabla\theta(s)\|_H.\end{aligned}$$

Hence, dividing the above inequality by $t-s$ and letting $s\uparrow t$ we have

$$\frac{\mathrm d}{\mathrm dt}\varphi(\eta(t);\theta(t))-(\theta^*(t),\theta'(t))\leqslant L(\alpha)\|\eta'(t)\|_H\|\nabla\theta(t)\|_H \quad \text{for a.a. } t\in(0,T). \tag{3.24}$$

The inequality (3.24) holds if $\varphi(\eta(t);\theta(t))$ is differentiable at t and t is a Lebesgue point of the function $\|\nabla\theta(t)\|_H$. Similarly, we obtain

$$\frac{\mathrm d}{\mathrm dt}\varphi(\eta(t);\theta(t))-(\theta^*(t),\theta'(t))\geqslant -L(\alpha)\|\eta'(t)\|_H\|\nabla\theta(t)\|_H \quad \text{for a.a. } t\in(0,T). \tag{3.25}$$

Combining (3.24) and (3.25), we get (3.23). □

We now give a proof of Proposition 3.2, by using Lemma 3.1.

Proof of (c) of Proposition 3.2. Choose a sequence $\{\tilde\alpha_{0,n}\}\subset C^2(\overline{Q_T})$ which is bounded in $L^\infty(Q_T)$ and such that $\tilde\alpha_{0,n}\geqslant\delta_0$ on Q_T for all n and $\tilde\alpha_{0,n}\to\tilde\alpha_0$ in $L^2(Q_T)$ as $n\to\infty$. Then, by virtue of Lemma 3.1, for each $n=1,2,\ldots$, the problem $(\mathrm{P2};\tilde\alpha_{0,n},\eta)$ has one and only one solution θ_n in $W^{1,2}(0,T;H)\cap L^\infty(0,T;H^1_0)$

such that $\varphi(\eta(t);\theta_n(t))$ is absolutely continuous in $t\in[0,T]$ and the following energy inequality holds:

$$\begin{aligned}(3.26)\qquad &\delta_0\|\theta_n'(t)\|_H^2+\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\eta(t);\theta_n(t))\leqslant L(\alpha)\|\eta'(t)\|_H\|\nabla\theta_n(t)\|_H\\ &(\leqslant R_2\|\eta'(t)\|_H(\varphi(\eta(t);\theta_n(t))+1))\quad \text{for a.a. } t\in(0,T).\end{aligned}$$

From (3.26) it follows that $\{\theta_n\}$ is bounded in $W^{1,2}(0,T;H)\cap L^\infty(0,T;H_0^1)$ and relatively compact in $C([0,T];H)$, so that there exist a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ and a function θ in $W^{1,2}(0,T;H)\cap L^\infty(0,T;H_0^1)$ such that

$$\begin{aligned}(3.27)\qquad \theta_{n_k}\to\theta \ \text{ in } C([0,T];H),\ \text{ weakly in } W^{1,2}(0,T;H)\\ \text{and weakly}^* \text{ in } L^\infty(0,T;H_0^1)\end{aligned}$$

as $k\to\infty$. Also, taking account of (3.27) and the L^∞-boundedness of $\tilde\alpha_{0,n_k}$ for all k, we see that

$$\tilde\alpha_{0,n_k}\theta_{n_k}'\to\tilde\alpha_0\theta'\quad\text{weakly in } L^2(0,T;H)$$

as $k\to\infty$. Since θ_{n_k} is the solution of (P2; $\tilde\alpha_{0,n_k},\eta$), it follows from (3.16) that

$$\begin{aligned}(3.28)\quad &\int_0^T(\tilde\alpha_{0,n_k}(t)\theta_{n_k}'(t),\theta_{n_k}(t)-w(t))\,\mathrm{d}t\\ &\quad+\nu\int_0^T(\nabla\theta_{n_k}(t),\nabla(\theta_{n_k}(t)-w(t)))\,\mathrm{d}t+\int_0^T\int_\Omega\alpha(\eta(t))|\nabla\theta_{n_k}(t)|\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant\int_0^T\int_\Omega\alpha(\eta(t))|\nabla w(t)|\,\mathrm{d}x\,\mathrm{d}t,\quad\forall\, w\in L^2(0,T;H_0^1).\end{aligned}$$

Here, note from (3.27) that

$$\liminf_{k\to\infty}\int_0^T\|\nabla\theta_{n_k}(t)\|_H^2\,\mathrm{d}t\geqslant\int_0^T\|\nabla\theta(t)\|_H^2\,\mathrm{d}t$$

and

$$\liminf_{k\to\infty}\int_0^T\int_\Omega\alpha(\eta(t))|\nabla\theta_{n_k}(t)|\,\mathrm{d}x\,\mathrm{d}t\geqslant\int_0^T\int_\Omega\alpha(\eta(t))|\nabla\theta(t)|\,\mathrm{d}x\,\mathrm{d}t.$$

