On the structure of finite loop capable nilpotent groups

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Abstract. In this paper we consider finite loops and discuss the problem which nilpotent groups are isomorphic to the inner mapping group of a loop. We recall some earlier results and by using connected transversals we transform the problem into a group theoretical one. We will get some new answers as we show that a nilpotent group having either $C_{p^k} \times C_{p^l}$, $k > l \geq 0$ as the Sylow p-subgroup for some odd prime p or the group of quaternions as the Sylow 2-subgroup may not be loop capable.

Keywords: loop, group, connected transversals

Classification: 20D10, 20N05

1. Introduction

A loop is a quasigroup with a neutral element. For each element a of a loop Q we have two permutations $L_a : Q \to Q, x \mapsto ax$ (left translation) and $R_a : Q \to Q, x \mapsto xa$ (right translation). We define a permutation group $M(Q) = \langle L_a, R_a \mid a \in Q\rangle$ to be the multiplication group of Q. The stabilizer of the neutral element is denoted by $I(Q)$ and is called the inner mapping group of Q. These groups were introduced by Bruck [2] in 1946 and they have been an important tool in the study of loops.

When Q is a group, $I(Q)$ is the group of inner automorphisms of Q. A group that is isomorphic to an inner automorphism group of some group is called capable. A characterization for capable finite abelian groups was given by Baer in 1934 [1]. For this paper, a loop capable group is a group which is isomorphic to an inner mapping loop of some finite loop. The main question in this paper is to study which nilpotent groups may be loop capable. Previous research has shown that $I(Q)$ is cyclic if and only if Q is an abelian group (this is true also in infinite case, see [8]) and in these cases $I(Q) = 1$. Using the theory of permutation groups, Drápal proved that a generalized quaternion group Q_{2^i} may never appear as an inner mapping group of a loop [5]. Niemenmaa has shown that the group $C_{k,l} = C_{p^k} \times C_{p^l}$, where p is an odd prime and $k > l \geq 0$ is not loop capable [10]. Recently, Niemenmaa has also proved that neither an abelian group with a Sylow p-subgroup $C_{k,1}$ ($k > 1$ and p is an odd prime) nor a nilpotent group with a cyclic Hall-subgroup may be loop capable [11], [12]. The purpose of this paper is to generalize these results.

The notion of connected transversals is very important to the study of multiplication groups of loops, since they can be used to characterize the structure of multiplication groups. Connected transversals were introduced by Kepka and Niemenmaa [7] in 1990. Section 2 contains basic information about them. Section 3 has the main results and the final section translates them into loop theoretical interpretations.

2. Connected transversals and loops

Let H be a subgroup of a group G and let A and B left transversals to H in G. We say that A and B are H-connected if $a^{-1}b^{-1}ab \in H$ for all $a \in A$ and $b \in B$. Connected transversals are both left and right transversals ([7, Lemmas 2.1 and 2.2]). The core of H in G (i.e. the largest normal subgroup of G contained in H) is denoted by H_G.

Now assume that Q is a loop. If we write $A = \{L_a \mid a \in Q\}$ and $B = \{R_a \mid a \in Q\}$, then A and B are both left and right transversals to $I(Q)$ in $M(Q)$. More importantly, they are $I(Q)$-connected. Also by definition $M(Q) = \langle A, B\rangle$ and the core of $I(Q)$ in $M(Q)$ is trivial. These facts characterize the necessary and sufficient structure in multiplication groups of loops [7, Theorem 4.1].

Theorem 2.1. *A group G is isomorphic to a multiplication group of a loop if and only if G has a subgroup H, $H_G = 1$, and there exist H-connected transversals A, B such that $G = \langle A, B\rangle$.*

Let G be finite group, $H \leq G$, and let there exist H-connected transversals A and B.

