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On relations between f−density and (R)−density

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Abstract. In this paper it is discus a relation between f−density and (R)−density. A generalization of Šalát's result concerning this relation in the case of asymptotic density is proved.

1. Introduction

Asymptotic density is a well known means used for measuring of size of sets of positive integers. We remind that the lower and the upper asymptotic densities are special cases of a more general concept of weighted density or (f)−density which is defined as follows.

Denote $\mathbb{R}_0^+$, $\mathbb{N}$ the set of all nonnegative real numbers and positive integers, respectively and let $f : \mathbb{N} \to \mathbb{R}_0^+$ be a (weight) function with $f(1) > 0$ which satisfies

$$\sum_{n=1}^{\infty} f(n) = \infty \tag{D}$$

and

$$\lim_{n\to\infty} \frac{f(n)}{\sum\limits_{i=1}^{n} f(i)} = 0 \tag{L}$$

For $A \subset \mathbb{N}$ we define the lower and upper f−densities of A (these densities are also known as densities with respect to the weight function f or simply as weighted densities):

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$$\underline{d}_f(A) = \liminf_{n\to\infty} \frac{\sum\limits_{m\in A, m\le n} f(m)}{\sum\limits_{m\in \mathbb{N}, m\le n} f(m)}, \quad \overline{d}_f(A) = \limsup_{n\to\infty} \frac{\sum\limits_{m\in A, m\le n} f(m)}{\sum\limits_{m\in \mathbb{N}, m\le n} f(m)}.$$

If $\underline{d}_f(A) = \overline{d}_f(A)$, then we say that the set A has $(f)-$density and we denote this common value as $d_f(A)$. There are two well known special f-densities. The first, when $f(n) = 1$ for each $n \in \mathbb{N}$, is called asymptotic density and their values are denoted as $\underline{d}, \overline{d}$ and d for the lower asymptotic density, upper asymptotic density and asymptotic density , respectively. The second one, when $f(n) = \frac{1}{n}$ for each $n \in \mathbb{N}$, is called logarithmic density and their values are denoted as $\underline{\delta}, \overline{\delta}$ and δ for the lower logarithmic density, upper logarithmic density and logarithmic density , respectively.

Now let us remind the notion of $(R)-$density. For $A \subset \mathbb{N}$ we put $R(A) = \{\frac{a}{b};\ a, b \in A\}$. We say that the set A is $(R)-$dense, if the set $R(A)$ is dense in $\mathbb{R}_0^+$. This concept was introduced in papers [5] and [6] where there were also proved the following relations between $(R)-$density and values of asymptotic density:

(a) $\qquad d(A) > 0 \quad \Rightarrow \quad A$ is (R) – dense,

(b) $\qquad \overline{d}(A) = 1 \quad \Rightarrow \quad A$ is (R) – dense.

These results were later completed in [3] proving

(c) $\qquad \underline{d}(A) > \frac{1}{2} \quad \Rightarrow \quad A$ is (R) – dense.

Notice also that no constant on the left sides of the above three implications can be decreased. A natural question arises whether similar implications hold, perhaps with different constants on the left sides of implications, also for other kinds of f-densities. This question was completely solved for logarithmic densities in [2]. Perhaps a bit surprising result says that all three implications for logarithmic densities hold with constants equal to $\frac{1}{2}$ each, and no one of them can be decreased. As a simple corollary one can see that there is a small chance that the implication (a) holds for some large general class of f-densities. On the other hand, we will see that this is not true in the case of implication (b). Relations between (R)-density and asymptotic densities were also studied, among others, in papers [1] and [4].

Finally, let us notice that the result (b) was in fact proved in a stronger form

(b*) $\qquad \overline{d}(A) = 1 \quad \Rightarrow \quad A$ is a strong quotient base.

Recall that a set $A \subset \mathbb{N}$ is called a strong quotient base if for every rational $\frac{p}{q} \in \mathbb{R}_0^+$ there are infinitely many pairs $(a, b) \in A \times A$ such that $\frac{a}{b} = \frac{p}{q}$.

The aim of this article is to prove a generalization of (b*) for a large class of f-densities and to give some comments to this case.