Hence, passing to the limit as $k\to\infty$ in (3.28), we see that the limit θ satisfies the same inequality as (3.28), namely

$$\begin{aligned}&\int_0^T(\tilde\alpha_0(t)\theta'(t),\theta(t)-w(t))\,\mathrm{d}t\\ &\quad+\nu\int_0^T(\nabla\theta(t),\nabla(\theta(t)-w(t)))\,\mathrm{d}t+\int_0^T\int_\Omega\alpha(\eta(t))|\nabla\theta(t)|\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant\int_0^T\int_\Omega\alpha(\eta(t))|\nabla w(t)|\,\mathrm{d}x\,\mathrm{d}t,\quad\forall\, w\in L^2(0,T;H_0^1),\end{aligned}$$

which is equivalent to $-\tilde{\alpha}_0(t)\theta'(t) \in \partial\varphi(\eta(t);\theta(t))$ in H for a.a. $t \in (0,T)$. This shows that θ is a solution of (P2; $\tilde{\alpha}_0, \eta$) on $[0,T]$. The energy inequality (3.11) is also obtained just as seen in the proof of Lemma 3.1.

Moreover, assume that $\partial\tilde{\alpha}_0/\partial t \in L^\infty(Q_T)$. Then, the solution of (P2; $\tilde{\alpha}_0, \eta$) is unique. In fact, let θ_1 and θ_2 be two solutions of (P2; $\tilde{\alpha}_0, \eta$) in the class $W^{1,2}(0,T;H) \cap L^\infty(0,T;H_0^1)$. We then have

$$\tilde{\alpha}_0(t)(\theta_1'(t) - \theta_2'(t)) + \theta_1^*(t) - \theta_2^*(t) = 0 \quad \text{in } H \text{ for a.a. } t \in (0,T),$$

where $\theta_i^*(t) = -\tilde{\alpha}_0(t)\theta_i'(t) \in \partial\varphi(\eta(t);\theta_i(t))$ in H for a.a. $t \in (0,T)$, $i = 1,2$. Now, multiply the above relation by $\theta_1 - \theta_2$ to get

$$\int_\Omega \tilde{\alpha}_0(t)(\theta_1(t) - \theta_2(t))'(\theta_1(t) - \theta_2(t))\,\mathrm{d}x + \nu\|\nabla(\theta_1(t) - \theta_2(t))\|_H^2 \leqslant 0 \qquad \text{for a.a. } t \in (0,T),$$

whence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tilde{\alpha}_0(t)|\theta_1(t) - \theta_2(t)|^2\,\mathrm{d}x + \nu\|\nabla(\theta_1(t) - \theta_2(t))\|_H^2 \leqslant \frac{1}{2}\Big\|\frac{|\tilde{\alpha}_0'|}{\tilde{\alpha}_0}\Big\|_\infty \int_\Omega \tilde{\alpha}_0(t)|\theta_1(t) - \theta_2(t)|^2\,\mathrm{d}x \quad \text{for a.a. } t \in (0,T).$$

By applying Gronwall's lemma to the above inequality, we conclude that $\theta_1(t) = \theta_2(t)$ in H for all $t \in [0,T]$, that is, the solution of (P2; $\tilde{\alpha}_0, \eta$) is unique. Thus, the assertion (c) has been completely proved. □

P r o o f of (d) of Proposition 3.2. Prior to the proof of (d), we recall a general result on subdifferentials. We define proper, l.s.c. and convex functions Φ_n and Φ on $L^2(0,T;H)$ by

$$\Phi_n(w) := \int_0^T \varphi(\eta_n(t);w(t))\,\mathrm{d}t, \quad \Phi(w) := \int_0^T \varphi(\eta(t);w(t))\,\mathrm{d}t, \quad \forall\, w \in L^2(0,T;H).$$

It is clear that $D(\Phi_n) = D(\Phi) = L^2(0,T;H_0^1)$. We denote by $\partial\Phi_n$ and $\partial\Phi$ the subdifferentials of Φ_n and Φ in $L^2(0,T;H)$, respectively. It is well known that for $w, w^* \in L^2(0,T;H)$, $w^* \in \partial\Phi_n(w)$ if and only if $w^*(t) \in \partial\varphi(\eta_n(t);w(t))$ in H for a.a. $t \in (0,T)$. Furthermore we note from (A2) and (3.13) that $\Phi_n(w)$ converges to $\Phi(w)$ for every $w \in L^2(0,T;H_0^1)$. Therefore, by the general theory of subdifferentials (cf. [4], [15]), $\partial\Phi_n$ converges to $\partial\Phi$ in the graph sense, namely if $w_n^* \in \partial\Phi_n(w_n)$, $w_n^* \to w^*$ weakly in $L^2(0,T;H)$ and $w_n \to w$ in $L^2(0,T;H)$, then, $w^* \in \partial\Phi(w)$.

Now, we give a proof of (d). With the same notation as in the statement of (d), we note from the energy inequality (3.11) that $\{\theta_n\}$ is bounded in $W^{1,2}(0,T;H)$ and $L^\infty(0,T;H^1_0)$, so that it is possible to extract a subsequence $\{\theta_{n_k}\}$ from $\{\theta_n\}$ such that $\theta_{n_k} \to \theta$ weakly in $W^{1,2}(0,T;H)$ and weakly* in $L^\infty(0,T;H^1_0)$, whence $\theta_{n_k} \to \theta$ in $C([0,T];H)$ as $k \to \infty$; namely, (3.14) is satisfied.