Lemma 2.2. *If $C \subseteq A \cup B$ and $M = \langle H, C\rangle$, then $C \subseteq M_G$.*

Lemma 2.3. *If $H_G = 1$, then $N_G(H) = H \times Z(G)$.*

Lemma 2.4. *Suppose that N is a normal subgroup of G. If the core of HN in G is K, then AK/K and BK/K are HK/K-connected transversals in G/K.*

Lemma 2.5. *If $H_G = 1$, then $Z(G) \subseteq A \cap B$.*

Lemma 2.6. *If H is nilpotent, then G is solvable.*

For the proofs, see [7, Lemma 2.5, Proposition 2.7 and Lemma 2.8], [8, Lemma 1.4] and [9, Theorem 1]. Furthermore, suppose that $G = \langle A, B\rangle$.

Lemma 2.7. *Let $p_1, p_2, \dots, p_r$ be different primes. If $H = H_1 \times H_2 \times \ldots \times H_r$ and H_i is a subgroup of $C_{p_i} \times C_{p_i} \times C_{p_i}$ for any $1 \leq i \leq r$, then $G' \leq N_G(H)$.*

Lemma 2.8. *If H is nilpotent, then H is subnormal in G.*

Lemma 2.9. *If $H \cong C_n \times E$, where E is a nilpotent group and* g.c.d.$(n, |E|) = 1$, *then $H_G > 1$.*

For the proofs, see [3, Theorem 3.9], [12, Theorem 2.8 and Theorem 3.1].

Lemma 2.10. *Suppose that H is nilpotent. If $H_G = 1$, then the core of $HZ(G)$ properly contains $Z(G)$.*

Proof: By Lemmas 2.3 and 2.8, $N_G(H) = H \times Z(G)$ and $Z(G) > 1$. Suppose that the core of $HZ(G)$ in G is $Z(G)$. Thus the core of $HZ(G)/Z(G)$ in $G/Z(G)$ is trivial. By Lemma 2.3, the normalizer of $HZ(G)/Z(G)$ in $G/Z(G)$ is $HZ(G)/Z(G) \times M/Z(G)$, where $M/Z(G)$ is the center of $G/Z(G)$. Take $m \in M$. By Lemmas 2.5 and 2.4, $m = az_1$, where $a \in A$ and $z_1 \in Z(G)$. Thus $a = mz_1^{-1} \in M$ and if $d \in B$, then $a^{-1}d^{-1}ad \in H \cap M = 1$. Hence $a \in C_G(B)$. Similarly we know that $a = mz_1^{-1} = bz_2$, where $b \in B$ and $z_2 \in Z(G)$, and therefore $b \in C_G(A)$ and $a \in Z(G)$. But now $M = Z(G)$, which is a contradiction with Lemma 2.8. $\square$

Finally, we need some more general results about loops and groups. The normal closure D^G of D in G is the subgroup $\langle D^g \mid g \in G \rangle$, i.e. the smallest normal subgroup of G containing D.

Lemma 2.11. *If x and y commute with $[x,y]$, then $(xy)^n = x^n y^n \, [x,y]^{\binom{n}{2}}$.*

Lemma 2.12. *Let G be a finite group with a subnormal subgroup D. If D is nilpotent, then D^G is nilpotent.*

Lemma 2.13. *The inner mapping group of a loop is never a generalized group of quaternions.*

For the proofs, see [6, pp. 253–254], [4, Theorem 8.8, p. 29] and [5, Corollary 1.3]. The centre $Z(Q)$ of a loop Q is the set of all elements a that satisfy $ax = xa$, $a(xy) = (ax)y$, $x(ay) = (xa)y$ and $x(ya) = (xy)a$ for all $x, y \in Q$. As with groups, we can define nilpotency for loops using centers. We write $Z_0 = 1$ and $Z_{i+1}/Z_i = Z(Q/Z_i)$ and obtain a series of normal subloops. If $Z_n = Q$ and $Z_{n-1} \neq Q$, then we say that the loop Q is centrally nilpotent of class n. Denote $I_0 = I(Q)$ and $I_{n+1} = N_{M(Q)}(I_n)$. Bruck proved in [2, p. 281] the following characterization for the nilpotency class of a loop.