2. Results

Theorem 2.1. *Let $A \subset \mathbb{N}$ and $\overline{d}_f(A) = 1$ with f non-increasing (and satisfying conditions* (D) *and* (L)*). Then the set A is a strong quotient base.*

Proof: Suppose the contrary, i.e. there exists a rational number $x = \frac{p}{q} \in (0, 1)$ with only finitely many possibilities of expressions of x as a fraction with both denominator and numerator belonging to the set A. Denote $\frac{p}{q} = x = \frac{p_1}{q_1} = \frac{p_2}{q_2} =$

$\dots = \frac{p_n}{q_n}$ all these possibilities having $p < p_1 < p_2 < \dots < p_n$. Then there exists a number $i_0 \in \mathbb{N}$ such that for all $i > i_0$ holds $p_n < q_n < ip < iq$. Obviously for all $i > i_0$ we have

$$ip \notin A \text{ or } iq \notin A. \tag{1}$$

Using conditions (D) and (L) one can easily see that

$$\overline{d}_f(A) = \limsup_{k\to\infty} \frac{\sum\limits_{m\in A, m\le kq} f(m)}{\sum\limits_{m\le kq} f(m)}. \tag{2}$$

Now we will estimate the upper bound of $\overline{d}_f(A)$. For this purpose there is enough to have some convenient estimation of $\sum\limits_{m\in A, m\le kq} f(m)$. We will start this estimation from "the opposite side", i.e. by estimating the sum of values of f of the numbers which are not in the set A. Taking into account that f is non-increasing and (1), we obtain the following inequalities in which we assume that $iq \notin A$ holds for all $i \in \mathbb{N}$, not only for $i > i_0$ (remember that changing the set A in finitely many elements does not affect the value of $\overline{d}_f(A)$).

$$\sum_{m\notin A, m\le kq} f(m) \ge \sum_{i=1}^{k} f(iq) \ge \sum_{i=1}^{k} f(iq+1) \tag{3}$$

Using again the inequalities $f(iq+1) \ge f(iq+2) \ge \dots \ge f(iq+q-1)$, we obtain that the estimation

$$f(iq+1) \ge \frac{1}{q-1} \sum_{m=iq+1}^{iq+q-1} f(m) \tag{4}$$

holds for every $i = 0, 1, \dots$. Denote $S = \sum\limits_{j=1}^{q-1} f(j)$ and realize that (1) yields

$$\sum_{i=1}^{k} \sum_{m=iq+1}^{iq+q-1} f(m) = \sum_{i=0}^{k} \sum_{m=iq+1}^{iq+q-1} f(m) - S \ge \sum_{m\in A, m\le kq+q-1} f(m) - S. \tag{5}$$

All the estimations (3), (4) and (5) together give

$$\sum_{m\notin A, m\le kq} f(m) \ge \frac{1}{q-1} \left(\sum_{m\in A, m\le kq+q-1} f(m) - S \right).$$

This inequality together with (2) yields

$$\overline{d}_f(A) = \limsup_{k\to\infty} \frac{\sum\limits_{m\in A, m\le kq} f(m)}{\sum\limits_{m\notin A, m\le kq} f(m) + \sum\limits_{m\in A, m\le kq} f(m)} \le$$

$$\leq \limsup_{k\to\infty} \frac{\sum\limits_{m\in A, m\leq kq} f(m)}{\frac{1}{q-1}(\sum\limits_{m\in A, m\leq kq} f(m) - S) + \sum\limits_{m\in A, m\leq kq} f(m)} =$$

$$= \frac{q-1}{q} < 1,$$

a contradiction to the assumption $\overline{d}_f(A) = 1$.

□

Remark 2.1. *The theorem would not hold if we assumed non-decreasing f instead of non-increasing f. In this remark we will give an example of $A \subset \mathbb{N}$ which is not a strong quotient base with $\overline{d}_f(A) = 1$ for a non-decreasing f satisfying* (D) *and* (L).

Let the greatest common divisor of $p, q \in \mathbb{N}$ be 1 and $q > 2p$. We will construct a set $A \subset \mathbb{N}$ such that $\frac{p}{q} \notin R(A)$ simply by assuring that (1) holds for all $i \in \mathbb{N}$. When constructing this set, we will need a sequence $(k_n)_{n\in\mathbb{N}\cup\{0\}}$ of integers with the following properties. Let $k_0 = 1$ and

$$(k_n)p > (k_{n-1})q \qquad n = 1, 2, \dots. \tag{6}$$

Notice that this condition assures that $k_n > k_{n-1}$ holds in general, which implies

$$\lim_{n\to\infty} k_n = \infty.$$

We will determine the set A by giving the list of all numbers which are in its complement:

$$p \notin A,$$

$$(k_{2n}+1)p \notin A,\ (k_{2n}+2)p \notin A, \dots (k_{2n+1})p \notin A,$$

$$(k_{2n+1}+1)q \notin A,\ (k_{2n}+2)q \notin A, \dots (k_{2n+2})q \notin A$$

for $n = 0,\ 1,\ 2, \dots.$, i.e.

$$A = \mathbb{N} - \{p\} - \bigcup_{n=0}^{\infty} \left[\left(\bigcup_{i=k_{2n}+1}^{k_{2n+1}} \{ip\} \right) \cup \left(\bigcup_{i=k_{2n+1}+1}^{k_{2n+2}} \{iq\} \right) \right].$$