Also, it follows from (3.12), (3.14) and the L^∞-boundedness of $\tilde\alpha_{0,n_k}$ for all k that

$$\theta_{n_k} \to \theta \text{ in } L^2(0,T;H) \quad \text{and} \quad \tilde\alpha_{0,n_k}\theta'_{n_k} \to \tilde\alpha_0\theta' \text{ weakly in } L^2(0,T;H)$$

as $k \to \infty$.

Since θ_{n_k} is the solution of $(\mathrm{P2};\tilde\alpha_{0,n_k},\eta)$, we see that

$$-\tilde\alpha_{0,n_k}\theta'_{n_k} \in \partial\Phi_{n_k}(\theta_{n_k}) \quad \text{in } L^2(0,T;H),\ \forall\, k.$$

Therefore, it follows from the above general theory that $-\tilde\alpha_0\theta' \in \partial\Phi(\theta)$ in $L^2(0,T;H)$. This shows that θ is a solution of $(\mathrm{P2};\tilde\alpha_0,\eta)$ on $[0,T]$.

Now, we proceed to the proof of $\theta_{n_k} \to \theta$ in $L^2(0,T;H^1_0)$ as $k \to \infty$. By the definition of subdifferential, we have for any k, j,

$$\begin{aligned}(3.29)\quad &\int_0^T (\tilde\alpha_{0,n_k}\theta'_{n_k}, \theta_{n_k} - \theta_{n_j})\,\mathrm{d}t \\ &\qquad + \nu\int_0^T (\nabla\theta_{n_k}, \nabla(\theta_{n_k} - \theta_{n_j}))\,\mathrm{d}t + \int_0^T\int_\Omega \alpha(\eta_{n_k})|\nabla\theta_{n_k}|\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \leqslant \int_0^T\int_\Omega \alpha(\eta_{n_k})|\nabla\theta_{n_j}|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Now, add (3.29) and the inequality obtained by exchanging n_k for n_j in (3.29) to get

$$\begin{aligned}\int_0^T (\tilde\alpha_{0,n_k}\theta'_{n_k} - \tilde\alpha_{0,n_j}\theta'_{n_j}, \theta_{n_k} - \theta_{n_j})\,\mathrm{d}t + \nu\int_0^T \|\nabla(\theta_{n_k} - \theta_{n_j})\|_H^2\,\mathrm{d}t \\ \leqslant L(\alpha)\int_0^T \|\eta_{n_k} - \eta_{n_j}\|_H \|\nabla(\theta_{n_k} - \theta_{n_j})\|_H\,\mathrm{d}t.\end{aligned}$$

From this it follows that

$$\begin{aligned}\frac{\nu}{2}\int_0^T \|\nabla(\theta_{n_k} - \theta_{n_j})\|_H^2\,\mathrm{d}t \leqslant \int_0^T \|\tilde\alpha_{0,n_k}\theta'_{n_k} - \tilde\alpha_{0,n_j}\theta'_{n_j}\|_H \|\theta_{n_k} - \theta_{n_j}\|_H\,\mathrm{d}t \\ + \frac{L(\alpha)^2}{2\nu}\int_0^T \|\eta_{n_k} - \eta_{n_j}\|_H^2\,\mathrm{d}t.\end{aligned}$$

Letting $k, j \to \infty$ in the above inequality, we infer from (3.13) that $\nabla(\theta_{n_k} - \theta_{n_j}) \to 0$ in $L^2(0,T;H)$ as $k, j \to \infty$. This implies that $\theta_{n_k} \to \theta$ in $L^2(0,T;H^1_0)$, so that (3.15) is obtained.

Moreover, if $\partial\tilde{\alpha}_0/\partial t \in L^\infty(Q_T)$, then θ is the unique solution of $(\mathrm{P2}; \tilde{\alpha}_0, \eta)$ on $[0,T]$, whence (3.14) and (3.15) hold without extracting any subsequence from $\{\theta_n\}$.

Thus, the proof of (d) is accomplished. □

4. Solvability of approximate problems

In this section, assuming that (A1)–(A4) are satisfied, for each $\varepsilon \in (0,1]$ we consider the approximate problem $(\mathrm{P})_\varepsilon$, formulated in Section 2.

Step 1: Local existence

The first step is to construct a local (in time) solution of $(\mathrm{P})_\varepsilon$. To do so, we employ the fixed point argument for continuous operators in compact convex sets. We consider a (non-empty) compact convex subset X of $C([0,T];H)$ defined by

$$X := \left\{ \eta \in C([0,T];H) \;\middle|\; \begin{array}{l} \eta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1), \\ 0 \leqslant \eta \leqslant 1 \text{ a.e. on } Q_T, \;\; \eta(0) = \eta_0 \text{ in } H, \\ \displaystyle\int_0^t \|\eta'(\tau)\|_H^2 \,\mathrm{d}\tau + \kappa\|\nabla\eta(t)\|_H^2 + 2\int_\Omega \hat{g}(\eta(t))\,\mathrm{d}x \\ \qquad \displaystyle\leqslant \kappa\|\nabla\eta_0\|_H^2 + 2\int_\Omega \hat{g}(\eta_0)\,\mathrm{d}x + 1, \;\; \forall\, t \in [0,T] \end{array} \right\};$$

for simplicity we put

$$M_0 := \left\{ \kappa\|\nabla\eta_0\|_H^2 + 2\int_\Omega \hat{g}(\eta_0)\,\mathrm{d}x + 1 \right\}^{1/2}.$$