Lemma 2.14. *A loop Q is centrally nilpotent of class n if and only if $I_n = M(Q)$ and $I_{n-1} \neq M(Q)$.*

3. Results

In this section we assume that H is a subgroup of a finite group G and there exist H-connected transversals A and B such that $G = \langle A, B \rangle$.

Theorem 3.1. *Let $H = C \times E$, where $C \cong C_{p^k} \times C_{p^l}$, p is an odd prime, $k > l \geq 0$, E is nilpotent and p does not divide $|E|$. Then $H_G > 1$.*

Proof: The proof will be done by induction on k. The case $k = 1$ is covered by Lemma 2.9. So we may suppose the claim is true for any positive integer smaller than $k > 1$. If the claim is not true for k, then let G be a smallest counterexample with $H_G = 1$. By Lemmas 2.3 and 2.8, $Z(G) > 1$. Now pick $z \in Z(G)$ with a prime order and consider groups $G/\langle z\rangle$ and $H\langle z\rangle/\langle z\rangle$. The core K of $H\langle z\rangle$ in G must properly contain $\langle z\rangle$. Hence, $K = H_1\langle z\rangle$, where $1 < H_1 \leq H$. As K is nilpotent, it follows that K is a group with prime power order. Now consider groups G/K and HK/K. It is clear that $|z| = p$ and K is abelian. The Frattini subgroup of K must be contained in H and therefore K is elementary abelian. If $|K| = p^3$, then $HK/K \cong C_{p^{k-1}} \times C_{p^{l-1}} \times E$, which leads to contradiction with the induction assumption. Therefore $K = H_1 \times \langle z\rangle$, $|H_1| = p$, $HK/K \cong C_{p^l} \times C_{p^l} \times E$ and $k - 1 = l$.

Put $M/K = Z(G/K)$. By Lemma 2.10, the core of $HK/K \times M/K$ in G/K is $NK/K \times M/K$, where $H_1 < N \leq H$. Now $F = NM$ is normal in G and consider the groups G/F and HF/F. Since the core of $HF = HM$ in G is F, we know that $N = C^* \times E_1$, where $C^* = C_{p^{t+1}} \times C_{p^t}$, $E_1 \leq E$ and $0 \leq t \leq l$. Denote $C_G(H_1) \cap M$ by T and $C_G(H_1) \cap NM = NT$ by W. As $C_G(H_1) = C_G(K)$ is normal in G, we have that T and W are normal in G. Now $W/K = NK/K \times T/K$ and therefore the Sylow p-subgroup P/K of W/K is normal and $P' \leq K \leq Z(P)$. Clearly, P is a normal subgroup of G. By Lemma 2.11 and the fact that p is odd, we have that $(zy)^p = z^p y^p \, [z, y]^{\binom{p}{2}} = 1$ for all $z, y \in \{g \in P \mid g^p = 1\} = S$. Thus S is a normal subgroup of G.

First, suppose that $t > 0$ and consider groups G/S and HS/S. The assumption $t > 0$ means that $HS/S \cong C_{p^{k-1}} \times C_{p^{k-2}} \times E$. By induction assumption, we may suppose that the core R of HS in G properly contains S and R contains an element of order p^2 from the group H. Let P_1 be a Sylow p-subgroup of R containing K. Since $HS/K \leq HT/K = HK/K \times T/K$, P_1/K is abelian and normal in R/K and therefore P_1 is normal in G. Thus $P_1' \leq K \leq Z(P_1)$. Denote $P_1 = H_2 S$, where $1 < H_2 \leq H$. Now by using Lemma 2.11 again, we have $(hs)^p = h^p s^p \, [h, s]^{\binom{p}{2}} = h^p$ for all $h \in H_2$ and $s \in S$. Thus $1 < P_1^p \leq H_G$, a contradiction.