Properties of the function f are following: firstly, f is constant on $[1,\ (k_1)p] \cap \mathbb{N}$ and on P_l for each $l \in \mathbb{N}$, where P_l is defined as follows:

$$P_l = [(k_{2l-1})p + 1,\ (k_{2l+1})p] \cap \mathbb{N}.$$

This gives us a possibility to compute easily the value of

$$\frac{\sum\limits_{m\in A, m\leq (k_{2n+1})p} f(m)}{\sum\limits_{m\in \mathbb{N}, m\leq (k_{2n+1})p} f(m)}$$

for $n \in \mathbb{N}$ arbitrary. Take into account that in $[1,\ (k_1)p] \cap \mathbb{N}$ there is exactly k_1 numbers which does not belong to the set A. Similarly we obtain that in the set

P_n there are no more than $k_{2n+1} - k_{2n-1}$ positive integers missing in A. Together with the fact that f is constant on P_n for each $n \in \mathbb{N}$, we conclude

$$\frac{\sum\limits_{m \in A, m \leq (k_{2n+1})p} f(m)}{\sum\limits_{m \in \mathbb{N}, m \leq (k_{2n+1})p} f(m)} \geq \frac{(1 - \frac{1}{p})S_n}{S_n}, \tag{7}$$

where

$$S_l = \sum_{m \in \mathbb{N}, m \leq (k_{2l+1})p} f(m)$$

for $l \in \mathbb{N} \cup \{0\}$. Further we set for each $n \in \mathbb{N}$ for each $m \in P_n$

$$f(m) = \frac{1}{ln(k_{2n-1})} S_{n-1}. \tag{8}$$

These selection of values $f(m)$ ensures the requirement of $\overline{d}_f(A) = 1$ and that both (L) and (D) holds. Indeed, concerning the value of $\overline{d}_f(A)$ notice that the interval $[(k_{2n-1})p+1,\ (k_{2n-1})q]$ includes at least k_{2n-1} integers which are all elements of A. This means that the value of

$$\frac{\sum\limits_{m \in A, m \leq n} f(m)}{\sum\limits_{m \in \mathbb{N}, m \leq n} f(m)}$$

as a function of variable n increases on the interval $[(k_{2n-1})p + 1,\ (k_{2n-1})q]$. The value of f on this interval defined in (8) together with the estimation (7) allow us to prove that $\overline{d}_f(A) = 1$:

$$\frac{\sum\limits_{m \in A, m \leq (k_{2n-1})q} f(m)}{\sum\limits_{m \in \mathbb{N}, m \leq (k_{2n-1})q} f(m)} = \frac{\sum\limits_{m \in A, m \leq (k_{2n-1})p} f(m) + \sum\limits_{(k_{2n-1})p+1 \leq m \leq (k_{2n-1})q} f(m)}{\sum\limits_{m \in \mathbb{N}, m \leq (k_{2n-1})p} f(m) + \sum\limits_{(k_{2n-1})p+1 \leq m \leq (k_{2n-1})q} f(m)} \geq$$

$$\geq \frac{(1 - \frac{1}{p})S_{n-1} + \frac{k_{2n-1}}{\ln(k_{2n-1})} S_{n-1}}{S_{n-1} + \frac{k_{2n-1}}{\ln(k_{2n-1})} S_{n-1}},$$

which implies

$$\lim_{n \to \infty} \frac{\sum\limits_{m \in A, m \leq (k_{2n-1})q} f(m)}{\sum\limits_{m \in \mathbb{N}, m \leq (k_{2n-1})q} f(m)} = 1,$$

thus $\overline{d}_f(A) = 1$.

As the next step, we will verify that (L) holds. Due to the fact that f is constant on P_l for each $l \in \mathbb{N}$ there is enough to prove that

$$\lim_{n\to\infty} \frac{f((k_{2n-1})p+1)}{\sum_{i=1}^{(k_{2n-1})p+1} f(i)} = 0.$$

This follows from (8):

$$\frac{f((k_{2n-1})p+1)}{\sum_{i=1}^{(k_{2n-1})p+1} f(i)} \le \frac{f((k_{2n-1})p+1)}{S_{n-1}} = \frac{1}{ln(k_{2n-1})}.$$

Now we shall check whether the function f is non-decreasing. We will compare values $f(m)$ and $f(m+1)$ for $m = (k_{2n+1})p$, where $n \in \mathbb{N}$ using (6), (8) and the assumption that $q > 2p$:

$$f(m) = \frac{S_n - S_{n-1}}{p(k_{2n+1} - k_{2n-1})} \le \frac{S_n}{k_{2n+1} - \frac{1}{4}k_{2n+1}} \le \frac{S_n}{\ln(k_{2n+1})} = f(m+1).$$

Notice finally that (D) holds simply because $f(1) > 0$ and f is non-decreasing.

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