Now, for each $\overline{\eta} \in X$, consider the problem $(\mathrm{P2}; \tilde{\alpha}_0, \overline{\eta})$ with $\tilde{\alpha}_0 = \alpha_0(\varrho_\varepsilon * \overline{\eta})$. Then, we infer from (A1) and (2.6) that $\tilde{\alpha}_0 \in L^\infty(Q_T)$ and $\partial\tilde{\alpha}_0/\partial t \in L^\infty(Q_T)$. Therefore, by (c) of Proposition 3.2, this problem has one and only one solution θ in $W^{1,2}(0,T;H) \cap L^\infty(0,T;H_0^1)$, and θ satisfies

$$\delta_0\|\theta'(t)\|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\overline{\eta}(t);\theta(t)) \leqslant R_2\|\overline{\eta}'(t)\|_H(\varphi(\overline{\eta}(t);\theta(t)) + 1) \quad \text{for a.a. } t \in (0,T),$$

which is rearranged in the form

$$\begin{aligned} &\delta_0 \exp\left\{-R_2\int_0^t \|\overline{\eta}'(\tau)\|_H \,\mathrm{d}\tau\right\}\|\theta'(t)\|_H^2 \\ &\qquad + \frac{\mathrm{d}}{\mathrm{d}t}\left\{\exp\left\{-R_2\int_0^t \|\overline{\eta}'(\tau)\|_H \,\mathrm{d}\tau\right\}\varphi(\overline{\eta}(t);\theta(t))\right\} \\ &\quad \leqslant \exp\left\{-R_2\int_0^t \|\overline{\eta}'(\tau)\|_H \,\mathrm{d}\tau\right\}R_2\|\overline{\eta}'(t)\|_H. \end{aligned}$$

Therefore, by integrating this inequality in time, we obtain that

$$\begin{aligned}
\delta_0 \int_0^t \|\theta'(\tau)\|_H^2 \,\mathrm{d}\tau + \varphi(\overline{\eta}(t);\theta(t)) \\
&\leqslant \exp\Big\{R_2 \int_0^t \|\overline{\eta}'(\tau)\|_H \,\mathrm{d}\tau\Big\}\varphi(\eta_0;\theta_0) \\
&\quad + \exp\Big\{R_2 \int_0^T \|\overline{\eta}'(\tau)\|_H \,\mathrm{d}\tau\Big\} R_2 \int_0^t \|\overline{\eta}'(\tau)\|_H \,\mathrm{d}\tau \\
&\leqslant \exp\{R_2 M_0 \sqrt{t}\}\varphi(\eta_0;\theta_0) + \exp\{R_2 M_0 \sqrt{T}\} R_2 M_0 \sqrt{t}.
\end{aligned}$$

From the last inequality it follows that there exists a small positive time T_0 with $0 < T_0 \leqslant T$, independent of $\overline{\eta} \in X$, such that

$$\delta_0 \int_0^t \|\theta'(\tau)\|_H^2 \,\mathrm{d}\tau + \varphi(\overline{\eta}(t);\theta(t)) \leqslant 2\varphi(\eta_0;\theta_0) + 1, \quad \forall\, t \in [0, T_0],$$

and hence

$$\delta_0 \int_0^t \|\theta'(\tau)\|_H^2 \,\mathrm{d}\tau + \frac{\nu}{2}\|\nabla\theta(t)\|_H^2 \leqslant 2\varphi(\eta_0;\theta_0) + 1, \quad \forall\, t \in [0, T_0]. \tag{4.1}$$

Next, for the function θ constructed above, consider the problem (P1; θ). By virtue of results mentioned in paragraph (1) of Section 3, the problem (P1; θ) has one and only one solution η in $W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$, and by (3.2) and (4.1) it holds that

$$\begin{aligned}
\|\eta'(t)\|_H^2 + \kappa \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\eta(t)\|_H^2 + 2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \hat{g}(\eta(t)) \,\mathrm{d}x \\
\leqslant R_1 \|\nabla\theta(t)\|_H^2 \leqslant R_1'(\varphi(\eta_0;\theta_0) + 1) \quad \text{for a.a. } t \in (0, T_0),
\end{aligned} \tag{4.2}$$

where $R_1' := 4\nu^{-1}R_1$. Choose a small positive time T_1 so that $0 < T_1 \leqslant T_0$ and $R_1'(\varphi(\eta_0;\theta_0) + 1)T_1 \leqslant 1$. Then we see from (4.2) that

$$\int_0^t \|\eta'(\tau)\|_H^2 \,\mathrm{d}\tau + \kappa\|\nabla\eta(t)\|_H^2 + 2\int_\Omega \hat{g}(\eta(t)) \,\mathrm{d}x \leqslant M_0^2, \quad \forall\, t \in [0, T_1]. \tag{4.3}$$