Hence we may suppose that $t = 0$, $N = H_1 E_1$ and $W = E_1 T$. As $M \cap H = H_1$, we know that $T = A_1 H_1 = B_1 H_1$, where $A_1 \subseteq A$ and $B_1 \subseteq B$. Since H_1 is normal in T and $H_G = 1$, we know that $T' \leq H_1$ and T is abelian. Let $Q > 1$ be a Sylow q-subgroup of T for a prime $q \neq p$. Now Q is normal in G. Consider groups HQ/Q and G/Q. By the choice of G and Lemma 2.4, we may suppose that the core L of HQ in G properly contains Q. Now $L \cap K = 1$, as otherwise $H_1 = L \cap K$ would be normal in G. We denote $L = H_2 Q$, where $1 < H_2 < H$. Let P_2 be a Sylow p-subgroup of L with $P_2 \leq H_2$. By considering group LK/K we know that $P_2 K$ is normal in LK and G. Thus $P_2 = P_2 K \cap L \leq H_G = 1$. Now $HL = HQ$ and if we consider groups HL/L and G/L, then we get a contradiction with the choice of G.

Therefore we may suppose that $Q = 1$ and T is an abelian p-group. So E_1 is a Hall-subgroup of $W = E_1 T$. As E_1 is the unique Hall-subgroup of $E_1 K$, we may deduce that E_1 normal in W. Therefore E_1 is also normal in G and $E_1 \leq H_G = 1$. But this is a contradiction with $H_1 < N$. So the claim is true for all k. $\square$

For every non-negative integer m, let $\mathcal{C}(m)$ be the class of abelian groups H satisfying the following conditions: H is of odd order and if $p_1 < p_2 < \ldots < p_k$ are the primes dividing the order $|H|$, then for every $1 \leq i \leq k$ the Sylow p_i-subgroup is of the form $C_{p_i^{m_i}} \times C_{p_i^{m_i}}$ and $m = m_1 + m_2 + \ldots + m_k$.

Denote also that $N_G^0(H) = H$ and $N_G^{i+1}(H) = N_G(N_G^i(H))$.

Corollary 3.2. *If $H \in \mathcal{C}(m)$, then $G' \leq N_G^m(H)$.*

Proof: The proof is done by induction on m. The case $m = 1$ is done by Lemma 2.7. Suppose that $m > 1$ and the claim is true for integers less than m. Assume that G is a smallest counterexample. First, suppose that $H_G > 1$. Then consider groups G/H_G and H/H_G. By Theorem 3.1, $H/H_G \in \mathcal{C}(n)$, where $n < m$. By induction assumption and the choice of G we know that $G' \leq N_G^n(H) \leq N_G^m(H)$. So we may assume that $H_G = 1$. By Lemmas 2.3 and 2.10, $N_G(H) = H \times Z(G)$ and the core K of $HZ(G)$ properly contains $Z(G)$. Consider groups G/K and HK/K. By Theorem 3.1, $HK/K \in \mathcal{C}(n)$, where $n < m$. Thus $G'K/K \leq N_{G/K}^n(HK/K)$, meaning that $G' \leq N_G^n(HZ(G)) = N_G^{n+1}(H) \leq N_G^m(H)$, a contradiction. $\square$

We shall also generalize Lemma 2.13 and for this purpose we need the following two results.