Now, we define an operator $S\colon X \to X$ as follows. For each $\overline{\eta} \in X$, we denote by θ the unique solution of

$$\begin{cases} \alpha_0((\varrho_\varepsilon * \overline{\eta})(t))\theta'(t) + \partial\varphi(\overline{\eta}(t);\theta(t)) \ni 0 \ \text{ in } H \ \text{ for a.a. } t \in (0,T), \\ \theta(0) = \theta_0 \ \text{ in } H. \end{cases} \tag{4.4}$$

As was remarked above, the inequality (4.1) is satisfied. Next, corresponding to this function θ, we denote by η the unique solution of

$$\begin{cases} \eta'(t) - \kappa\Delta_N\eta(t) + g(\eta(t)) + \alpha'(\eta(t))|\nabla\theta(t)| = 0 \ \text{ in } H \ \text{ for a.a. } t \in (0,T), \\ \eta(0) = \eta_0 \ \text{ in } H. \end{cases} \tag{4.5}$$

This solution satisfies (4.3). Here, given $\overline{\eta}$ in X, put by using indirectly the solution θ of (4.4)

$$[S\overline{\eta}](t) = \begin{cases} \eta(t) & \text{for } t \in [0,T_1], \\ \eta(T_1) & \text{for } t \in [T_1,T]. \end{cases} \tag{4.6}$$

Then it is easy to check that S maps X into itself. Moreover, on account of the convergence results mentioned in (b) of Proposition 3.1 and (d) of Proposition 3.2, S is continuous in X with respect to the topology of $C([0,T];H)$. In fact, let $\{\overline{\eta}_n\} \subset X$, $\overline{\eta} \in X$, and suppose $\overline{\eta}_n \to \overline{\eta}$ in $C([0,T];H)$ as $n \to \infty$. Then it follows from (A1) and (2.6) that $\{\tilde{\alpha}_{0,n} := \alpha_0(\varrho_\varepsilon * \overline{\eta}_n)\}$ is a bounded sequence in $L^\infty(Q_T)$, $\partial\tilde{\alpha}_0/\partial t = (\partial/\partial t)\alpha_0(\varrho_\varepsilon * \overline{\eta}) \in L^\infty(Q_T)$ and $\tilde{\alpha}_{0,n} \to \tilde{\alpha}_0$ in $L^2(Q_T)$ as $n \to \infty$, so that we can apply (d) of Proposition 3.2, and, hence, (b) of Proposition 3.1 to see the continuity of S.

Therefore, Schauder's fixed point theorem guarantees that S has at least one fixed point η in X. The pair of functions $[\eta,\theta]$, with the solution θ of (4.4) corresponding to $\overline{\eta} = \eta$, is a solution of $(\mathrm{P})_\varepsilon$ on the time interval $[0,T_1]$. Thus, we have shown that the approximate problem $(\mathrm{P})_\varepsilon$ has a local (in time) solution $[\eta,\theta]$.

Step 2: Global existence

The second step is to show the global existence of a solution of $(\mathrm{P})_\varepsilon$. Now, we put

$$E = \{T_1 \in [0,T];\ (\mathrm{P})_\varepsilon \ \text{ has a solution on } \ [0,T_1]\}.$$

Our aim is to show that E is non-empty, closed and open in $[0,T]$. As was seen in Step 1, $E \neq \emptyset$. Let T_1 be any number in E and $[\eta,\theta]$ be a solution of $(\mathrm{P})_\varepsilon$ on $[0,T_1]$. Then, by virtue of the local existence result in Step 1, this solution can be extended onto a bigger interval than $[0,T_1]$. Hence, E is open in $[0,T]$.

Next, assume that $\{T_n\}$ is any strictly increasing sequence in $[0,T]$ and put $T_0 := \lim_{n\to\infty} T_n$. Also, let $[\eta_n,\theta_n]$ be a solution of $(\mathrm{P})_\varepsilon$ on $[0,T_n]$ for each n.

On account of the energy inequalities (3.2) and (3.11), each pair of functions $[\eta_n,\theta_n]$ satisfies

$$\|\eta_n'(t)\|_H^2 + \kappa\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\eta_n(t)\|_H^2 + 2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \hat{g}(\eta_n(t))\,\mathrm{d}x \leqslant R_1\|\nabla\theta_n(t)\|_H^2 \quad \text{for a.a. } t \in (0,T_n) \tag{4.7}$$

and

$$\delta_0\|\theta_n'(t)\|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\eta_n(t);\theta_n(t)) \leqslant L(\alpha)\|\eta_n'(t)\|_H\|\nabla\theta_n(t)\|_H \quad \text{for a.a. } t\in(0,T_n). \tag{4.8}$$

Therefore, adding (4.7) and (4.8) and using Young's inequality, we get

$$\begin{aligned}
&\frac{1}{2}\|\eta_n'(t)\|_H^2 + \delta_0\|\theta_n'(t)\|_H^2 \\
&\qquad + \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\kappa\|\nabla\eta_n(t)\|_H^2 + 2\int_\Omega \hat{g}(\eta_n(t))\,\mathrm{d}x + \varphi(\eta_n(t);\theta_n(t))\Big\} \\
&\quad \leqslant \Big\{R_1 + \frac{L(\alpha)^2}{2}\Big\}\|\nabla\theta_n(t)\|_H^2 \\
&\quad \leqslant \Big\{R_1 + \frac{L(\alpha)^2}{2}\Big\}\cdot\frac{2}{\nu}\varphi(\eta_n(t);\theta_n(t)) \\
&\quad \leqslant \frac{2}{\nu}\Big\{R_1 + \frac{L(\alpha)^2}{2}\Big\}\Big\{\kappa\|\nabla\eta_n(t)\|_H^2 + 2\int_\Omega \hat{g}(\eta_n(t))\,\mathrm{d}x + \varphi(\eta_n(t);\theta_n(t))\Big\}
\end{aligned}$$

for a.a. $t\in(0,T_n)$. By Gronwall's lemma, the last inequality implies that there is a positive constant R_4, which is independent of n, such that

$$\begin{aligned}\|\eta_n'\|_{L^2(0,T_n;H)} + \|\theta_n'\|_{L^2(0,T_n;H)} + \|\nabla\eta_n\|_{L^\infty(0,T_n;H)}& \\ + \|\theta_n\|_{L^\infty(0,T_n;H_0^1)} \leqslant R_4.&\end{aligned} \tag{4.9}$$

Furthermore, we note that $0\leqslant\eta_n\leqslant 1$ a.e. on Q_T for all n. Therefore, using the uniform estimate (4.9), we can extract a subsequence $\{[\eta_{n_k},\theta_{n_k}]\}$ and find a pair of functions $[\eta,\theta]$ on $\Omega\times[0,T_0)$ such that for every time T' with $0<T'<T_0$

$$\begin{aligned}
\eta_{n_k}\to\eta \ \text{ weakly in } W^{1,2}(0,T';H) \ \text{ and weakly}^* \ \text{ in } L^\infty(0,T';H^1),\\
\theta_{n_k}\to\theta \ \text{ weakly in } W^{1,2}(0,T';H) \ \text{ and weakly}^* \ \text{ in } L^\infty(0,T';H_0^1)
\end{aligned}$$

as $k\to\infty$. Clearly, $\eta\in W^{1,2}(0,T_0;H)\cap L^\infty(0,T_0;H^1)$, $\theta\in W^{1,2}(0,T_0;H)\cap L^\infty(0,T_0;H_0^1)$ and

$$\tilde{\alpha}_{0,n_k} := \alpha_0(\varrho_\varepsilon * \eta_{n_k}) \to \alpha_0(\varrho_\varepsilon * \eta) =: \tilde{\alpha}_0 \quad \text{in } L^2(Q_{T'})$$

for every $T'\in(0,T_0)$ as $k\to\infty$. Now, it is easy to verify by making use of the convergence results (b) of Proposition 3.1 and (d) of Proposition 3.2 that $[\eta,\theta]$ is a solution of $(\mathrm{P})_\varepsilon$ on $[0,T_0)$. Then, by virtue of the local existence result in Step 1, this solution can be extended onto the interval $[0,T_0]$, that is, $T_0\in E$. Thus, E is non-empty, open and closed in $[0,T]$. Accordingly $E=[0,T]$ must hold, which shows that $(\mathrm{P})_\varepsilon$ has at least one solution on the whole interval $[0,T]$.

5. Proof of theorems

In this section we give the proofs of Theorem 2.1 and 2.2.

Proof of Theorem 2.1. Let $[\eta_\varepsilon, \theta_\varepsilon]$ be a solution of $(\mathrm{P})_\varepsilon$ on $[0,T]$ for each $\varepsilon \in (0,1]$ as constructed in the previous section. Then we have (cf. (4.9))

$$\|\eta'_\varepsilon\|_{L^2(0,T;H)} + \|\theta'_\varepsilon\|_{L^2(0,T;H)} + \|\nabla\eta_\varepsilon\|_{L^\infty(0,T;H)} + \|\theta_\varepsilon\|_{L^\infty(0,T;H^1_0)} \leqslant R_4, \tag{5.1}$$

where R_4 is the same constant as in (4.9) and hence it is independent of $\varepsilon \in (0,1]$. Furthermore, we note that $0 \leqslant \eta_\varepsilon \leqslant 1$ a.e. on Q_T for all $\varepsilon \in (0,1]$. Therefore, there are a sequence $\{\varepsilon_n\}$ in $(0,1]$ with $\varepsilon_n \downarrow 0$ and functions η, θ such that

$$\begin{aligned} \eta_n &:= \eta_{\varepsilon_n} \to \eta \ \text{ weakly in } W^{1,2}(0,T;H) \text{ and weakly}^* \text{ in } L^\infty(0,T;H^1),\\ \theta_n &:= \theta_{\varepsilon_n} \to \theta \ \text{ weakly in } W^{1,2}(0,T;H) \text{ and weakly}^* \text{ in } L^\infty(0,T;H^1_0); \end{aligned}$$

hence, $\eta_n \to \eta$ in $C([0,T];H)$ and weakly in $L^2(0,T;H^2)$, $\theta_n \to \theta$ in $C([0,T];H)$,

$$\tilde\alpha_{0,n} := \alpha_0(\varrho_{\varepsilon_n} * \eta_n) \to \alpha_0(\eta) =: \tilde\alpha_0 \quad \text{in } L^2(Q_T) \tag{5.2}$$

and

$$\tilde\alpha_{0,n}\theta'_n \to \tilde\alpha_0\theta' \quad \text{weakly in } L^2(0,T;H)$$

as $n \to \infty$. Here, we note that

$$\eta'_n(t) - \kappa\Delta_N\eta_n(t) + g(\eta_n(t)) + \alpha'(\eta_n(t))|\nabla\theta_n(t)| = 0 \quad \text{in } H \text{ for a.a. } t \in (0,T) \tag{5.3}$$

and

$$\tilde\alpha_{0,n}(t)\theta'_n(t) + \partial\varphi(\eta_n(t);\theta_n(t)) \ni 0 \quad \text{in } H \text{ for a.a. } t \in (0,T). \tag{5.4}$$