Lemma 3.3. *Let p be a prime and $H = P \times D$, where $P \cong C_p \times C_p$ and $D > 1$ is a nilpotent group not divisible by p. If* g.c.d.$(|Z(G)|, |D|) = 1$*, then $H_G > 1$.*

Proof: Suppose that $H_G = 1$. By Lemmas 2.3, 2.8 and 2.10, it follows that $N_G(H) = H \times Z(G)$ and the core K of $HZ(G)$ in G properly contains $Z(G) > 1$. Since K is nilpotent it is clear that g.c.d.$(|D|, |K|) = 1$. By Lemma 2.9, $K = PZ(G)$. Now G is solvable by Lemma 2.6 and therefore we may pick $Q \geq D$ be a Hall-subgroup of G for primes dividing $|D|$. Let $R = KN_G(Q)$ and if $R < G$, then take $M \geq R$ to be a maximal subgroup of G. By Lemma 2.2, $M_G > 1$ and as M/M_G is nilpotent, we know that M is normal in G. Using Frattini-argument, we get $G = MN_G(Q) = M$, which is a contradiction. Hence, $G = KN_G(Q)$ and therefore $D^G = \langle D^g \mid g \in G \rangle \leq Q$ and D^G is normal in G. Now it is clear that the elements from K and D^G commute with each other. Since $H_G = 1$, we know that $D < D^G$. As H is subnormal in G, D is also subnormal in G. By Lemma 2.12, D^G is nilpotent and therefore $N_{D^G}(D) > D$. Now $HZ(G) < KN_{D^G}(D) \leq N_G(H)$, a contradiction. $\square$

Theorem 3.4. *If $H = Q_8 \times D$, where D is a nilpotent group of odd order, then $H_G > 1$.*

Proof: The case $D = 1$ is done by Theorem 2.1 and Lemma 2.13, so we may suppose that $D > 1$. Let G be a smallest counterexample with $H_G = 1$. By Lemmas 2.3 and 2.8, $Z(G) > 1$. Pick $z \in Z(G)$ with a prime order and consider groups $G/\langle z \rangle$ and $H\langle z \rangle/\langle z \rangle$. The choice of G implies that the core K of the group $H\langle z \rangle$ in G properly contains $\langle z \rangle$. Now we consider groups G/K and HK/K. It is clear that K is a 2-group. If $K = Q_8 \times \langle z \rangle$, then $Z(Q_8)$ is characteristic in K, which is a contradiction with $H_G = 1$. By Lemma 2.9 we may deduce that $K = Z(Q_8) \times \langle z \rangle$. Thus $HK/K \cong P \times D$, where $P \cong C_2 \times C_2$.

Denote the center of G/K by M/K. Let $R > 1$ be a Sylow q-subgroup of M for a prime dividing $|D|$. It is clear that RK is normal in M. Now $R\langle z \rangle \leq N_{RK}(R)$. The case $N_{RK}(R) = R\langle z \rangle$ is not possible, thus R is normal in RK and G. Now consider groups G/R and HR/R and we get that the core C of HR in G properly contains R. If $1 < C \cap K$, then $1 < Z(Q_8) \leq H_G$, a contradiction. Thus C is of odd order and if we consider groups G/C and HC/C, then we get a contradiction with the choice of G. Therefore $R = 1$ and g.c.d.$(|M/K|, |DK/K|) = 1$. Now Lemma 3.3 gives the final contradiction. $\square$

4. Loop theoretical consequences

So far the results have been purely group theoretical. By using Theorem 2.1 we can translate them into statements about loops. First we get the following loop theoretical interpretations for Theorems 3.1 and 3.4.

Corollary 4.1. *Let p be an odd prime and $H = C \times E$, where $C \cong C_{p^k} \times C_{p^l}$, where $k > l \geq 0$, E is a nilpotent group and p does not divide $|E|$. Then H is not loop capable.*

Corollary 4.2. *Let $H = Q_8 \times D$, where D is a nilpotent group of odd order. Then H is not loop capable.*

We can also combine Lemma 2.14 and Corollary 3.2 in order to get

Corollary 4.3. *Let Q be a finite loop. If $I(Q) \in \mathcal{C}(m)$, then the nilpotency class of Q is at most $m + 1$.*

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(*Received* October 12, 2009, *revised* February 8, 2010)