By the same argument as in the proof of Proposition 3.2 (d), it follows from (5.2) that $\theta_n \to \theta$ in $L^2(0,T;H^1_0)$. Hence, letting $n \to \infty$ in (5.3) and (5.4), we see that

$$\eta'(t) - \kappa\Delta_N\eta(t) + g(\eta(t)) + \alpha'(\eta(t))|\nabla\theta(t)| = 0 \quad \text{in } H \text{ for a.a. } t \in (0,T)$$

and

$$\alpha_0(\eta(t))\theta'(t) + \partial\varphi(\eta(t);\theta(t)) \ni 0 \quad \text{in } H \text{ for a.a. } t \in (0,T).$$

This shows that $[\eta,\theta]$ is a solution of (P) on $[0,T]$. □

P r o o f of Theorem 2.2. Let $[\eta_i, \theta_i]$ $(i = 1, 2)$ be two solutions of (P) on $[0, T]$. Then, we multiply the difference

$$\eta_1' - \eta_2' - \kappa\Delta_N(\eta_1 - \eta_2) + g(\eta_1) - g(\eta_2) + \alpha'(\eta_1)|(\theta_1)_x| - \alpha'(\eta_2)|(\theta_2)_x| = 0$$

by $\eta_1 - \eta_2$ and integrate the resultant in space to obtain

$$\begin{aligned}(5.5)\quad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta_1(t) - \eta_2(t)\|_H^2 + \kappa\|(\eta_1)_x(t) - (\eta_2)_x(t)\|_H^2 \\ &\leqslant L(g)\|\eta_1(t) - \eta_2(t)\|_H^2 \\ &\quad + (-\alpha'(\eta_1(t))|(\theta_1)_x(t)| + \alpha'(\eta_2(t))|(\theta_2)_x(t)|, \eta_1(t) - \eta_2(t)) \\ &= L(g)\|\eta_1(t) - \eta_2(t)\|_H^2 \\ &\quad + (-\alpha'(\eta_1(t))|(\theta_1)_x(t)| + \alpha'(\eta_1(t))|(\theta_2)_x(t)|, \eta_1(t) - \eta_2(t)) \\ &\quad + (-\alpha'(\eta_1(t))|(\theta_2)_x(t)| + \alpha'(\eta_2(t))|(\theta_2)_x(t)|, \eta_1(t) - \eta_2(t)) \\ &\leqslant L(g)\|\eta_1(t) - \eta_2(t)\|_H^2 + L(\alpha)\|(\theta_1)_x(t) - (\theta_2)_x(t)\|_H\|\eta_1(t) - \eta_2(t)\|_H \\ &\leqslant R_5(\|\eta_1(t) - \eta_2(t)\|_H^2 + \|(\theta_1)_x(t) - (\theta_2)_x(t)\|_H\|\eta_1(t) - \eta_2(t)\|_H)\end{aligned}$$

for a.a. $t \in (0, T)$, where $R_5 := L(g) + L(\alpha)$.

Next, we multiply the difference

$$\alpha_0(\eta_1(t))\theta_1'(t) - \alpha_0(\eta_2(t))\theta_2'(t) + \theta_1^*(t) - \theta_2^*(t) = 0 \quad \text{in } H$$

by $\theta_1(t) - \theta_2(t)$, where $\theta_i^*(t) \in \partial\varphi(\eta_i(t); \theta_i(t))$ in H for a.a. $t \in (0, T)$, $i = 1, 2$. Then, by the same arguments as above, we get:

$$\begin{aligned}(5.6)\quad &(\alpha_0(\eta_1(t))\theta_1'(t) - \alpha_0(\eta_2(t))\theta_2'(t), \theta_1(t) - \theta_2(t)) + \nu\|(\theta_1)_x(t) - (\theta_2)_x(t)\|_H^2 \\ &\leqslant \int_{-L}^{L} (\alpha(\eta_1(x,t)) - \alpha(\eta_2(x,t)))(|(\theta_2)_x(x,t)| - |(\theta_1)_x(x,t)|)\,\mathrm{d}x \\ &\leqslant L(\alpha)\|\eta_1(t) - \eta_2(t)\|_H\|(\theta_1)_x(t) - (\theta_2)_x(t)\|_H\end{aligned}$$

for a.a. $t \in (0, T)$. Furthermore, note that

$$\begin{aligned}(5.7)\quad &(\alpha_0(\eta_1(t))\theta_1'(t) - \alpha_0(\eta_2(t))\theta_2'(t), \theta_1(t) - \theta_2(t)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\int_{-L}^{L} \alpha_0(\eta_1(x,t))\frac{1}{2}|\theta_1(x,t) - \theta_2(x,t)|^2\,\mathrm{d}x\Big\} \\ &\quad - \int_{-L}^{L} \frac{\mathrm{d}}{\mathrm{d}t}[\alpha_0(\eta_1(x,t))]\frac{1}{2}|\theta_1(x,t) - \theta_2(x,t)|^2\,\mathrm{d}x \\ &\quad + (\alpha_0(\eta_1(t))\theta_2'(t) - \alpha_0(\eta_2(t))\theta_2'(t), \theta_1(t) - \theta_2(t))\end{aligned}$$

$$\geqslant \frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{-L}^{L} \alpha_0(\eta_1(x,t))\frac{1}{2}|\theta_1(x,t)-\theta_2(x,t)|^2\,\mathrm{d}x\right\}$$
$$- \frac{L(\alpha_0)}{2}\|\theta_1(t)-\theta_2(t)\|_\infty \|\eta_1'(t)\|_H \|\theta_1(t)-\theta_2(t)\|_H$$
$$- L(\alpha_0)\|\eta_1(t)-\eta_2(t)\|_\infty \|\theta_2'(t)\|_H \|\theta_1(t)-\theta_2(t)\|_H$$

for a.a. $t \in (0,T)$. Therefore, it follows from (5.6) and (5.7) that

$$\begin{aligned}
(5.8)\qquad & \frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{-L}^{L} \alpha_0(\eta_1(x,t))\frac{1}{2}|\theta_1(x,t)-\theta_2(x,t)|^2\,\mathrm{d}x\right\} \\
& \quad + \nu\|(\theta_1)_x(t)-(\theta_2)_x(t)\|_H^2 \\
& \leqslant R_6(\|\eta_1(t)-\eta_2(t)\|_H \|(\theta_1)_x(t)-(\theta_2)_x(t)\|_H \\
& \quad + \|\theta_1(t)-\theta_2(t)\|_{H_0^1}\|\eta_1'(t)\|_H\|\theta_1(t)-\theta_2(t)\|_H \\
& \quad + \|\eta_1(t)-\eta_2(t)\|_{H^1}\|\theta_2'(t)\|_H\|\theta_1(t)-\theta_2(t)\|_H)
\end{aligned}$$

for a.a. $t \in (0,T)$, where $R_6 > 0$ is some constant depending only on $L(\alpha)$, $L(\alpha_0)$ and the constants of the embedding $H_0^1 \hookrightarrow L^\infty$ and $H^1 \hookrightarrow L^\infty$.

By adding (5.5) and (5.8), and applying the Schwarz inequality, we have:

$$\begin{aligned}
(5.9)\quad & \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\|\eta_1(t)-\eta_2(t)\|_H^2 + \int_{-L}^{L} \alpha_0(\eta_1(x,t))\frac{1}{2}|\theta_1(x,t)-\theta_2(x,t)|^2\,\mathrm{d}x\right\} \\
& \quad + \frac{1}{2}\kappa\|(\eta_1)_x(t)-(\eta_2)_x(t)\|_H^2 + \frac{\nu}{2}\|(\theta_1)_x(t)-(\theta_2)_x(t)\|_H^2 \\
& \leqslant R_7(1+\|\eta_1'(t)\|_H^2 + \|\theta_2'(t)\|_H^2) \\
& \quad \times \left\{\frac{1}{2}\|\eta_1(t)-\eta_2(t)\|_H^2 + \int_{-L}^{L} \alpha_0(\eta_1(x,t))\frac{1}{2}|\theta_1(x,t)-\theta_2(x,t)|^2\,\mathrm{d}x\right\}
\end{aligned}$$

for a.a. $t \in (0,T)$, where $R_7 > 0$ is some constant depending on R_5, R_6, κ, ν and δ_0.

Now, since $\|\eta_1'\|_H \in L^2(0,T)$ and $\|\theta_2'\|_H \in L^2(0,T)$, we infer from (5.9) by Growall's lemma that

$$\eta_1(t)-\eta_2(t) = 0, \quad \theta_1(t)-\theta_2(t) = 0 \quad \text{in } H \text{ for all } t \in [0,T],$$

which implies the uniqueness of the solution to (P) on $[0,T]$. Thus, the proof of Theorem 2.2 has been completed. □

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Authors' addresses: *A. Ito*, Department of Electronic Engineering and Computer Science, School of Engineering, Kinki University, 1, Takayaumenobe, Higashihiroshima, Hiroshima, 739-2116, Japan, e-mail: `aito@hiro.kindai.ac.jp`; *N. Kenmochi*, Department of Mathematics, Faculty of Education, Chiba University, 1-33 Yayoi-chō, Inage-ku, Chiba, 263-8522, Japan, e-mail: `kenmochi@faculty.chiba-u.jp`; *N. Yamazaki*, Department of Mathematical Science, Common Subject Division, Muroran Institute of Technology, 27-1 Mizumoto-chō, Muroran, 050-8585, Japan, e-mail: `noriaki@mmm.muroran-it.ac.jp`.